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Abstract

We consider the complete system of equations governing the motion of a general compressible, viscous, electrically and heat conductive fluid driven by non-conservative boundary conditions. We show the existence of a bounded absorbing set in the energy space and asymptotic compactness of trajectories. As a corollary, the set of all entire globally bounded solutions is identified as a natural attractor. Examples of boundary conditions giving rise to unbounded solutions are also discussed.

Keywords: compressible MHD system, Levinson dissipativity, asymptotic compactness, attractor

1 Introduction

Dissipative systems are thermodynamically open systems often in the out of equilibrium regime. They are examples of the most interesting real world phenomena developing patterns, structures, or behaviours that they did not have when first formed. The dissipative systems exchange energy and matter with their environment, while dissipating mechanical and other forms of energy in accordance with the Second law of thermodynamics. In continuum fluid mechanics, dissipative systems are modelled by a family of field equations accompanied with *inhomogeneous* boundary conditions.

Many spectacular events occurring in geophysics or astrophysics, among which the dynamics of the solar convection zone, are attributed to the action of the magnetic field, see e.g. Thompson

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and Christensen-Dalsgaard [35]. As direct measurements and data collecting are often hampered by obvious technical limitations, reliable mathematical models are indispensable for understanding the observed phenomena. As pointed out by Gough [23], a correct choice of boundary conditions plays a crucial role.

The state of a viscous, compressible, electrically and heat conducting fluid is characterized by its mass density $\varrho = \varrho(t, x)$, the (absolute) temperature $\vartheta = \vartheta(t, x)$, the velocity $\mathbf{u} = \mathbf{u}(t, x)$ and the magnetic field $\mathbf{B} = \mathbf{B}(t, x)$. The time evolution is governed by the system of field equations of *compressible magnetohydrodynamics (MHD)*, see e.g. Weiss and Proctor [36]:

Equation of continuity:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0. \quad (1.1)$$

Momentum equation:

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + (\boldsymbol{\omega} \times \varrho \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \operatorname{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \operatorname{curl}_x \mathbf{B} \times \mathbf{B} + \varrho \nabla_x M. \quad (1.2)$$

Induction equation:

$$\partial_t \mathbf{B} + \operatorname{curl}_x(\mathbf{B} \times \mathbf{u}) + \operatorname{curl}_x(\zeta(\vartheta) \operatorname{curl}_x \mathbf{B}) = 0, \quad \operatorname{div}_x \mathbf{B} = 0. \quad (1.3)$$

Internal energy balance:

$$\begin{aligned} \partial_t(\varrho e(\varrho, \vartheta)) + \operatorname{div}_x(\varrho e(\varrho, \vartheta) \mathbf{u}) + \operatorname{div}_x \mathbf{q}(\vartheta, \nabla_x \vartheta) \\ = \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \zeta(\vartheta) |\operatorname{curl}_x \mathbf{B}|^2 - p(\varrho, \vartheta) \operatorname{div}_x \mathbf{u}. \end{aligned} \quad (1.4)$$

The system is written in a general *rotating frame* frequently used in astro/geophysics, where $\boldsymbol{\omega} \times \varrho \mathbf{u}$ is the Coriolis force and the potential M ,

$$M = G + \frac{1}{2} |\boldsymbol{\omega} \times x|^2,$$

includes the gravitational component $G = G(t, x)$ as well as the centrifugal force. Note that G may depend on the time t in the rotating frame if the source of gravitation is located outside the fluid domain.

We consider a *Newtonian fluid*, with the viscous stress tensor

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \operatorname{div}_x \mathbf{u} \mathbb{I}, \quad (1.5)$$

where the viscosity coefficients $\mu > 0$ and $\eta \geq 0$ are continuously differentiable functions of the temperature. Similarly, the heat flux obeys *Fourier's law*,

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta, \quad (1.6)$$

where the heat conductivity coefficient $\kappa > 0$ is a continuously differentiable function of the temperature.

As pointed out in the introduction, the boundary conditions play a crucial role in the long time behaviour of the system. We suppose that the fluid occupies a bounded domain $\Omega \subset R^3$ with a smooth boundary,

$$\begin{aligned} \partial\Omega &= \Gamma_D^{\mathbf{u}} \cup \Gamma_N^{\mathbf{u}} = \Gamma_D^{\vartheta} \cup \Gamma_N^{\vartheta} = \Gamma_D^{\mathbf{B}} \cup \Gamma_N^{\mathbf{B}}, \\ \Gamma_D^{\mathbf{u}}, \Gamma_N^{\mathbf{u}}, \Gamma_D^{\vartheta}, \Gamma_N^{\vartheta}, \Gamma_D^{\mathbf{B}}, \Gamma_N^{\mathbf{B}} &\text{ compact,} \\ \Gamma_D^{\mathbf{u}} \cap \Gamma_N^{\mathbf{u}} &= \emptyset, \Gamma_D^{\vartheta} \cap \Gamma_N^{\vartheta} = \emptyset, \Gamma_D^{\mathbf{B}} \cap \Gamma_N^{\mathbf{B}} = \emptyset. \end{aligned} \quad (1.7)$$

Accordingly, each Γ_D^* , Γ_N^* is either empty or coincides with a finite union of connected components of $\partial\Omega$. We impose the following boundary conditions:

Boundary velocity:

$$\mathbf{u}|_{\Gamma_D^{\mathbf{u}}} = 0, \quad (1.8)$$

$$\mathbf{u} \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{u}}} = 0, \quad [\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\Gamma_N^{\mathbf{u}}} = 0. \quad (1.9)$$

Boundary temperature/heat flux:

$$\vartheta|_{\Gamma_D^{\vartheta}} = \vartheta_B, \quad (1.10)$$

$$\nabla_x \mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n}|_{\Gamma_N^{\vartheta}} = 0. \quad (1.11)$$

Boundary magnetic field:

$$\mathbf{B} \times \mathbf{n}|_{\Gamma_D^{\mathbf{B}}} = \mathbf{b}_\tau, \quad (1.12)$$

$$\mathbf{B} \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{B}}} = b_\nu, \quad [(\mathbf{B} \times \mathbf{u}) + \zeta \mathbf{curl}_x \mathbf{B}] \times \mathbf{n}|_{\Gamma_N^{\mathbf{B}}} = 0. \quad (1.13)$$

Clearly, the boundary conditions prescribed on Γ_D^* are of Dirichlet type, while those on Γ_N^* are of Neumann type. The fluxes on Γ_N^* are set to be zero for the sake of simplicity. More general fluxes are shortly discussed in Section 5.3. In accordance with (1.8), (1.9), the boundary is impermeable, meaning the fluid velocity is always tangent to the boundary. In particular, the total mass

$$m_0 = \int_{\Omega} \varrho(t, \cdot) \, dx \quad (1.14)$$

is a constant of motion. Accordingly, there is no exchange of mass with the outer world and the system is therefore driven by the imposed boundary temperature and/or magnetic field.

1.1 Levinson dissipativity

The compressible MHD system admits a natural energy

$$E(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) = \underbrace{\frac{1}{2}\varrho|\mathbf{u}|^2}_{\text{kinetic energy}} + \underbrace{\varrho e(\varrho, \vartheta)}_{\text{internal energy}} + \underbrace{\frac{1}{2}|\mathbf{B}|^2}_{\text{magnetic energy}}.$$

Our goal is to show that the system is dissipative in the sense of Levinson in terms of the total energy

$$\mathcal{E} = \int_{\Omega} E(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) \, dx.$$

Specifically, there exists a universal constant \mathcal{E}_{∞} such that

$$\limsup_{\tau \rightarrow \infty} \int_{\Omega} E(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau, \cdot) \, dx < \mathcal{E}_{\infty} \quad (1.15)$$

for *any* solution $(\varrho, \vartheta, \mathbf{u}, \mathbf{B})$ of the compressible MHD system defined on (T, ∞) . We point out that \mathcal{E}_{∞} depends only on the “data” but it is the same for any global trajectory. In particular, it is independent of the initial energy of the system.

The data are:

- the total mass m_0 of the fluid, cf. (1.14);
- the gravitational potential G ;
- the rotation vector $\boldsymbol{\omega}$;
- the boundary temperature ϑ_B ;
- the boundary tangential magnetic field \mathbf{b}_{τ} , the boundary normal magnetic field b_{ν} .

We shall use the symbol $\|(\text{data})\|$ to denote the norm of the above data in suitable function spaces specified below.

Besides the total mass m_0 , there might be other conserved quantities. As shown e.g. by Bauer, Pauly, and Schomburg [4], see also Kozono and Yanagisawa [25], any vector field \mathbf{b} defined on Ω admits a decomposition

$$\mathbf{b} = \nabla_x P + \mathbf{h} + \mathbf{curl}_x \mathbf{A}, \quad (1.16)$$

where

$$\mathbf{h} \in \mathcal{H}(\Omega) = \left\{ \mathbf{h} \in L^2(\Omega; R^3) \mid \mathbf{curl}_x \mathbf{h} = \mathbf{div}_x \mathbf{h} = 0, \mathbf{h} \times \mathbf{n}|_{\Gamma_{\mathbb{B}}} = 0, \mathbf{h} \cdot \mathbf{n}|_{\Gamma_{\mathbb{N}}} = 0 \right\}. \quad (1.17)$$

It follows from the induction equation (1.3) and our choice of the boundary conditions (1.12), (1.13) that

$$\frac{d}{dt} \int_{\Omega} \mathbf{B} \cdot \mathbf{h} \, dx = 0 \text{ for any } \mathbf{h} \in \mathcal{H}(\Omega).$$

For the sake of simplicity, we shall assume

$$\int_{\Omega} \mathbf{B} \cdot \mathbf{h} \, dx = 0 \text{ for all } \mathbf{h} \in \mathcal{H}(\Omega). \quad (1.18)$$

The general case can be handled in a similar way provided the projection of \mathbf{B} on the space $\mathcal{H}(\Omega)$ is included in the “data”. Note that the space $\mathcal{H}(\Omega)$ is always finite dimensional, see [4, Section 5.1]. Moreover, it is easy to see that $\mathcal{H} = \{0\}$ whenever Ω is simply connected and $\Gamma_D^{\mathbf{B}}$ connected. If $\Gamma_N^{\mathbf{B}} = \partial\Omega$ or $\Gamma_D^{\mathbf{B}} = \partial\Omega$, the dimension of $\mathcal{H}(\Omega)$ coincides with the first and second Betti number of Ω , respectively, see Kozono and Yanagisawa [26].

In view of (1.15), the boundary conditions must allow the outflow of the thermal energy. We suppose the pressure $p = p(\varrho, \vartheta)$ and the internal energy $e = e(\varrho, \vartheta)$ are interrelated through *Gibbs’ equation*

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right), \quad (1.19)$$

where $s = s(\varrho, \vartheta)$ is the entropy. Consequently, the internal energy balance (1.4) may be reformulated in the form of *entropy balance equation*

$$\begin{aligned} \partial_t(\varrho s(\varrho, \vartheta)) + \operatorname{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \operatorname{div}_x\left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta}\right) \\ = \frac{1}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} + \zeta(\vartheta) |\operatorname{curl}_x \mathbf{B}|^2 \right), \end{aligned} \quad (1.20)$$

see e.g. Weiss and Proctor [36]. The quantity on the right-hand side of (1.20) represents the entropy production rate, and, in accordance with the Second law of thermodynamics, it is always non-negative. Consequently, all forms of energy are eventually transformed to heat that must be allowed to leave through $\partial\Omega$. Thus, necessarily,

$$\Gamma_D^{\vartheta} \neq \emptyset. \quad (1.21)$$

Next, to avoid development of rapidly rotating fluid in a rotationally symmetric domain, we impose

$$\Gamma_D^{\mathbf{u}} \neq \emptyset. \quad (1.22)$$

Finally, we impose a technical condition to control the boundary magnetic oscillations. First, we introduce the class of stationary magnetic boundary data.

Definition 1.1. [Stationary magnetic field]

We say that the boundary data b_ν, \mathbf{b}_τ are *stationary*, if there exists a continuously differ-

entiable vector field \mathbf{B}_B such that

$$\operatorname{div}_x \mathbf{B}_B = 0, \quad \operatorname{curl}_x \mathbf{B}_B = 0 \text{ in } \Omega, \quad \mathbf{B}_B \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{B}}} = b_\nu, \quad \mathbf{B}_B \times \mathbf{n}|_{\Gamma_D^{\mathbf{B}}} = \mathbf{b}_\tau.$$

Stationarity imposes certain restrictions to be satisfied by \mathbf{b}_τ ,

$$\operatorname{div}_\tau \mathbf{b}_\tau = 0 \text{ on } \Gamma_D^{\mathbf{B}}, \text{ where } \operatorname{div}_\tau \text{ denotes the tangential divergence,} \quad (1.23)$$

see Auchmuty and Alexander [2]. To establish the existence of a bounded absorbing set, we need an extra hypothesis imposed on the boundary data if the magnetic boundary field *is not* stationary, namely

$$\Gamma_D^{\mathbf{B}} \subset \Gamma_D^{\mathbf{u}}. \quad (1.24)$$

In accordance with Kirchhoff's rule for currents, we have

$$\operatorname{div}_\tau \mathbf{b}_\tau + \operatorname{curl}_x \mathbf{B} \cdot \mathbf{n} = 0 \text{ on } \Gamma_D^{\mathbf{B}},$$

whenever

$$\operatorname{div}_x \mathbf{B} = 0 \text{ in } \Omega, \quad \mathbf{B} \times \mathbf{n}|_{\Gamma_D^{\mathbf{B}}} = \mathbf{b}_\tau,$$

see Alexander and Auchmuty [2]. Thus prescribing a non-stationary vector field \mathbf{b}_τ yields an inhomogeneous boundary condition satisfied by the normal component of the electric current \mathbf{J} proportional to $\operatorname{curl}_x \mathbf{B}$, cf. Section 5.4.

1.2 Absence of static solutions - an example from astrophysics

Although all global in time solutions eventually enter a bounded absorbing ball, it is worth noting that the system in question may not admit a static stationary solutions, meaning

$$\mathbf{u} = 0, \quad \varrho = \varrho(x), \quad \vartheta = \vartheta(x), \quad \mathbf{B} = \mathbf{B}(x).$$

Indeed consider a simple model of a solar convective zone discussed by Tao et al. [34]. The domain boundary is determined by two concentric balls,

$$\Omega = \left\{ x \in R^3 \mid 0 < r_1 < |x| < r_2 \right\}, \quad (1.25)$$

with the gravitational potential $G = \bar{g} \frac{1}{|x|}$. The boundary temperature distribution is determined by two positive constants $\Theta_{\text{int}}, \Theta_{\text{ext}}$,

$$\vartheta_B = \Theta_{\text{int}} \text{ if } |x| = r_1, \quad \vartheta_B = \Theta_{\text{ext}} \text{ if } |x| = r_2, \quad 0 < \Theta_{\text{ext}} < \Theta_{\text{int}}. \quad (1.26)$$

Finally, the boundary magnetic field is oriented in the normal direction,

$$\Gamma_N^{\mathbf{B}} = \emptyset, \quad \mathbf{b}_\tau = 0. \quad (1.27)$$

The whole system rotates with the rotation axis $\boldsymbol{\omega} \neq 0$.

Suppose the system (1.1)–(1.4) endowed with the above boundary conditions admits a static solution, meaning

$$\begin{aligned}\nabla_x p(\varrho, \vartheta) &= \varrho \nabla_x M + \mathbf{curl}_x \mathbf{B} \times \mathbf{B}, \\ \mathbf{curl}_x (\zeta(\vartheta) \mathbf{curl}_x \mathbf{B}) &= 0, \quad \operatorname{div}_x \mathbf{B} = 0, \\ \operatorname{div}_x \mathbf{q}(\vartheta, \nabla_x \vartheta) &= \zeta(\vartheta) |\mathbf{curl}_x \mathbf{B}|^2.\end{aligned}\tag{1.28}$$

The first observation is

$$\mathbf{curl}_x \mathbf{B} = 0,$$

which follows easily via multiplying the induction equation on \mathbf{B} and integrating by parts. Consequently, the system reduces to

$$\nabla_x p(\varrho, \vartheta) = \varrho \nabla_x M, \quad \operatorname{div}_x \mathbf{q}(\vartheta, \nabla_x \vartheta) = 0.\tag{1.29}$$

Applying \mathbf{curl}_x to the first equation, we get

$$\nabla_x \varrho \times \nabla_x M = 0.\tag{1.30}$$

Similarly, writing

$$\nabla_x p(\varrho, \vartheta) = \partial_\varrho p(\varrho, \vartheta) \nabla_x \varrho + \partial_\vartheta p(\varrho, \vartheta) \nabla_x \vartheta = \varrho \nabla_x M$$

we deduce from (1.30) $\nabla_x \varrho \times \nabla_x \vartheta = 0$ anticipating the condition

$$\frac{\partial p(\varrho, \vartheta)}{\partial \vartheta} \neq 0.\tag{1.31}$$

We conclude

$$\nabla_x M \times \nabla_x \vartheta = 0.\tag{1.32}$$

Since \mathbf{q} is given by Fourier's law $\mathbf{q} = -\kappa(\vartheta) \nabla_x \vartheta$, the unique solution ϑ of $\operatorname{div}_x \mathbf{q}(\varrho, \nabla_x \vartheta) = 0$ with the boundary conditions (1.26) admits a non-zero gradient $\nabla_x \vartheta$ parallel to the gravitational force

$$\nabla_x G \times \nabla_x \vartheta = 0.$$

Consequently, condition (1.32) implies

$$\nabla_x \vartheta \times \nabla_x |\boldsymbol{\omega} \times x|^2 = 0,$$

which is impossible as soon as $\boldsymbol{\omega} \neq 0$.

1.3 Asymptotic compactness

The existence of global-in-time *classical* solutions for the compressible MHD system in the far from equilibrium regime is not known. Here, we adopt the concept of *weak solution* developed in [14]. Unfortunately, the weak solutions may not be uniquely determined by the initial/boundary data. Adapting the approach of Sell [31], see also Málek and Nečas [29], we consider solutions of the compressible MHD system in the space of *trajectories* with a natural group operation of forward time shift:

$$(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) \mapsto (\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t + \tau), \quad \tau \geq 0.$$

Anticipating the existence of a bounded absorbing set, we consider a sequence of global in time solutions $(\varrho_n, \vartheta_n, \mathbf{u}_n, \mathbf{B}_n)$ along with a sequence of time shifts $\tau_n \rightarrow \infty$. We say the system is *asymptotically compact* if

$$(\varrho_n, \vartheta_n, \mathbf{u}_n, \mathbf{B}_n)(\cdot + \tau_n) \rightarrow (\varrho, \vartheta, \mathbf{u}, \mathbf{B}) \text{ locally on compact time intervals } (-L, L),$$

where $(\varrho, \mathbf{u}, \vartheta, \mathbf{B})$ is a solution of the same problem with suitable data defined for $t \in (-\infty, \infty)$. Here “suitable data” means limits of the corresponding time shifts of the original data. In particular, the problem is the same if the data are independent of time.

Solutions defined for all $t \in (-\infty, \infty)$ are called *entire solutions*. The set of all bounded entire solutions,

$$\mathcal{A} = \left\{ (\varrho, \vartheta, \mathbf{u}, \mathbf{B}) \mid \begin{array}{l} (\varrho, \vartheta, \mathbf{u}, \mathbf{B}) - \text{solution of the compressible (MHD) system for } t \in (-\infty, \infty) \\ \int_{\Omega} E(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) \, dx < \mathcal{E}_{\infty} \end{array} \right\}$$

is shift invariant and captures the essential features of the long time behaviour of the system.

1.4 Organization of the paper

The results of the present paper can be seen as a continuation of an ongoing research programme [12], [18], [19] extending the classical theory of the dynamics of incompressible viscous fluid flows (see e.g. Constantin et al. [6], [7], Foias et al. [20]) to the compressible case. To the best of our knowledge, the present paper is the first result addressing the problem of long-time behaviour of the compressible MHD system far from equilibrium. There is a large number of literature concerning the incompressible case, see e.g. Duvaut and Lions [10], Eden and Libin [11], Sermange and Temam [32], Jiang and Jiang [24] and the references therein.

The concept of weak solution and other preliminaries are introduced in Section 2. The bulk of the paper consists of three parts. In Section 3, we show the existence of a bounded absorbing set. Section 4 contains the proof of asymptotic compactness. Applications and possible extensions are discussed in Section 5.

2 Main hypothesis, weak solutions

Before introducing the concept of weak solution, let us state the main hypotheses concerning the structural properties of the constitutive relations.

2.1 Equation of state

The hypotheses imposed on the form of the equations of state are the same as in [15, Chapter 4]. They are based on the Second law of thermodynamics enforced through Gibbs' relation (1.19) and the *hypothesis of thermodynamics stability*

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0. \quad (2.1)$$

We suppose the pressure obeys the *thermal equation of state*

$$p(\varrho, \vartheta) = p_M(\varrho, \vartheta) + p_R(\vartheta), \quad p_R(\vartheta) = \frac{a}{3}\vartheta^4, \quad a > 0, \quad (2.2)$$

where p_R is the radiation pressure. Augmenting the pressure by a radiation component is not only technically convenient but also relevant to problems in astrophysics, cf. Battaner [3]. The internal energy satisfies the *caloric equation of state*

$$e(\varrho, \vartheta) = e_M(\varrho, \vartheta) + e_R(\varrho, \vartheta), \quad e_R(\varrho, \vartheta) = \frac{a}{\varrho}\vartheta^4. \quad (2.3)$$

The gas pressure components p_M and e_M satisfy the relation characteristic for monoatomic gases:

$$p_M(\varrho, \vartheta) = \frac{2}{3}\varrho e_M(\varrho, \vartheta). \quad (2.4)$$

Now, it follows from Gibbs' relation (1.19) applied to p_M , e_M that they must take a general form

$$p_M(\varrho, \vartheta) = \vartheta^{\frac{5}{2}}P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right), \quad e_M(\varrho, \vartheta) = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho}P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) \quad (2.5)$$

for some the function $P \in C^1[0, \infty)$. In addition, the hypothesis of thermodynamics stability (2.1) applied to p_M , e_M yields

$$P(0) = 0, \quad P'(Z) > 0, \quad \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} > 0 \text{ for all } Z \geq 0. \quad (2.6)$$

In accordance with (2.5), the entropy takes the form

$$s(\varrho, \vartheta) = s_M(\varrho, \vartheta) + s_R(\varrho, \vartheta), \quad s_M(\varrho, \vartheta) = \mathcal{S}\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right), \quad s_R(\varrho, \vartheta) = \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \quad (2.7)$$

where

$$\mathcal{S}'(Z) = -\frac{3\frac{5}{3}P(Z) - P'(Z)Z}{2Z^2}. \quad (2.8)$$

Finally, we impose two technical but physically grounded hypotheses in the degenerate area

$$\frac{\varrho}{\vartheta^{\frac{3}{2}}} = Z \gg 1.$$

First, it follows from (2.6) that the function $Z \mapsto P(Z)/Z^{\frac{5}{3}}$ is decreasing, and we suppose

$$\lim_{Z \rightarrow \infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \quad (2.9)$$

Second, by the same token, the function $Z \mapsto \mathcal{S}(Z)$ decreasing, and we suppose

$$\lim_{Z \rightarrow \infty} \mathcal{S}(Z) = 0. \quad (2.10)$$

Hypothesis (2.9) reflects the effect of the electron pressure in the degenerate area, while (2.10) is nothing other than the Third law of thermodynamics.

It follows from (2.5)–(2.10) that

$$\begin{aligned} p(\varrho, \vartheta) &\approx \varrho e(\varrho, \vartheta), \\ \varrho^{\frac{5}{3}} + \vartheta^4 &\lesssim p(\varrho, \vartheta) \lesssim \varrho^{\frac{5}{3}} + \vartheta^4 + 1, \\ 0 \leq \varrho s(\varrho, \vartheta) &\lesssim \vartheta^3 + \varrho (1 + [\log \varrho]^+ + [\log \vartheta]^+). \end{aligned} \quad (2.11)$$

Here and hereafter, the symbol $a \lesssim b$ means there is a positive constant c such that $a \leq cb$, the symbol $a \approx b$ is used to denote $a \lesssim b$ and $b \lesssim a$. The reader may consult [15, Chapter 4] for the physical background of the above hypotheses as well as the proof of (2.11).

2.2 Diffusion and the transport coefficients

The transport coefficients appearing in (1.5), (1.6) satisfy

$$\begin{aligned} 0 < \underline{\mu}(1 + \vartheta) &\leq \mu(\vartheta) \leq \bar{\mu}(1 + \vartheta), \quad |\mu'(\vartheta)| \leq c \text{ for all } \vartheta \geq 0, \\ 0 &\leq \eta(\vartheta) \leq \bar{\eta}(1 + \vartheta), \end{aligned} \quad (2.12)$$

and

$$0 < \underline{\kappa}(1 + \vartheta^\beta) \leq \kappa(\vartheta) \leq \bar{\kappa}(1 + \vartheta^\beta) \text{ for some } \beta > 6. \quad (2.13)$$

Finally, we suppose the coefficient of magnetic diffusion $\zeta = \zeta(\vartheta)$ is a continuously differentiable function of the temperature,

$$0 < \underline{\zeta}(1 + \vartheta) \leq \zeta(\vartheta) \leq \bar{\zeta}(1 + \vartheta), \quad |\zeta'(\vartheta)| \leq c \text{ for all } \vartheta \geq 0. \quad (2.14)$$

2.3 Boundary data

It is convenient to assume the boundary data are given as restrictions of functions defined on Ω and for any $t \in R$. Specifically,

$$\vartheta_B = \tilde{\vartheta}|_{\Gamma_D^\vartheta}, \quad \mathbf{b}_\tau = \mathbf{B}_B \times \mathbf{n}|_{\Gamma_D^\mathbf{B}}, \quad b_\nu = \mathbf{B}_B \cdot \mathbf{n}|_{\Gamma_N^\mathbf{B}}$$

for suitable extensions $\tilde{\vartheta}$, \mathbf{B}_B . Accordingly, certain regularity of the boundary data is necessary.

We suppose the boundary temperature $\vartheta_B = \vartheta_B(t, x)$ belongs to the class

$$\begin{aligned} \vartheta_B &\in BC(R; C^2(\Gamma_D^\vartheta)), \quad \partial_t \vartheta_B \in BC(R; C^1(\Gamma_D^\vartheta)), \\ \inf_{R \times \Gamma_D^\vartheta} \vartheta_B &> 0. \end{aligned} \quad (2.15)$$

Similarly, we suppose

$$\mathbf{b}_\tau \in BC(R; C^2(\Gamma_D^\mathbf{B}; R^3)), \quad \partial_t \mathbf{b}_\tau \in BC(R; C^1(\Gamma_D^\mathbf{B}; R^3)) \quad (2.16)$$

$$b_\nu \in BC(R; C^2(\Gamma_N^\mathbf{B})), \quad \partial_t b_\nu \in BC(R; C^1(\Gamma_N^\mathbf{B})). \quad (2.17)$$

Moreover, we impose the necessary compatibility conditions

$$\mathbf{b}_\tau \cdot \mathbf{n}|_{\Gamma_D^\mathbf{B}} = 0, \quad \text{and} \quad \int_{\partial\Omega} b_\nu \, d\sigma = 0 \quad \text{if} \quad \Gamma_D^\mathbf{B} = \emptyset. \quad (2.18)$$

Finally, we introduce the space

$$H_{0,\sigma} = \left\{ \mathbf{b} \in L^2(\Omega; R^3) \mid \mathbf{curl}_x \mathbf{b} \in L^2(\Omega; R^3), \operatorname{div}_x \mathbf{b} = 0, \mathbf{b} \times \mathbf{n}|_{\Gamma_D^\mathbf{B}} = 0, \mathbf{b} \cdot \mathbf{n}|_{\Gamma_N^\mathbf{B}} = 0 \right\}. \quad (2.19)$$

The space $H_{0,\sigma}$ is a Hilbert space with the norm $\|\mathbf{b}\|_{H_0}^2 = \|\mathbf{curl}_x \mathbf{b}\|_{L^2(\Omega; R^3)}^2 + \|\mathbf{b}\|_{L^2(\Omega; R^3)}^2$. The space \mathcal{H} introduced in (1.17) is a closed subspace of $H_{0,\sigma}$, and the following version of Poincaré inequality

$$\|\mathbf{b}\|_{W^{1,2}(\Omega; R^3)} \lesssim \|\mathbf{curl}_x \mathbf{b}\|_{L^2(\Omega; R^3)} \quad \text{holds for all} \quad \mathbf{b} \in \mathcal{H}^\perp \cap H_{0,\sigma}, \quad (2.20)$$

see Csató, Kneuss and Rajendran [8, Theorem 2.1].

The specific extensions of the boundary data will be constructed in Section 3.1.

2.4 Weak solutions

As we are interested in the long-time behaviour when the system “forgets” its initial state, the choice of initial data plays no role in the analysis. Accordingly, it is convenient to introduce the concept of global in time weak solutions defined for $t \in (T, \infty)$.

Definition 2.1. (Global in time weak solutions)

A quantity $(\varrho, \vartheta, \mathbf{u}, \mathbf{B})$ is termed *weak solution* of the compressible MHD system (1.1)–(1.4), with the boundary conditions (1.8) – (1.13) in the time–space cylinder $(T, \infty) \times \Omega$ if the following holds:

- **Equation of continuity.** $\varrho \in L_{\text{loc}}^\infty(T, \infty; L^{\frac{5}{3}}(\Omega))$, $\varrho \geq 0$, and the integral identity

$$\int_T^\infty \int_\Omega \left(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right) dx dt = 0 \quad (2.21)$$

holds for any $\varphi \in C_c^1((T, \infty) \times \bar{\Omega})$. In addition, the renormalized version of (2.21)

$$\int_T^\infty \int_\Omega \left(b(\varrho) \partial_t \varphi + b(\varrho) \mathbf{u} \cdot \nabla_x \varphi + \left(b(\varrho) - b'(\varrho) \varrho \right) \text{div}_x \mathbf{u} \varphi \right) dx dt = 0 \quad (2.22)$$

holds for any $\varphi \in C_c^1((T, \infty) \times \bar{\Omega})$ and any $b \in C^1(R)$, $b' \in C_c(R)$.

- **Momentum equation.**

$\varrho \mathbf{u} \in L_{\text{loc}}^\infty(T, \infty; L^{\frac{5}{4}}(\Omega; R^3))$, $\mathbf{u} \in L_{\text{loc}}^2(T, \infty; W^{1,2}(\Omega; R^3))$, $\mathbf{u}|_{\Gamma_D^{\mathbf{u}}} = 0$, $\mathbf{u} \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{u}}} = 0$, and the integral identity

$$\begin{aligned} & \int_T^\infty \int_\Omega \left(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + (\varrho \mathbf{u} \times \boldsymbol{\omega}) \cdot \boldsymbol{\varphi} + p(\varrho, \vartheta) \text{div}_x \boldsymbol{\varphi} \right) dx \\ &= \int_0^\tau \int_\Omega \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} dx dt - \int_T^\infty \int_\Omega \left(\mathbf{B} \otimes \mathbf{B} - \frac{1}{2} |\mathbf{B}|^2 \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} dx dt \\ & - \int_T^\infty \int_\Omega \varrho \nabla_x M \cdot \boldsymbol{\varphi} dx dt \end{aligned} \quad (2.23)$$

for any $\boldsymbol{\varphi} \in C_c^1((T, \infty) \times \bar{\Omega}; R^3)$, $\boldsymbol{\varphi}|_{\Gamma_D^{\mathbf{u}}} = 0$, $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{u}}} = 0$.

- **Induction equation.** $\mathbf{B} \in L_{\text{loc}}^\infty((T, \infty); L^2(\Omega; R^3))$,

$$\text{div}_x \mathbf{B}(\tau, \cdot) = 0 \text{ for any } \tau \in (T, \infty), \quad (2.24)$$

$$(\mathbf{B} - \mathbf{B}_B) \in L_{\text{loc}}^2(T, \infty; H_{0,\sigma}(\Omega; R^3)). \quad (2.25)$$

The integral identity

$$\int_T^\infty \int_\Omega \left(\mathbf{B} \cdot \partial_t \boldsymbol{\varphi} - (\mathbf{B} \times \mathbf{u}) \cdot \text{curl}_x \boldsymbol{\varphi} - \zeta(\vartheta) \text{curl}_x \mathbf{B} \cdot \text{curl}_x \boldsymbol{\varphi} \right) dx dt = 0 \quad (2.26)$$

holds for any $\varphi \in C_c^1((T, \infty) \times \bar{\Omega}; R^3)$,

$$\boldsymbol{\varphi} \times \mathbf{n}|_{\Gamma_D^{\mathbf{B}}} = 0, \quad \boldsymbol{\varphi} \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{B}}} = 0. \quad (2.27)$$

- **Entropy inequality.** $\vartheta \in L_{\text{loc}}^\infty(T, \infty; L^4(\Omega)) \cap L_{\text{loc}}^2(T, \infty; W^{1,2}(\Omega))$, $\vartheta > 0$ a.a. in $(T, \infty) \times \Omega$, $\log(\vartheta) \in L_{\text{loc}}^2(T, \infty; W^{1,2}(\Omega))$,

$$\vartheta|_{\Gamma_D^\vartheta} = \vartheta_B.$$

The integral inequality

$$\begin{aligned} & \int_T^\infty \int_\Omega \left(\varrho s(\varrho, \vartheta) \partial_t \varphi + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \varphi \right) dx dt \\ & \leq - \int_T^\infty \int_\Omega \frac{\varphi}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} + \zeta(\vartheta) |\mathbf{curl}_x \mathbf{B}|^2 \right) dx dt \end{aligned} \quad (2.28)$$

holds for any $\varphi \in C_c^1((T, \infty) \times \bar{\Omega})$, $\varphi \geq 0$, $\varphi|_{\Gamma_D^\vartheta} = 0$.

- **Ballistic energy inequality.** The inequality

$$\begin{aligned} & \int_T^\infty \partial_t \psi \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + \frac{1}{2} |\mathbf{B}|^2 - \tilde{\vartheta} \varrho s(\varrho, \vartheta) - \mathbf{B}_B \cdot \mathbf{B} \right) dx dt \\ & - \int_T^\infty \psi \int_\Omega \frac{\tilde{\vartheta}}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} + \zeta(\vartheta) |\mathbf{curl}_x \mathbf{B}|^2 \right) dx dt \\ & \geq \int_T^\infty \psi \int_\Omega \left(\varrho s(\varrho, \vartheta) \partial_t \tilde{\vartheta} + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \tilde{\vartheta} + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \tilde{\vartheta} \right) dx dt \\ & + \int_T^\infty \psi \int_\Omega \left(\mathbf{B} \cdot \partial_t \mathbf{B}_B - (\mathbf{B} \times \mathbf{u}) \cdot \mathbf{curl}_x \mathbf{B}_B - \zeta(\vartheta) \mathbf{curl}_x \mathbf{B} \cdot \mathbf{curl}_x \mathbf{B}_B \right) dx dt \\ & - \int_T^\infty \psi \int_\Omega \varrho \nabla_x M \cdot \mathbf{u} dx dt \end{aligned} \quad (2.29)$$

holds for any $\psi \in C_c^1(T, \infty)$, $\psi \geq 0$, and any $\tilde{\vartheta} \in C_c^1((T, \infty) \times \bar{\Omega})$, $\tilde{\vartheta} > 0$, $\tilde{\vartheta}|_{\Gamma_D^\vartheta} = \vartheta_B$.

Remark 2.2. Note carefully that the ballistic energy inequality (2.29) remains valid if we replace \mathbf{B}_B by any other extension $\tilde{\mathbf{B}}$ such that

$$\text{div}_x \tilde{\mathbf{B}} = 0, \quad \tilde{\mathbf{B}} \times \mathbf{n}|_{\Gamma_D^{\mathbf{B}}} = \mathbf{B}_B \times \mathbf{n}|_{\Gamma_D^{\mathbf{B}}}, \quad \tilde{\mathbf{B}} \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{B}}} = \mathbf{B}_B \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{B}}},$$

respectively. Indeed the difference $\mathbf{B}_B - \tilde{\mathbf{B}}$ becomes an eligible test function for the weak formulation of the induction equation (2.26), (2.27).

The existence of global-in-time weak solutions as well as the weak-strong uniqueness property for any finite energy initial data was shown in [14] under more restrictive hypotheses on the boundary data.

3 Bounded absorbing set

We are ready to state the first main result of the present paper concerning the existence of a bounded absorbing set. It is convenient to suppose that both the potential G as well as the boundary data ϑ_B , \mathbf{b}_τ , and b_ν are defined for all $t \in R$. Accordingly, we introduce

$$\begin{aligned}
\|(\text{data})\| &= m_0 + m_0^{-1} + \|G\|_{W^{1,\infty}(R \times \Omega)} + |\boldsymbol{\omega}| + \|\vartheta_B^{-1}\|_{L^\infty(R \times \partial\Gamma_D^\vartheta)} \\
&+ \sup_{t \in R} \|\vartheta_B(t, \cdot)\|_{W^{2,\infty}(\Gamma_D^\vartheta)} + \sup_{t \in R} \|\partial_t \vartheta_B\|_{W^{1,\infty}(\Gamma_D^\vartheta)} \\
&+ \sup_{t \in R} \|\mathbf{b}_\tau(t, \cdot)\|_{W^{2,\infty}(\Gamma_D^{\mathbf{B}}; R^3)} + \sup_{t \in R} \|b_\nu\|_{W^{2,\infty}(\Gamma_N^{\mathbf{B}})} \\
&+ \sup_{t \in R} \|\partial_t \mathbf{b}_\tau(t, \cdot)\|_{W^{1,\infty}(\Gamma_D^{\mathbf{B}}; R^3)} + \sup_{t \in R} \|\partial_t b_\nu\|_{W^{1,\infty}(\Gamma_N^{\mathbf{B}})}. \tag{3.1}
\end{aligned}$$

In what follows, we shall use the symbol $c(\|(\text{data})\|)$ to denote a generic positive function of $\|(\text{data})\|$ bounded for bounded arguments.

Theorem 3.1. [Bounded absorbing set]

Let $\Omega \subset R^3$ be a bounded domain of class $C^{2+\nu}$, the boundary of which admits the decomposition (1.7). Let the pressure p be related to the internal energy e through the equations of state (2.2), (2.3), (2.4), where the functions P and \mathcal{S} satisfy (2.9), (2.10). Let the transport coefficients μ , η , κ , and ζ be continuously differentiable functions of the temperature satisfying (2.12)–(2.14). Finally, let $G \in BC^1(R \times \bar{\Omega})$ and let the boundary data belong to the class (2.15)–(2.18), where

$$\Gamma_D^{\mathbf{u}} \neq \emptyset, \Gamma_D^\vartheta \neq \emptyset. \tag{3.2}$$

If, in addition, the boundary magnetic field $b_\nu(\tau_n, \cdot)$, $\mathbf{b}_\tau(\tau_n, \cdot)$ is not stationary for some sequence of times $\tau_n \rightarrow \infty$, we assume

$$\Gamma_D^{\mathbf{B}} \subset \Gamma_D^{\mathbf{u}}. \tag{3.3}$$

Then there exists a positive constant $\mathcal{E}_\infty(\|(\text{data})\|)$ depending only on the amplitude of the data specified in (3.1) such that following holds. For any weak solution $(\varrho, \vartheta, \mathbf{u}, \mathbf{B})$ of the compressible MHD system in $(T, \infty) \times \Omega$ satisfying

$$\int_\Omega \varrho \, dx = m_0, \int_\Omega \mathbf{B} \cdot \mathbf{h} \, dx = 0 \text{ for all } \mathbf{h} \in \mathcal{H}(\Omega),$$

there exists a time $\tau > 0$ such that

$$\int_{\Omega} E(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) \, dx < \mathcal{E}_{\infty} \text{ for all } t > T + \tau.$$

Moreover, the length of the time τ depends only on $\|(\text{data})\|$ and on the “initial energy”

$$\mathcal{E}_T \equiv \limsup_{t \rightarrow T^-} \int_{\Omega} E(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) \, dx.$$

The rest of this section is devoted to the proof of Theorem 3.1. A similar result in the context of the Rayleigh–Bénard convection problem was shown in [19]. The influence of the magnetic field on the fluid motion, however, requires essential modifications of the proof presented in [19]. In particular, we construct a two component extension of the boundary magnetic field, where the first component is solenoidal, irrotational and satisfies the Neumann boundary condition, while the second one is solenoidal and small in a suitable Lebesgue norm in Ω with bounded rotation, see Section 3.1.2 below.

As already pointed out, the condition $\Gamma_D^{\vartheta} \neq \emptyset$ is essential, and, in fact, necessary for the existence of a universal absorbing set. Indeed we show an example of a flow driven by an external magnetic field with thermally insulated boundary, where the total energy becomes unbounded for $t \rightarrow \infty$, see Section 5.

3.1 Extension of the boundary data

We start by specifying the extension of the boundary data for the temperature and the magnetic field.

3.1.1 Extension of the boundary temperature

Given ϑ_B we consider its harmonic extension $\tilde{\vartheta}$,

$$\Delta_x \tilde{\vartheta}(\tau, \cdot) = 0, \quad \tilde{\vartheta}(\tau, \cdot)|_{\Gamma_D^{\vartheta}} = \vartheta_B(\tau, \cdot), \quad \nabla_x \tilde{\vartheta}(\tau, \cdot) \cdot \mathbf{n}|_{\Gamma_N^{\vartheta}} = 0, \quad \tau \in R. \quad (3.4)$$

We shall use the same symbol ϑ_B for the boundary data and their extension $\tilde{\vartheta}$ hereafter.

3.1.2 Extension of the boundary magnetic field

If the boundary data b_{ν} , \mathbf{b}_{τ} are stationary, we simply consider the solenoidal irrotational extension \mathbf{B}_B claimed in Definition 1.1.

Extension of the *non-stationary* boundary magnetic field is more delicate. First, we find an extension $\tilde{\mathbf{B}}_N$ satisfying

$$\operatorname{div}_x \tilde{\mathbf{B}}_N = 0, \quad \operatorname{curl}_x \tilde{\mathbf{B}}_N = 0, \quad \tilde{\mathbf{B}}_N \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{B}}} = b_{\nu}, \quad \tilde{\mathbf{B}}_N \times \mathbf{n}|_{\Gamma_B^{\mathbf{B}}} = 0. \quad (3.5)$$

Indeed we may consider $\tilde{\mathbf{B}}_N = \nabla_x \Phi_N$, where Φ_N is the unique solution of the mixed boundary value problem

$$\Delta_x \Phi_N = 0, \quad \Phi_N|_{\Gamma_D^{\mathbf{B}}} = 0, \quad \nabla_x \Phi_N \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{B}}} = b_\nu.$$

To find a suitable extension of the Dirichlet data \mathbf{b}_τ , we introduce the so-called *Bogovskii operator* \mathcal{B} enjoying the following properties, see e.g. Galdi [21], Geißert, Heck, and Hieber [22]:

- \mathcal{B} is a bounded linear operator mapping $L_0^q(Q)$ to $W_0^{1,q}(Q; R^3)$, where Q is a bounded Lipschitz domain in R^d and the symbol L_0^q denotes the space of L^q -integrable functions with zero mean:

$$\begin{aligned} \operatorname{div}_x \mathcal{B}[f] &= f \text{ in } Q \text{ for any } f \in L^q(Q), \quad \int_Q f = 0, \quad \mathcal{B}[f]|_{\partial Q} = 0, \\ \|\mathcal{B}[f]\|_{W^{1,q}(Q; R^3)} &\lesssim \|f\|_{L^q(Q)}, \quad 1 < q < \infty. \end{aligned} \quad (3.6)$$

- If, moreover, $f = \operatorname{div}_x \mathbf{g}$, $\mathbf{g} \in L^r(Q; R^3)$, $\mathbf{g} \cdot \mathbf{n}|_{\partial Q} = 0$, then

$$\|\mathcal{B}[\operatorname{div}_x \mathbf{g}]\|_{L^r(Q; R^3)} \lesssim \|\mathbf{g}\|_{L^r(Q; R^3)}, \quad 1 < r < \infty. \quad (3.7)$$

Now, consider the boundary vector

$$\mathbf{B}_D = \mathbf{n} \times \mathbf{b}_\tau \text{ on } \Gamma_D^{\mathbf{B}} \Rightarrow \mathbf{B}_D \times \mathbf{n} = \mathbf{b}_\tau \text{ on } \Gamma_D^{\mathbf{B}}. \quad (3.8)$$

First, let $\tilde{\mathbf{B}}_D$ be an arbitrary, not necessarily solenoidal, Lipschitz extension of \mathbf{B}_D inside Ω . Consider a cut-off function

$$\chi \in C_c^\infty[0, 1), \quad 0 \leq \chi \leq 1, \quad \chi(z) = 1 \text{ for } z \in [0, d), \quad 0 < d < 1,$$

together with

$$\mathbf{B}_{\delta, D}(t, x) = \chi\left(\frac{\operatorname{dist}[x, \Gamma_D^{\mathbf{B}}]}{\delta}\right) \tilde{\mathbf{B}}_D(t, x), \quad \delta > 0.$$

It follows from (3.8) that

$$\begin{aligned} \mathbf{B}_{\delta, D} \times \mathbf{n} &= \mathbf{b}_\tau \text{ on } \Gamma_D^{\mathbf{B}}, \\ \operatorname{supp}[\mathbf{B}_{\delta, D}] &\subset \mathcal{U}_{\delta_0}(\Gamma_D^{\mathbf{B}}) \text{ for all } 0 < \delta \leq \delta_0, \end{aligned} \quad (3.9)$$

where \mathcal{U}_{δ_0} denotes the δ_0 -neighbourhood.

Next, fix δ_0 so that $\mathcal{U}_{\delta_0}(\Gamma_D^{\mathbf{B}}) \cap \Gamma_N^{\mathbf{B}} = \emptyset$ and consider

$$\tilde{\mathbf{B}}_{\delta, D} = \begin{cases} \mathbf{B}_{\delta, D} - \mathcal{B}_{\delta_0}[\operatorname{div}_x \mathbf{B}_{\delta, D}] \text{ in } \mathcal{U}_{\delta_0}(\Gamma_D^{\mathbf{B}}) \cap \Omega, \\ 0 \text{ in } \Omega \setminus \mathcal{U}_{\delta_0}(\Gamma_D^{\mathbf{B}}) \end{cases}, \quad (3.10)$$

where \mathcal{B}_{δ_0} is the Bogovskii operator defined on each component of the open set $\mathcal{U}_{\delta_0}(\Gamma_D^{\mathbf{B}}) \cap \Omega$. It is easy to check that $\tilde{\mathbf{B}}_{\delta,D}$ enjoys the following properties:

$$\begin{aligned} \tilde{\mathbf{B}}_{\delta,D} \times \mathbf{n}|_{\Gamma_D^{\mathbf{B}}} &= \mathbf{b}_\tau, \quad \tilde{\mathbf{B}}_{\delta,D}|_{\Gamma_N^{\mathbf{B}}} = 0, \\ \operatorname{div}_x \tilde{\mathbf{B}}_{\delta,D} &= 0 \text{ in } \Omega. \end{aligned} \quad (3.11)$$

In addition, as a consequence of boundedness of the Bogovskii operator (3.6),

$$\|\tilde{\mathbf{B}}_{\delta,D}\|_{W^{1,p}(\Omega; \mathbb{R}^3)} \leq c(p, \delta) \|\mathbf{b}_\tau\|_{W^{1,\infty}(\Gamma_D^{\mathbf{B}})}, \quad 1 < p < \infty. \quad (3.12)$$

Note we might have $c(p, \delta) \rightarrow \infty$ for $\delta \rightarrow 0$. Finally, by virtue of (3.7),

$$\|\tilde{\mathbf{B}}_{\delta,D}\|_{L^r(\Omega; \mathbb{R}^3)} \leq o(r, \delta) \|\mathbf{b}_\tau\|_{W^{1,\infty}(\Gamma_D^{\mathbf{B}})}, \quad \text{where } o(r, \delta) \rightarrow 0 \text{ as } \delta \rightarrow 0 \text{ for any } 1 < r < \infty. \quad (3.13)$$

We consider the extension of the boundary data in the form

$$\mathbf{B}_B = \mathbf{B}_{\delta,B} = \tilde{\mathbf{B}}_N + \tilde{\mathbf{B}}_{\delta,D} \quad (3.14)$$

for a suitable $0 < \delta < \delta_0$ to be fixed below. By virtue of (3.5),

$$\operatorname{div}_x \mathbf{B}_{\delta,B} = 0, \quad \operatorname{curl}_x \mathbf{B}_{\delta,B} = \operatorname{curl}_x \tilde{\mathbf{B}}_{\delta,D}, \quad \mathbf{B}_{\delta,B} \times \mathbf{n}|_{\Gamma_D^{\mathbf{B}}} = \mathbf{b}_\tau, \quad \mathbf{B}_{\delta,B} \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{B}}} = b_\nu. \quad (3.15)$$

Remark 3.2. A similar ansatz for the function $\tilde{\mathbf{B}}_{\delta,D}$ was used by Zhang and Zhao [37] based on the construction proposed by Alekseev [1].

3.2 Ballistic energy inequality

The proof of Theorem 3.1 is based on the ballistic energy inequality (2.29) evaluated for $\tilde{\vartheta} = \vartheta_B$, $\mathbf{B}_B = \mathbf{B}_{\delta,B}$ specified in Section 3.1. In accordance with (3.15), we get:

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + \frac{1}{2} |\mathbf{B}|^2 - \vartheta_B \varrho s(\varrho, \vartheta) - \mathbf{B}_{\delta,B} \cdot \mathbf{B} \right) dx \\ & + \int_{\Omega} \frac{\vartheta_B}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} + \zeta(\vartheta) |\operatorname{curl}_x \mathbf{B}|^2 \right) dx \\ & \leq - \int_{\Omega} \left(\varrho s(\varrho, \vartheta) \partial_t \vartheta_B + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \vartheta_B + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \vartheta_B \right) dx \\ & - \int_{\Omega} \left(\mathbf{B} \cdot \partial_t \mathbf{B}_{\delta,B} - (\mathbf{B} \times \mathbf{u}) \cdot \operatorname{curl}_x \tilde{\mathbf{B}}_{\delta,D} - \zeta(\vartheta) \operatorname{curl}_x \mathbf{B} \cdot \operatorname{curl}_x \tilde{\mathbf{B}}_{\delta,D} \right) dx \\ & + \int_{\Omega} \varrho \nabla_x M \cdot \mathbf{u} dx. \end{aligned} \quad (3.16)$$

in $\mathcal{D}'(T, \infty)$. Note that we may suppose $\operatorname{curl}_x \tilde{\mathbf{B}}_{\delta,D} = 0$ if the boundary magnetic field is stationary.

Remark 3.3. Strictly speaking, the pointwise values of the ballistic energy are defined only for a.a. $\tau \in (T, \infty)$. However, thanks to the inequality (3.16), we may identify the ballistic energy with its càglàd representative

$$\begin{aligned} & \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + \frac{1}{2} |\mathbf{B}|^2 - \vartheta_B \varrho s(\varrho, \vartheta) - \mathbf{B}_{\delta, B} \cdot \mathbf{B} \right) (\tau, \cdot) \, dx \\ &= \operatorname{ess\,lim}_{t \rightarrow \tau^-} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + \frac{1}{2} |\mathbf{B}|^2 - \vartheta_B \varrho s(\varrho, \vartheta) - \mathbf{B}_{\delta, B} \cdot \mathbf{B} \right) (t, \cdot) \, dx \end{aligned} \quad (3.17)$$

defined for any $\tau > T$.

Our goal is to show that all terms on the right-hand side of (3.16) can be controlled by the dissipation rate appearing on the left-hand side. This is relatively straightforward for the integrals independent of the density.

3.2.1 Coercivity of the dissipation rate

It follows from coercivity of the transport coefficients stated in the hypotheses (2.12)–(2.14), and hypothesis (3.2) $\Gamma_D^{\mathbf{u}} \neq \emptyset$ that

$$\begin{aligned} & \int_{\Omega} \frac{\vartheta_B}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} + \zeta(\vartheta) |\operatorname{curl}_x \mathbf{B}|^2 \right) \, dx \\ & \gtrsim \inf \vartheta_B \left[\|\mathbf{u}\|_{W^{1,2}(\Omega; R^3)}^2 + \|\nabla_x \vartheta^{\frac{\beta}{2}}\|_{L^2(\Omega; R^3)}^2 + \|\nabla_x \log \vartheta\|_{L^2(\Omega; R^3)}^2 + \|\operatorname{curl}_x \mathbf{B}\|_{L^2(\Omega; R^3)}^2 \right]. \end{aligned}$$

In addition, as $\Gamma_D^{\vartheta} \neq \emptyset$, we may apply Poincaré inequality obtaining

$$\|\nabla_x \vartheta^{\frac{\beta}{2}}\|_{L^2(\Omega; R^3)}^2 + \|\nabla_x \log \vartheta\|_{L^2(\Omega; R^3)}^2 \gtrsim \|\vartheta^{\frac{\beta}{2}}\|_{W^{1,2}(\Omega)}^2 + \|\log \vartheta\|_{W^{1,2}(\Omega)}^2 - c(\|(\text{data})\|).$$

Similarly, by virtue of inequality (2.20),

$$\|\operatorname{curl}_x \mathbf{B}\|_{L^2(\Omega; R^3)}^2 \gtrsim \|\mathbf{B}\|_{W^{1,2}(\Omega; R^3)}^2 - c(\|(\text{data})\|).$$

Indeed we may use the extension $\mathbf{B}_{\delta_0, B}$ of the boundary field constructed in Section 3.1.2, together with its projection $\widehat{\mathbf{B}}_{\delta_0, B}$ on the space \mathcal{H}^{\perp} to deduce

$$\begin{aligned} \|\operatorname{curl}_x \mathbf{B}\|_{L^2(\Omega; R^3)}^2 &= \|\operatorname{curl}_x (\mathbf{B} - \widehat{\mathbf{B}}_{\delta_0, B})\|_{L^2(\Omega; R^3)}^2 - c_1(\|(\text{data})\|) \\ &\gtrsim \|\mathbf{B} - \widehat{\mathbf{B}}_{\delta_0, B}\|_{W^{1,2}(\Omega; R^3)}^2 - c_1(\|(\text{data})\|) \gtrsim \|\mathbf{B}\|_{W^{1,2}(\Omega; R^3)}^2 - c_2(\|(\text{data})\|). \end{aligned}$$

It is important to notice that the above estimate is independent of the value of the parameter δ to be fixed later in the course of the proof.

Summing up the previous estimates we may infer that

$$\left[\|\mathbf{u}\|_{W^{1,2}(\Omega; R^3)}^2 + \|\vartheta^{\frac{\beta}{2}}\|_{W^{1,2}(\Omega)}^2 + \|\log \vartheta\|_{W^{1,2}(\Omega)}^2 + \|\mathbf{B}\|_{W^{1,2}(\Omega; R^3)}^2 \right]$$

$$\leq c(\|(\text{data})\|) \left[1 + \int_{\Omega} \frac{\vartheta_B}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} + \zeta(\vartheta) |\mathbf{curl}_x \mathbf{B}|^2 \right) dx \right]. \quad (3.18)$$

We point out that hypothesis (3.2) plays a crucial role in this step.

3.2.2 Estimates of the integrals containing the magnetic field

We use again hypothesis (3.2) together with (3.10) to perform integration by parts

$$\int_{\Omega} (\mathbf{B} \times \mathbf{u}) \cdot \mathbf{curl}_x \tilde{\mathbf{B}}_{\delta, D} dx = \int_{\Omega} \mathbf{curl}_x (\mathbf{B} \times \mathbf{u}) \cdot \tilde{\mathbf{B}}_{\delta, D} dx,$$

where

$$\begin{aligned} \left| \int_{\Omega} \mathbf{curl}_x (\mathbf{B} \times \mathbf{u}) \cdot \tilde{\mathbf{B}}_{\delta, D} dx \right| &\leq \|\tilde{\mathbf{B}}_{\delta, D}\|_{L^3(\Omega; R^3)} \|\mathbf{curl}_x (\mathbf{B} \times \mathbf{u})\|_{L^{\frac{3}{2}}(\Omega; R^3)} \\ &\lesssim \|\tilde{\mathbf{B}}_{\delta, D}\|_{L^3(\Omega; R^3)} \left(\|\mathbf{B}\|_{L^6(\Omega; R^3)} \|\nabla_x \mathbf{u}\|_{L^2(\Omega; R^{3 \times 3})} + \|\nabla_x \mathbf{B}\|_{L^2(\Omega; R^{3 \times 3})} \|\mathbf{u}\|_{L^6(\Omega; R^3)} \right). \end{aligned}$$

Consequently, by virtue of (3.13) and the Sobolev embedding $W^{1,2} \hookrightarrow L^6$,

$$\left| \int_{\Omega} (\mathbf{B} \times \mathbf{u}) \cdot \mathbf{curl}_x \tilde{\mathbf{B}}_{\delta, D} dx \right| \leq o(\delta, \|(\text{data})\|) \left(\|\mathbf{u}\|_{W^{1,2}(\Omega; R^3)}^2 + \|\mathbf{B}\|_{W^{1,2}(\Omega; R^3)}^2 \right), \quad (3.19)$$

where $o(\delta, \|(\text{data})\|) \rightarrow 0$ as $\delta \rightarrow 0$ uniformly for $t \in R$.

Similarly, using (3.12) for $p = 2$, we get

$$\left| \int_{\Omega} \mathbf{B} \cdot \partial_t \mathbf{B}_{\delta, B} dx \right| \leq \delta \|\mathbf{B}\|_{L^2(\Omega; R^3)}^2 + c(\delta, \|(\text{data})\|) \text{ for any } \delta > 0, \quad (3.20)$$

and, by the same token,

$$\begin{aligned} \left| \int_{\Omega} \zeta(\vartheta) \mathbf{curl}_x \mathbf{B} \cdot \mathbf{curl}_x \tilde{\mathbf{B}}_{\delta, D} dx \right| \\ \leq \delta \|\mathbf{curl}_x \mathbf{B}\|_{L^2(\Omega; R^3)}^2 + c(\delta, \|(\text{data})\|, r) \|\zeta(\vartheta)\|_{L^r(\Omega)}^2 \text{ for some } r > 2. \end{aligned}$$

Furthermore, as $\frac{\beta}{2} > 3 (> 2)$ and the growth of ζ is restricted by (2.14), we may infer that

$$\left| \int_{\Omega} \zeta(\vartheta) \mathbf{curl}_x \mathbf{B} \cdot \mathbf{curl}_x \tilde{\mathbf{B}}_{\delta, D} dx \right| \leq \delta \left(\|\mathbf{curl}_x \mathbf{B}\|_{L^2(\Omega; R^3)}^2 + \|\vartheta^{\frac{\beta}{2}}\|_{W^{1,2}(\Omega)}^2 \right) + c(\delta, \|(\text{data})\|) \quad (3.21)$$

for any $0 < \delta \leq \delta_0$.

Thus fixing $\delta > 0$ sufficiently small we may use (3.18), together with the bounds (3.19)–(3.21) to rewrite inequality (3.16) in the form

$$\frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + \frac{1}{2} |\mathbf{B}|^2 - \vartheta_B \varrho s(\varrho, \vartheta) - \mathbf{B}_B \cdot \mathbf{B} \right) dx$$

$$\begin{aligned}
& + \frac{3}{4} \int_{\Omega} \frac{\vartheta_B}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} + \zeta(\vartheta) |\mathbf{curl}_x \mathbf{B}|^2 \right) dx \\
& \leq - \int_{\Omega} \left(\varrho s(\varrho, \vartheta) \partial_t \vartheta_B + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \vartheta_B + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \vartheta_B \right) dx \\
& + \int_{\Omega} \varrho \nabla_x M \cdot \mathbf{u} dx + c(\|(\text{data})\|), \tag{3.22}
\end{aligned}$$

where we have set $\mathbf{B}_B = \mathbf{B}_{\delta, B}$.

3.2.3 Estimates of the integrals containing the density and temperature

First, write

$$\int_{\Omega} \varrho \nabla_x M \cdot \mathbf{u} dx = \frac{d}{dt} \int_{\Omega} \varrho M dx - \int_{\Omega} \varrho \partial_t M dx. \tag{3.23}$$

Next, in accordance with the choice of ϑ_B in (3.4),

$$\int_{\Omega} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \vartheta_B dx = - \int_{\Omega} \frac{\kappa(\vartheta)}{\vartheta} \nabla_x \vartheta \cdot \nabla_x \vartheta_B dx = - \int_{\Gamma_B^{\vartheta}} \mathcal{K}(\vartheta_B) \nabla_x \vartheta_B \cdot \mathbf{n} d\sigma, \tag{3.24}$$

where $\mathcal{K}'(\vartheta) = \frac{\kappa(\vartheta)}{\vartheta}$.

In view of (3.23), (3.24), the inequality (3.22) can be rewritten in the form

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + \frac{1}{2} |\mathbf{B}|^2 - \vartheta_B \varrho s(\varrho, \vartheta) - \mathbf{B}_B \cdot \mathbf{B} - \varrho M \right) dx \\
& + \frac{3}{4} \int_{\Omega} \frac{\vartheta_B}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} + \zeta(\vartheta) |\mathbf{curl}_x \mathbf{B}|^2 \right) dx \\
& \leq - \int_{\Omega} \left(\varrho s(\varrho, \vartheta) \partial_t \vartheta_B + \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \vartheta_B \right) dx + c(\|(\text{data})\|). \tag{3.25}
\end{aligned}$$

The integrals containing entropy can be handled in the same way as in [19, Section 4.4]. In accordance with hypothesis (2.7),

$$\varrho s(\varrho, \vartheta) = \varrho \mathcal{S} \left(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \right) + \frac{4a}{3} \vartheta^3.$$

On the one hand, as s complies with the Third law of thermodynamics enforced by the hypotheses (2.8), (2.10), we have

$$\varrho \mathcal{S} \left(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \right) \leq \varrho \mathcal{S}(r) \text{ provided } \frac{\varrho}{\vartheta^{\frac{3}{2}}} \geq r, \text{ where } \mathcal{S}(r) \rightarrow 0 \text{ as } r \rightarrow \infty. \tag{3.26}$$

On the other hand, if $\varrho < r \vartheta^{\frac{3}{2}}$, we get

$$0 \leq \varrho \mathcal{S} \left(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \right) \leq c \left(1 + r \vartheta^{\frac{3}{2}} \left[\log^+(r \vartheta^{\frac{3}{2}}) + \log^+(\vartheta) \right] \right), \tag{3.27}$$

cf. also [15, Chapter 12, formula (12.35)].

Thus we may conclude, exactly as in [19, Section 4.4],

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + \frac{1}{2} |\mathbf{B}|^2 - \vartheta_B \varrho s(\varrho, \vartheta) - \mathbf{B}_B \cdot \mathbf{B} - \varrho M \right) dx \\
& + \frac{1}{4} \int_{\Omega} \frac{\vartheta_B}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} + \zeta(\vartheta) |\mathbf{curl}_x \mathbf{B}|^2 \right) dx \\
& \leq \mathcal{S}(r) c_1(\|(\text{data})\|) \int_{\Omega} \varrho |\mathbf{u}| dx + c_2(r, \|(\text{data})\|), \quad \mathcal{S}(r) \rightarrow 0 \text{ as } r \rightarrow \infty,
\end{aligned} \tag{3.28}$$

where $c_2(r, \|(\text{data})\|) \rightarrow \infty$ as $r \rightarrow \infty$.

3.3 Pressure estimates

The integral on the right-hand side of (3.28) contains ϱ and as such is not directly controllable by the dissipation rate. Following the same strategy as in [19] we evoke the *pressure estimates* obtained by using the quantity

$$\varphi = \psi(t) \mathcal{B} \left[\varrho^\alpha - \frac{1}{|\Omega|} \int_{\Omega} \varrho^\alpha dx \right]$$

as a test function in the momentum balance (2.28), where \mathcal{B} is the Bogovskii operator introduced in Section 3.1.2. As a consequence of (3.6) and the Sobolev embedding $W^{1,p} \hookrightarrow L^\infty$, $p > 3$,

$$\left| \mathcal{B} \left[\varrho^\alpha - \frac{1}{|\Omega|} \int_{\Omega} \varrho^\alpha dx \right] \right| \leq c(m_0) \text{ as soon as } \alpha < \frac{1}{3}. \tag{3.29}$$

Following step by step the arguments of [19, Section 4.6.1] we deduce the inequality

$$\begin{aligned}
& \int_{\tau}^{\tau+1} \int_{\Omega} \varrho^{\frac{5}{3}+\alpha} dx dt \leq c(m_0) \left[1 + \int_{\tau}^{\tau+1} \|\vartheta\|_{L^4(\Omega)}^4 dt \right. \\
& \quad \left. \left(1 + \sup_{t \in (\tau, \tau+1)} \|\varrho(t, \cdot)\|_{L^{\frac{5}{3}}(\Omega)} \right) \int_{\tau}^{\tau+1} \|\mathbf{u}\|_{W^{1,2}(\Omega; \mathbb{R}^3)}^2 dt + \sup_{t \in (\tau, \tau+1)} \int_{\Omega} \varrho |\mathbf{u}| dx \right. \\
& \quad \left. + \int_{\tau}^{\tau+1} \int_{\Omega} |\mathbf{curl}_x \mathbf{B} \times \mathbf{B}| dx dt \right], \quad \alpha = \frac{1}{15},
\end{aligned} \tag{3.30}$$

which is nothing other than [19, Section 4, formula (4.28)] augmented by the contribution of the Lorentz force.

3.4 Proof of Theorem 3.1

The estimates (3.28), (3.30) are the main tools for proving Theorem 3.1. Denote

$$F(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + \frac{1}{2}|\mathbf{B}|^2 - \vartheta_B \varrho s(\varrho, \vartheta) - \mathbf{B}_B \cdot \mathbf{B} - \varrho M.$$

It follows from the structural restrictions imposed through the hypotheses (2.5)–(2.10) that

$$\frac{1}{\lambda}E(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) - c_1(\lambda, \|(\text{data})\|) \leq F(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) \leq \lambda E(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) + c_2(\lambda, \|(\text{data})\|) \quad (3.31)$$

for any $\lambda > 1$. Similarly to [19, Section 4.6], we show the following result.

Lemma 3.4. *Under the hypotheses of Theorem 3.1, suppose*

$$\int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau, \cdot) \, dx - \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau + 1, \cdot) \, dx \leq K \quad (3.32)$$

for some $\tau > T$, $K \geq 0$.

Then there is $L = L(K, \|(\text{data})\|)$ such that

$$\sup_{t \in (\tau, \tau+1)} \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) \, dx \leq L(K, \|(\text{data})\|). \quad (3.33)$$

Proof. Step 1:

If (3.32) holds, then the estimates (3.18), (3.28) give rise to

$$\begin{aligned} & \int_{\tau}^{\tau+1} \left(\|\mathbf{u}\|_{W^{1,2}(\Omega; R^3)}^2 + \|\vartheta^{\frac{\beta}{2}}\|_{W^{1,2}(\Omega)}^2 + \|\log(\vartheta)\|_{W^{1,2}(\Omega)}^2 + \|\mathbf{B}\|_{W^{1,2}(\Omega; R^3)}^2 \right) dt \\ & \leq c_1(\|(\text{data})\|) \mathcal{S}(r) \int_{\tau}^{\tau+1} \int_{\Omega} \varrho |\mathbf{u}| \, dx \, dt + c_2(r, K, \|(\text{data})\|). \end{aligned} \quad (3.34)$$

Next,

$$\begin{aligned} & \int_{\tau}^{\tau+1} \left(\|\vartheta\|_{L^4(\Omega)}^4 + \int_{\Omega} |\mathbf{curl}_x \mathbf{B} \times \mathbf{B}| \, dx \right) dt \\ & \lesssim \left(1 + \int_{\tau}^{\tau+1} \left(\|\vartheta^{\frac{\beta}{2}}\|_{W^{1,2}(\Omega)}^2 + \|\mathbf{B}\|_{W^{1,2}(\Omega; R^3)}^2 \right) dt \right) \\ & \leq c(K, \|(\text{data})\|) \left(1 + \int_{\tau}^{\tau+1} \int_{\Omega} \varrho |\mathbf{u}| \, dx \, dt \right), \end{aligned}$$

where we have used (3.34) with $r = 1$. Consequently, inequality (3.30) reduces to

$$\int_{\tau}^{\tau+1} \int_{\Omega} \varrho^{\frac{5}{3} + \alpha} \, dx \, dt$$

$$\leq c(K, \|(\text{data})\|) \left[1 + \left(1 + \sup_{t \in (\tau, \tau+1)} \|\varrho(t, \cdot)\|_{L^{\frac{5}{3}}(\Omega)} \right) \int_{\tau}^{\tau+1} \|\mathbf{u}\|_{W^{1,2}(\Omega; \mathbb{R}^3)}^2 dt + \sup_{t \in (\tau, \tau+1)} \int_{\Omega} \varrho |\mathbf{u}| dx \right],$$

$$\alpha = \frac{1}{15}.$$

In addition, using once more (3.34) with $r = 1$, together with the estimates (2.11) and (3.31), we may infer that

$$\begin{aligned} & \int_{\tau}^{\tau+1} \int_{\Omega} \varrho^{\frac{5}{3} + \alpha} dx dt \\ & \leq c(K, \|(\text{data})\|) \left[1 + \left(\sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| dx \right)^{\frac{3}{5}} \int_{\tau}^{\tau+1} \|\mathbf{u}\|_{W^{1,2}(\Omega; \mathbb{R}^3)}^2 dt \right. \\ & \quad \left. + \left(\sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| dx \right)^{\frac{1}{2}} \right], \\ & \alpha = \frac{1}{15}. \end{aligned} \tag{3.35}$$

Here, we have used Hölder's inequality to estimate

$$\int_{\Omega} \varrho |\mathbf{u}| dx \leq \|\sqrt{\varrho}\|_{L^2(\Omega)} \|\sqrt{\varrho} \mathbf{u}\|_{L^2(\Omega; \mathbb{R}^3)} \leq c(\|(\text{data})\|) \left[1 + \left(\int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| dx \right)^{\frac{1}{2}} \right]. \tag{3.36}$$

Step 2:

Similarly to [19, Section 4.6], under the hypothesis (3.32), we claim that the pointwise values of the functional $\int_{\Omega} F dx$ are controlled by its average:

$$\sup_{t \in (\tau, \tau+1)} \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) dx \leq c(K, \|(\text{data})\|) \left[1 + \int_{\tau}^{\tau+1} \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) dx dt \right]. \tag{3.37}$$

Indeed there is $\xi \in (\tau, \tau + 1)$ such that

$$\int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\xi, \cdot) dx \leq \int_{\tau}^{\tau+1} \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) dx dt.$$

Next, using (3.28) with $r = 1$ and (3.36), we get

$$\frac{d}{dt} \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) dx \leq c(\|(\text{data})\|) \left[1 + \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) dx \right].$$

Consequently,

$$\int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(s, \cdot) dx \leq \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\xi, \cdot) dx + \int_{\xi}^s \frac{d}{dt} \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) dx$$

$$\leq c(\|(\text{data})\|) \left[1 + \int_{\tau}^{\tau+1} \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) \, dx \, dt \right] \quad (3.38)$$

whenever $\xi \leq s \leq \tau + 1$. Finally, by virtue of hypothesis (3.32) and (3.38),

$$\begin{aligned} \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau, \cdot) \, dx &\leq K + \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau + 1, \cdot) \, dx \\ &\leq K + c(\|(\text{data})\|) \left[1 + \int_{\tau}^{\tau+1} \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) \, dx \, dt \right]. \end{aligned}$$

Consequently, we may repeat the same arguments as in (3.38) replacing ξ by τ , which yields (3.37).

Step 3:

We use Hölder's inequality and Sobolev embedding theorem $W^{1,2} \hookrightarrow L^6$ to bound the integral on the right-hand side of (3.34),

$$\int_{\Omega} \varrho |\mathbf{u}| \, dx \leq \|\sqrt{\varrho}\|_{L^2(\Omega)} \|\sqrt{\varrho}\|_{L^3(\Omega)} \|\mathbf{u}\|_{L^6(\Omega; \mathbb{R}^3)} \leq c\sqrt{m_0} \|\sqrt{\varrho}\|_{L^3(\Omega)} \|\mathbf{u}\|_{W^{1,2}(\Omega; \mathbb{R}^3)}.$$

Consequently, going back to (3.34) we obtain

$$\begin{aligned} &\int_{\tau}^{\tau+1} \left(\|\mathbf{u}\|_{W^{1,2}(\Omega; \mathbb{R}^3)}^2 + \|\vartheta^{\frac{\beta}{2}}\|_{W^{1,2}(\Omega)}^2 + \|\mathbf{B}\|_{W^{1,2}(\Omega; \mathbb{R}^3)}^2 \right) dt \\ &\leq c_1(\|(\text{data})\|) \mathcal{S}(r) \int_{\tau}^{\tau+1} \|\varrho\|_{L^{\frac{3}{2}}(\Omega)} + c_2(r, K, \|(\text{data})\|), \end{aligned} \quad (3.39)$$

where possibly $c_2(r, K, \|(\text{data})\|) \rightarrow \infty$ as $r \rightarrow \infty$. In particular, inequality (3.35) may be rewritten in the form

$$\begin{aligned} &\int_{\tau}^{\tau+1} \int_{\Omega} \varrho^{\frac{5}{3}+\alpha} \, dx \, dt \\ &\leq c_1(K, \|(\text{data})\|) \left[1 + \left(\sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| \, dx \right)^{\frac{3}{5}} \mathcal{S}(r) \int_{\tau}^{\tau+1} \|\varrho\|_{L^{\frac{3}{2}}(\Omega)} \, dt \right. \\ &\quad \left. + c_2(r, K, \|(\text{data})\|) \left(\sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| \, dx \right)^{\frac{3}{5}} \right], \quad \alpha = \frac{1}{15}. \end{aligned} \quad (3.40)$$

Step 4:

Using (3.39) we derive a bound on the kinetic energy:

$$\int_{\tau}^{\tau+1} \int_{\Omega} \varrho |\mathbf{u}|^2 \, dx \, dt \leq \sup_{t \in (\tau, \tau+1)} \|\varrho\|_{L^{\frac{3}{2}}(\Omega)} \int_{\tau}^{\tau+1} \|\mathbf{u}\|_{L^6(\Omega; \mathbb{R}^3)}^2 \, dt$$

$$\begin{aligned}
&\lesssim \sup_{t \in (\tau, \tau+1)} \|\varrho\|_{L^{\frac{3}{2}}(\Omega)} \int_{\tau}^{\tau+1} \|\mathbf{u}\|_{W^{1,2}(\Omega; \mathbb{R}^3)}^2 dt \\
&\leq c_1(r, K, \|(\text{data})\|) \sup_{t \in (\tau, \tau+1)} \|\varrho\|_{L^{\frac{3}{2}}(\Omega)} + c_2(K, \|(\text{data})\|) \mathcal{S}(r) \sup_{t \in (\tau, \tau+1)} \|\varrho\|_{L^{\frac{3}{2}}(\Omega)} \int_{\tau}^{\tau+1} \|\varrho\|_{L^{\frac{3}{2}}(\Omega)} dt.
\end{aligned}$$

Next, by interpolation,

$$\|\varrho\|_{L^{\frac{3}{2}}(\Omega)} \leq \|\varrho\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{6}} \|\varrho\|_{L^1(\Omega)}^{\frac{1}{6}};$$

whence

$$\begin{aligned}
&\int_{\tau}^{\tau+1} \int_{\Omega} \varrho |\mathbf{u}|^2 dx \\
&\leq c_1(r, K, \|(\text{data})\|) \sup_{t \in (\tau, \tau+1)} \|\varrho\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{6}} + c_2(K, \|(\text{data})\|) \mathcal{S}(r) \sup_{t \in (\tau, \tau+1)} \|\varrho\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{6}} \int_{\tau}^{\tau+1} \|\varrho\|_{L^{\frac{5}{3}}(\Omega)}^{\frac{5}{6}} dt \\
&\leq c_1(K, \|(\text{data})\|) \left[1 + c_2(r, K, \|(\text{data})\|) \left(\sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| dx \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left(\sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| dx \right)^{\frac{1}{2}} \mathcal{S}(r) \int_{\tau}^{\tau+1} \left(\int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| dx \right)^{\frac{1}{2}} dt \right] \\
&\leq c_1(K, \|(\text{data})\|) \left[1 + c_2(r, K, \|(\text{data})\|) \left(\sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| dx \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \mathcal{S}(r) \sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| dx \right]. \tag{3.41}
\end{aligned}$$

Step 5:

We estimate the pressure/internal energy term by interpolation between L^1 and $L^{\frac{5}{3}+\alpha}$,

$$\int_{\tau}^{\tau+1} \int_{\Omega} \varrho^{\frac{5}{3}} dx dt \leq c(m_0) \left(\int_{\tau}^{\tau+1} \int_{\Omega} \varrho^{\frac{5}{3}+\alpha} dx dt \right)^{\frac{10}{11}} \text{ for } \alpha = \frac{1}{15}.$$

Going back to (3.40) we conclude

$$\begin{aligned}
&\int_{\tau}^{\tau+1} \int_{\Omega} \varrho^{\frac{5}{3}} dx dt \\
&\leq c_1(K, \|(\text{data})\|) \left[1 + \left(\sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| dx \right)^{\frac{3}{5}} \mathcal{S}(r) \int_{\tau}^{\tau+1} \|\varrho\|_{L^{\frac{3}{2}}(\Omega)} dt \right]
\end{aligned}$$

$$\begin{aligned}
& + c_2(r, K, \|(data)\|) \left(\sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| \, dx \right)^{\frac{3}{5}} \Bigg]^{\frac{10}{11}} \\
& \quad \{ \text{similarly to (3.41)} \} \\
& \leq c_1(K, \|(data)\|) \left[1 + \mathcal{S}(r) \left(\sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| \, dx \right)^{\frac{3}{5} + \frac{1}{2}} \right. \\
& + c_2(r, K, \|(data)\|) \left(\sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| \, dx \right)^{\frac{3}{5}} \Bigg]^{\frac{10}{11}} \\
& \leq c_1(K, \|(data)\|) \left[1 + \mathcal{S}(r) \sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| \, dx \right. \\
& + c_2(r, K, \|(data)\|) \left(\sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| \, dx \right)^{\frac{6}{11}} \Bigg]. \tag{3.42}
\end{aligned}$$

Step 6:

Using inequality (3.37), and summing up (3.39) - (3.42) to estimate the integral on the right-hand side we have

$$\begin{aligned}
\sup_{t \in (\tau, \tau+1)} \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) \, dx & \leq \sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})|(t, \cdot) \, dx \\
& \leq c(K, \|(data)\|) \int_{\tau}^{\tau+1} \left[1 + \int_{\Omega} \varrho |\mathbf{u}|^2 \, dx + \|\vartheta^{\frac{\beta}{2}}\|_{W^{1,2}(\Omega)}^2 + \int_{\Omega} \varrho^{\frac{5}{3}} \, dx + \|\mathbf{B}\|_{W^{1,2}(\Omega; \mathbb{R}^3)}^2 \right] dt \\
& \leq c_1(K, \|(data)\|) \left[1 + c_2(r, K, \|(data)\|) \left(\sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| \, dx \right)^{\frac{6}{11}} \right. \\
& \left. + \mathcal{S}(r) \sup_{t \in (\tau, \tau+1)} \int_{\Omega} |F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})| \, dx \right]. \tag{3.43}
\end{aligned}$$

Thus choosing $r = r(K, \|(data)\|)$ so large that

$$c_1(K, \|(data)\|)\mathcal{S}(r) < 1$$

in (3.43) we obtain the desired conclusion (3.33). □

Having proved Lemma 3.4 we are ready to finish the proof of Theorem 3.1. Suppose that

$$\limsup_{t \rightarrow T^-} \int_{\Omega} E(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) \, dx = \mathcal{E}_T < \infty.$$

In view of (3.31),

$$\limsup_{t \rightarrow T^-} \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) \, dx \leq \mathcal{F}_T(\mathcal{E}_T, \|(\text{data})\|) < \infty.$$

Fix $K = 1$ in Lemma 3.4. First we claim there is $\tau \geq T$, where τ depends only on \mathcal{F}_T and $\|(\text{data})\|$ such that

$$\int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau, \cdot) \, dx \leq L(1, \|(\text{data})\|), \quad (3.44)$$

where $L(1, \|(\text{data})\|)$ is the bound in (3.33). More precisely,

$$\tau - T \leq [\mathcal{F}_T + |\Omega| \inf F] + 1$$

since otherwise, in accordance with Lemma 3.4, we would have

$$\int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau, \cdot) \, dx < \int_{\Omega} \inf F \, dx.$$

Next, we claim the implication

$$\int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau, \cdot) \, dx \leq L(1, \|(\text{data})\|) \Rightarrow \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau + 1, \cdot) \, dx \leq L(1, \|(\text{data})\|).$$

Indeed either

$$\int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau, \cdot) \, dx - \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau + 1, \cdot) \, dx \leq 1$$

and the desired conclusion follows from Lemma 3.4, or

$$\int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau + 1, \cdot) \, dx < \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau, \cdot) \, dx - 1 < L(1, \|(\text{data})\|).$$

Thus we have a sequence

$$\tau + n, \, n = 0, 1, \dots, \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau + n, \cdot) \, dx \leq L(1, \|(\text{data})\|)$$

and the desired uniform bound

$$\sup_{t \geq \tau} \int_{\Omega} F(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(t, \cdot) \, dx \leq \mathcal{F}_{\infty}(\|(\text{data})\|)$$

follows from (3.28) (with $r = 1$) by the standard Gronwall argument. Clearly, in view of (3.31), this yields a similar bound for $\int_{\Omega} E(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) \, dx$ as claimed in Theorem 3.1.

4 Asymptotic compactness

The proof of asymptotic compactness basically mimicks the proof of sequential stability of global in time weak solutions with the necessary modification to ensure compactness of the density.

We consider a family of data locally *precompact* in the appropriate function spaces. Specifically,

$$0 < \underline{m}_0 \leq m_{0,n} \leq \overline{m}_0, \quad |\boldsymbol{\omega}_n| \leq \omega, \quad (4.1)$$

$$\|G_n\|_{W^{1,\infty}(R \times \Omega)} \leq \overline{G}, \quad (G_n)_{n \geq 1} \text{ precompact in } W^{1,\infty}([-N, N] \times \Omega) \text{ for any } N > 0. \quad (4.2)$$

Similarly,

$$\begin{aligned} 0 < \underline{\vartheta} \leq \vartheta_{B,n}, \quad \sup_{t \in R} \left(\|\vartheta_{B,n}(t, \cdot)\|_{W^{2,\infty}(\Gamma_D^\vartheta)} + \|\partial_t \vartheta_{B,n}(t, \cdot)\|_{W^{1,\infty}(\Gamma_D^\vartheta)} \right) \leq \overline{\vartheta}, \\ (\vartheta_{B,n})_{n \geq 1} \text{ precompact in } C([-N, N]; C^2(\Gamma_D^\vartheta)) \text{ for any } N > 0, \\ (\partial_t \vartheta_{B,n})_{n \geq 1} \text{ precompact in } C([-N, N]; C^1(\Gamma_D^\vartheta)) \text{ for any } N > 0; \end{aligned} \quad (4.3)$$

$$\begin{aligned} \sup_{t \in R} \left(\|\mathbf{b}_{\tau,n}(t, \cdot)\|_{W^{2,\infty}(\Gamma_D^{\mathbf{B}}; R^3)} + \|\partial_t \mathbf{b}_{\tau,n}(t, \cdot)\|_{W^{1,\infty}(\Gamma_D^{\mathbf{B}}; R^3)} \right) \leq \overline{B}_\tau, \\ (\mathbf{b}_{\tau,n})_{n \geq 1} \text{ precompact in } C([-N, N]; C^2(\Gamma_D^{\mathbf{B}}; R^3)) \text{ for any } N > 0, \\ (\partial_t \mathbf{b}_{\tau,n})_{n \geq 1} \text{ precompact in } C([-N, N]; C^1(\Gamma_D^{\mathbf{B}}; R^3)) \text{ for any } N > 0; \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} \sup_{t \in R} \left(\|b_{\nu,n}(t, \cdot)\|_{W^{2,\infty}(\Gamma_N^{\mathbf{B}})} + \|\partial_t b_{\nu,n}(t, \cdot)\|_{W^{1,\infty}(\Gamma_N^{\mathbf{B}})} \right) \leq \overline{B}_\nu, \\ (b_{\nu,n})_{n \geq 1} \text{ precompact in } C([-N, N]; C^2(\Gamma_N^{\mathbf{B}})) \text{ for any } N > 0, \\ (\partial_t b_{\nu,n})_{n \geq 1} \text{ precompact in } C([-N, N]; C^1(\Gamma_N^{\mathbf{B}})) \text{ for any } N > 0. \end{aligned} \quad (4.5)$$

Theorem 4.1. [Asymptotic compactness]

Let the hypotheses of Theorem 3.1 hold. Suppose that $(\varrho_n, \vartheta_n, \mathbf{u}_n, \mathbf{B}_n)_{n \geq 1}$ is a sequence of weak solutions to the compressible MHD system on (T_n, ∞) , $T_n \geq -\infty$, $T_n \rightarrow -\infty$, with the data

$$(m_{0,n}, \omega_n, G_n, \vartheta_{B,n}, \mathbf{b}_{\tau,n}, b_{n,n})_{n \geq 1}$$

specified in (4.1)–(4.5),

$$\int_{\Omega} \varrho_n \, dx = m_{0,n}, \quad \int_{\Omega} \mathbf{B}_n \cdot \mathbf{h} \, dx = 0 \text{ for all } \mathbf{h} \in \mathcal{H}(\Omega; R^3).$$

In addition, suppose

$$\limsup_{\tau \rightarrow T_n^+} \int_{\Omega} E(\varrho_n, \vartheta_n, \mathbf{u}_n, \mathbf{B}_n)(\tau, \cdot) \, dx < \mathcal{E}_0 < \infty$$

uniformly for all $n = 1, 2, \dots$.

Then there is a subsequence (not relabelled here) such that

$$\begin{aligned} \varrho_n &\rightarrow \varrho \text{ in } C_{\text{weak}}([-N, N]; L^{\frac{5}{3}}(\Omega)) \text{ and in } C([-N, N]; L^1(\Omega)), \\ \vartheta_n &\rightarrow \vartheta \text{ in } L^q([-N, N]; L^4(\Omega)) \text{ and weakly in } L^2(-N, N; W^{1,2}(\Omega)), \quad 1 \leq q < \infty, \\ \mathbf{u}_n &\rightarrow \mathbf{u} \text{ weakly in } L^2(-N, N; W^{1,2}(\Omega; \mathbb{R}^3)), \\ \mathbf{B}_n &\rightarrow \mathbf{B} \text{ in } L^2(-N, N; L^2(\Omega; \mathbb{R}^3)) \text{ and weakly in } L^2(-N, N; W^{1,2}(\Omega; \mathbb{R}^3)), \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} m_{0,n} &\rightarrow m_0 > 0, \quad \boldsymbol{\omega}_n \rightarrow \boldsymbol{\omega}, \\ G_n &\rightarrow G \text{ in } W^{1,\infty}([-N, N] \times \Omega), \\ \vartheta_{n,B} &\rightarrow \vartheta_B \text{ in } C([-N, N]; C^2(\Gamma_D^\vartheta)), \quad \inf_{R \times \Omega} \vartheta_B > 0, \\ \mathbf{b}_{\tau,n} &\rightarrow \mathbf{b}_\tau \text{ in } C([-N, N]; C^2(\Gamma_D^{\mathbf{B}}; \mathbb{R}^3)), \\ b_{\nu,n} &\rightarrow b_\nu \text{ in } C([-N, N]; C^2(\Gamma_N^{\mathbf{B}})) \end{aligned} \quad (4.7)$$

for any $N > 0$, where $(\varrho, \vartheta, \mathbf{u}, \mathbf{B})$ is a weak solution of the compressible MHD system with the data $(m_0, \boldsymbol{\omega}, G, \vartheta_B, \mathbf{b}_\tau, b_\nu)$ defined in $R \times \Omega$ (entire solution) satisfying

$$\int_{\Omega} E(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau, \cdot) \, dx \leq \mathcal{E}_\infty \text{ for all } \tau \in R. \quad (4.8)$$

As already pointed out, the proof of Theorem 4.1 is based on ideas similar to the proof of sequential stability of weak solutions on a *compact* time interval $(-N, N)$. Here the necessary energy bounds are provided by Theorem 3.1; whence the only missing piece of information is the strong convergence of the densities at the “initial” time $t = -N$. Indeed compactness of the sequence $(\varrho_n)_{n \geq 1}$ in the existence proof proposed by Lions [28] or [13] is conditioned by compactness of the “initial” density distributions. In the present setting, however, the “initial” state is *a priori* not given.

Fortunately, this problem can be solved by careful analysis of propagation of the density oscillations in time elaborated in [16] and further developed in [12] in the context of inhomogeneous velocity boundary conditions. Adaptation to the present setting is straightforward as soon as we observe compactness of the Lorentz force in the momentum equation, specifically

$$\operatorname{curl}_x \mathbf{B}_n \times \mathbf{B}_n \rightarrow \operatorname{curl}_x \mathbf{B} \times \mathbf{B} \text{ weakly in } L^{\frac{3}{2}}((-N, N) \times \Omega; \mathbb{R}^3).$$

However, given the available energy bounds, this can be shown by the standard Aubin–Lions argument. The remaining compactness properties necessary to establish the convergence in (4.6) are well understood and elaborated in detail in [9].

5 Concluding discussion, applications

Suppose that the gravitational potential G as well as the boundary data ϑ_B , \mathbf{b}_τ , b_ν are independent of the time. Under the hypotheses of Theorem 3.1, the set

$$\mathcal{A} = \left\{ (\varrho, \vartheta, \mathbf{u}, \mathbf{B}) \mid (\varrho, \vartheta, \mathbf{u}, \mathbf{B}) \text{ -- solution of the compressible (MHD) system for } t \in (-\infty, \infty) \right. \\ \left. \int_{\Omega} E(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) \, dx < \mathcal{E}_\infty \right\}$$

is non–empty, time–shift invariant, meaning

$$(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) \in \mathcal{A} \Rightarrow (\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\cdot + T) \in \mathcal{A}, \quad T \in \mathbb{R},$$

and compact with respect to the topologies specified in (4.6).

5.1 Convergence to equilibria

A particular situation occurs when \mathcal{A} is a singleton containing the unique stationary solution of the problem. As a direct consequence of Theorem 4.1 we obtain unconditional convergence to equilibrium for *any* weak solution of the compressible MHD system.

As an example, consider the situation:

- $\vartheta_B > 0$ is a positive constant on $\Gamma_D^\vartheta \neq \emptyset$;
-

$$\mathbf{b}_\tau = \mathbf{B}_B \times \mathbf{n}|_{\Gamma_B}, \quad b_\nu = \mathbf{B}_B \cdot \mathbf{n}|_{\Gamma_N},$$

where

$$\mathbf{curl}_x \mathbf{B}_B = \operatorname{div}_x \mathbf{B}_B = 0 \text{ in } \Omega.$$

Accordingly, the ballistic energy inequality (3.16) yields

$$\frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) + \frac{1}{2} |\mathbf{B}|^2 - \vartheta_B \varrho s(\varrho, \vartheta) - \varrho M \right) \, dx \\ + \int_{\Omega} \frac{\vartheta_B}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} + \zeta(\vartheta) |\mathbf{curl}_x \mathbf{B}|^2 \right) \, dx \leq 0. \quad (5.1)$$

Integrating (5.1) in time we deduce that any solution belonging to \mathcal{A} necessarily satisfies

$$\mathbf{u} \equiv 0, \quad \vartheta \equiv \vartheta_B, \quad \partial_t \varrho = 0, \quad \operatorname{div}_x \mathbf{B} = \mathbf{curl}_x \mathbf{B} = 0.$$

Moreover, it turns out that \mathbf{B} is uniquely determined by the boundary conditions and the hypothesis

$$\int_{\Omega} \mathbf{B} \cdot \mathbf{h} \, dx = 0 \text{ for all } \mathbf{h} \in \mathcal{H}(\Omega).$$

Finally, $\varrho = \varrho(x)$ solves the static problem

$$\nabla_x p(\varrho, \vartheta_B) = \varrho \nabla_x M$$

that admits a unique solution determined by the mass constraint

$$\int_{\Omega} \varrho \, dx = m_0$$

as long as $P'(0) > 0$ in (2.6), see [17].

5.2 Systems with unbounded energy

As mentioned in the introduction, the result claimed in Theorem 3.1 cannot hold if the fluid system is thermally isolated, meaning

$$\Gamma_D^\vartheta = \emptyset. \quad (5.2)$$

Indeed it follows from the entropy inequality (2.28) that the total entropy

$$\tau \mapsto \int_{\Omega} \varrho s(\varrho, \vartheta)(\tau, \cdot) \, dx$$

is a non-decreasing function, which precludes the existence of a bounded absorbing set depending solely on the data specified in (3.1).

What is more, we show that (5.2) may give rise to trajectories with unbounded energy. Indeed suppose that there is a global-in-time solution $(\varrho, \vartheta, \mathbf{u}, \mathbf{B})$ and a sequence of times $\tau_n \rightarrow \infty$ such that

$$\sup_{n \geq 1} \int_{\Omega} E(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau_n, \cdot) \, dx < \infty. \quad (5.3)$$

It follows from the constitutive restrictions (2.11) that boundedness of total energy implies boundedness of the total entropy $\int_{\Omega} \varrho s(\varrho, \vartheta)(\tau_n, \cdot) \, dx$. However, by virtue of the no-flux boundary conditions induced by (5.2), the total entropy is a non-decreasing function of the time. Consequently, we may infer that

$$\int_T^\infty \int_{\Omega} \frac{1}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} + \zeta(\vartheta) |\mathbf{curl}_x \mathbf{B}|^2 \right) \, dx \, dt < \infty. \quad (5.4)$$

In particular, it follows from (5.4) that

$$\int_{\tau_n}^{\tau_n+1} \int_{\Omega} \frac{1}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \zeta(\vartheta) |\mathbf{curl}_x \mathbf{B}|^2 \right) \, dx \, dt \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (5.5)$$

Consequently, introducing the time shifts

$$\mathbf{u}_n = \mathbf{u}(\cdot + \tau_n), \quad \mathbf{B}_n = \mathbf{B}(\cdot + \tau_n),$$

we deduce

$$\mathbf{u}_n \rightarrow 0 \text{ in } L^2(0, 1; W^{1,2}(\Omega; R^3)), \quad \mathbf{B}_n \rightarrow \widehat{\mathbf{B}} \text{ in } L^2(0, 1; L^2(\Omega; R^3)) \text{ and weakly in } L^2(0, 1; W^{1,2}(\Omega; R^3)).$$

As the limits satisfy the induction equation (1.3), we conclude

$$\widehat{\mathbf{B}} = \widehat{\mathbf{B}}(x), \quad \operatorname{div}_x \widehat{\mathbf{B}} = 0, \quad \operatorname{curl}_x \widehat{\mathbf{B}} = 0 \text{ in } \Omega, \quad \widehat{\mathbf{B}} \times \mathbf{n}|_{\Gamma_D^{\mathbf{B}}} = \mathbf{b}_\tau, \quad \widehat{\mathbf{B}} \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{B}}} = b_\nu. \quad (5.6)$$

Thus the boundary data b_ν , \mathbf{b}_τ must be stationary in the sense of Definition 1.1. Otherwise, the conclusion contradicts (5.3).

Using the necessary condition for stationarity (1.23) we obtain the following result.

Theorem 5.1. [Systems with unbounded energy]

Suppose

$$\Gamma_D^{\mathbf{u}} \neq \emptyset, \quad \Gamma_D^{\vartheta} = \emptyset,$$

and b_ν , \mathbf{b}_τ independent of t ,

$$\Gamma_D^{\mathbf{B}} \neq \emptyset, \quad \text{where } \mathbf{b}_\tau \cdot \mathbf{n} = 0, \quad \operatorname{div}_\tau \mathbf{b}_\tau \neq 0.$$

Then

$$\int_{\Omega} E(\varrho, \vartheta, \mathbf{u}, \mathbf{B})(\tau, \cdot) \, dx \rightarrow \infty \text{ as } \tau \rightarrow \infty$$

for any global-in-time weak solution to the compressible MHD system.

5.3 General boundary fluxes

The results can be extended to general boundary fluxes. Prominent examples are the Navier–slip boundary condition for the velocity,

$$\mathbf{u} \cdot \mathbf{n}|_{\Gamma_N^{\mathbf{u}}} = 0, \quad [\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \cdot \mathbf{n} + d\mathbf{u}] \times \mathbf{n}|_{\Gamma_N^{\mathbf{u}}} = 0, \quad d > 0, \quad (5.7)$$

see e.g. Bulíček, Málek, and Rajagopal [5], and the radiative heat flux condition

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n} - d|\vartheta - \vartheta_0|^k(\vartheta - \vartheta_0)|_{\Gamma_D^{\vartheta}} = 0 \quad (5.8)$$

used in some models in astrophysics, cf. Weiss and Proctor [36].

If $d > 0$, then the conditions $\Gamma_D^{\mathbf{u}} \neq \emptyset$, $\Gamma_D^{\vartheta} \neq \emptyset$ can be dropped in Theorem 3.1. Hypothesis $\Gamma_D^{\mathbf{B}} \subset \Gamma_D^{\mathbf{u}}$, however, is still necessary. We leave the details to the interested reader.

5.4 “Do nothing” boundary conditions

Another interesting type of boundary conditions arises under a simple but physically grounded assumption that the Maxwell equations, here represented by the induction equation (1.3), are satisfied both in Ω and in the adjacent component Ω_{ext} with possibly different (discontinuous) transport coefficients. The boundary conditions on $\Gamma \subset \partial\Omega$ separating Ω and Ω_{ext} are then nothing other than the Rankine–Hugeniot compatibility conditions to ensure the satisfaction of the induction equation in the weak sense.

More specifically, we write the Maxwell’s equations yielding (1.3) in a more precise way:

$$\begin{aligned}\partial_t \mathbf{B} + \mathbf{curl}_x \mathbf{E} &= 0, \\ \operatorname{div}_x \mathbf{B} &= 0, \\ \mathbf{J} &= \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B}), \\ \mathbf{J} &= \mathbf{curl}_x \mathbf{H}, \quad \mathbf{B} = \bar{\mu} \mathbf{H},\end{aligned}\tag{5.9}$$

where \mathbf{E} denotes the electric field, \mathbf{J} the electric current density, \mathbf{H} the magnetic field intensity, σ is the electric conductivity and $\bar{\mu}$ the magnetic permitivity of the material. Both σ and $\bar{\mu}$ may experience jumps over Γ . Accordingly, the Lorentz force in the momentum equation reads

$$\mathbf{J} \times \mathbf{B} = \mathbf{curl}_x \mathbf{H} \times \mathbf{B}.$$

The associated Rankine-Hugeniot conditions on Γ are:

$$\begin{aligned}\mathbf{B} \cdot \mathbf{n} &= \mathbf{B}_{\text{ext}} \cdot \mathbf{n}, \\ \mathbf{H} \times \mathbf{n} &= \mathbf{H}_{\text{ext}} \times \mathbf{n}, \\ \mathbf{E} \times \mathbf{n} &= \mathbf{E}_{\text{ext}} \times \mathbf{n}.\end{aligned}\tag{5.10}$$

If the adjacent component Ω_{ext} happens to be a vacuum, we have $\sigma_{\text{ext}} = 0$, and (5.9), (5.10) give rise to a non-local condition for \mathbf{B} on Γ given by a Neumann–Dirichlet type map, see Ladyzhenskaya and Solonnikov [27], Sakhaev and Solonnikov [30], Solonnikov [33].

At the level of the energy estimates yielding the existence of a bounded absorbing set, the above situation does not create any additional difficulties. Indeed, in view of (5.10), the function

$$\bar{\mu}(\mathbf{B} - \mathbf{B}_B) = \mathbf{H} - \mathbf{H}_B, \quad \bar{\mu} = \bar{\mu}_{\text{int}} \mathbf{1}_\Omega + \bar{\mu}_{\text{ext}} \mathbf{1}_{\Omega_{\text{ext}}}$$

is still a legitimate test function for the induction equation (1.3) yielding

$$\begin{aligned}\frac{d}{dt} \int_{\Omega \cup \Omega_{\text{ext}}} \bar{\mu} \left(\frac{1}{2} |\mathbf{B}|^2 - \mathbf{B}_B \cdot \mathbf{B} \right) dx &= \int_{\Omega \cup \Omega_{\text{ext}}} (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{curl}_x (\mathbf{H} - \mathbf{H}_B) dx \\ &+ \int_{\Omega \cup \Omega_{\text{ext}}} \frac{1}{\sigma} \mathbf{curl}_x \mathbf{H} \cdot \mathbf{curl}_x (\mathbf{H} - \mathbf{H}_B) dx.\end{aligned}\tag{5.11}$$

If we set the velocity $\mathbf{u}_{\text{ext}} = 0$, all energy bounds derived in the proof of Theorem 3.1 remain valid; whence the existence of a bounded absorbing set follows from Theorem 3.1

The proof of asymptotic compactness claimed in Theorem 4.1 may be more delicate. Still we conjecture that the results of Bauer, Pauly, and Schomburg [4] may be used to show the asymptotic compactness for the sequence of magnetic fields to extend Theorem 4.1 to the case of “do nothing” boundary conditions.

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References

- [1] G. V. Alekseev. Solvability of control problems for stationary equations of the magnetohydrodynamics of a viscous fluid. *Sibirsk. Mat. Zh.*, **45**(2):243–263, 2004.
- [2] J. C. Alexander and G. Auchmuty. l^2 -well-posedness of 3d div-curl boundary value problems. *Quart. Appl. Math.*, **63**(3):479–508, 2005.
- [3] A. Battaner. *Astrophysical fluid dynamics*. Cambridge University Press, Cambridge, 1996.
- [4] S. Bauer, D. Pauly, and M. Schomburg. The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions. *SIAM J. Math. Anal.*, **48**(4):2912–2943, 2016.
- [5] M. Bulíček, J. Málek, and K.R. Rajagopal. Navier’s slip and evolutionary Navier-Stokes-Fourier-like systems with pressure, shear- rate and temperature dependent viscosity. 2006. Preprint, Nečas Center for Mathematical Modelling, Prague.
- [6] P. Constantin and C. Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.
- [7] P. Constantin, C. Foias, and R. an Temam. *Attractors representing turbulent flows*. Mem. Amer. Math. Soc. 53, Providence, 1985.
- [8] G. Csató, O. Kneuss, and D. Rajendran. On the boundary conditions in estimating $\nabla\omega$ by $\text{div } \omega$ and $\text{curl } \omega$. *Proc. Roy. Soc. Edinburgh Sect. A*, **149**(3):739–760, 2019.
- [9] B. Ducomet and E. Feireisl. The equations of magnetohydrodynamics: On the interaction between matter and radiation in the evolution of gaseous stars. *Commun. Math. Phys.*, **266**:595–629, 2006.

- [10] G. Duvaut and J.-L. Lions. Inéquations en thermoélasticité et magnétohydrodynamique. *Arch. Rational Mech. Anal.*, **46**:241–279, 1972.
- [11] A. Eden and A. Libin. Explicit dimension estimates of attractors for the MHD equations in three-dimensional space. *Phys. D*, **40**(3):338–352, 1989.
- [12] F. Fanelli, E. Feireisl, and M. Hofmanová. Ergodic theory for energetically open compressible fluid flows. *Phys. D*, **423**:Paper No. 132914, 25, 2021.
- [13] E. Feireisl. *Dynamics of viscous compressible fluids*. Oxford University Press, Oxford, 2004.
- [14] E. Feireisl, Y.-S. Kwon, P. Gwiazda, and A. Świerczewska-Gwiazda. Mathematical theory of compressible magnetohydrodynamics driven by non-conservative boundary conditions. Preprint IM ASCR Prague IM-2023-11, 2023.
- [15] E. Feireisl and A. Novotný. *Mathematics of open fluid systems*. Birkhäuser-Verlag, Basel, 2022.
- [16] E. Feireisl and H. Petzeltová. Asymptotic compactness of global trajectories generated by the Navier-Stokes equations of compressible fluid. *J. Differential Equations*, **173**:390–409, 2001.
- [17] E. Feireisl and H. Petzeltová. On the long-time behaviour of solutions to the Navier-Stokes-Fourier system with a time-dependent driving force. *J. Dynam. Differential Equations*, **19**(3):685–707, 2007.
- [18] E. Feireisl and D. Pražák. *Asymptotic behavior of dynamical systems in fluid mechanics*. AIMS, Springfield, 2010.
- [19] E. Feireisl and A. Świerczewska Gwiazda. The Rayleigh–Bénard Problem for Compressible Fluid Flows. *Arch. Ration. Mech. Anal.*, **247**(1):Paper No. 9, 2023.
- [20] C. Foias, O. Manley, R. Rosa, and R. Temam. *Navier-Stokes equations and turbulence*, volume 83 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001.
- [21] G. P. Galdi. *An introduction to the mathematical theory of the Navier - Stokes equations, Second Edition*. Springer-Verlag, New York, 2003.
- [22] M. Geißert, H. Heck, and M. Hieber. On the equation $\operatorname{div} u = g$ and Bogovskii’s operator in Sobolev spaces of negative order. In *Partial differential equations and functional analysis*, volume 168 of *Oper. Theory Adv. Appl.*, pages 113–121. Birkhäuser, Basel, 2006.
- [23] D. Gough. Towards understanding solar convection and activity. *Solar Physics*, **192**:3–26, 2000.

- [24] Fei Jiang and Song Jiang. On the stabilizing effect of the magnetic fields in the magnetic Rayleigh-Taylor problem. *SIAM J. Math. Anal.*, **50**(1):491–540, 2018.
- [25] H. Kozono and T. Yanagisawa. Leray’s problem on the stationary Navier-Stokes equations with inhomogeneous boundary data. *Math. Z.*, **262**(1):27–39, 2009.
- [26] H. Kozono and T. Yanagisawa. L^r -variational inequality for vector fields and the Helmholtz-Weyl decomposition in bounded domains. *Indiana Univ. Math. J.*, **58**(4):1853–1920, 2009.
- [27] O. A. Ladyženskaja and V. A. Solonnikov. Solution of some non-stationary problems of magnetohydrodynamics for a viscous incompressible fluid. *Trudy Mat. Inst. Steklov*, **59**:115–173, 1960.
- [28] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.2, Compressible models*. Oxford Science Publication, Oxford, 1998.
- [29] J. Málek and J. Nečas. A finite-dimensional attractor for the three dimensional flow of incompressible fluid. *J. Differential. Equations*, **127**:498–518, 1996.
- [30] Sh. Sakhaev and V. A. Solonnikov. On some stationary problems of magnetohydrodynamics in multiply connected domains. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, **397**(Kraevye Zadachi Matematicheskoi Fiziki i Smezhnye Voprosy Teorii Funktsii. 42):126–149, 174, 2011.
- [31] G. R. Sell. Global attractors for the three-dimensional Navier-Stokes equations. *J. Dynamics Differential Equations*, **8**(1):1–33, 1996.
- [32] M. Sermange and R. Temam. Some mathematical questions related to the MHD equations. *Comm. Pure Appl. Math.*, **36**(5):635–664, 1983.
- [33] V. A. Solonnikov. L_p -estimates for the solution of a linear problem that arises in magnetohydrodynamics. *Algebra i Analiz*, **23**(1):232–254, 2011.
- [34] L. Tao, N.O. Weiss, D.P. Brownjohn, and M.R.E. Proctor. Flux separation in stellar magnetoconvection. *The Astronomical Journal*, **496**:L39–L42, 1998.
- [35] M.J. Thompson and J. Christensen-Dalsgaard. *Stellar atrophysical fluid dynamics*. Cambridge University Press, Cambridge, 2003.
- [36] N. O. Weiss and M. R. E. Proctor. *Magnetoconvection*. Cambridge Monographs on Mechanics. Cambridge University Press, Cambridge, 2014.
- [37] Z. Zhang and C. Zhao. Solvability of an inhomogeneous boundary value problem for steady MHD equations. *Math. Methods Appl. Sci.*, **44**(1):1147–1154, 2021.