Analysis of open fluid systems with uncertain data

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Navier–Stokes–Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

Momentum balance (Newton' s second law)

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u}) + \varrho \nabla_x G$$

Internal energy balance (First law of thermodynamics)

$$\partial_t \varrho e(\varrho, \vartheta) + \mathrm{div}_x (\varrho e(\varrho, \vartheta) \mathbf{u}) + \mathrm{div}_x \mathbf{q}(\nabla_x \vartheta) = \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} - p(\varrho, \vartheta) \mathrm{div}_x \mathbf{u}$$

Newton's rheological law

$$\mathbb{S}(\mathbb{D}_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0$$

Fourier's law

$$\mathbf{q}(\nabla_x \vartheta) = -\kappa \nabla_x \vartheta, \ \kappa > 0$$

Thermodynamics

Gibbs' law, Second law of thermodynamics

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

Entropy balance equation (Second law of thermodynamics)

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S}(\mathbb{D}_x\mathbf{u}) : \mathbb{D}_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

Thermodynamic stability

$$(\varrho, S, \mathbf{m}) \mapsto \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + \varrho e(\varrho, S)\right] \text{ strictly convex, } S = \varrho s,\ \mathbf{m} = \varrho\mathbf{u}$$

Boyle-Mariotte equation of state

$$p(\varrho,\vartheta) = \varrho\vartheta,\ e(\varrho,\vartheta) = c_v\vartheta,\ c_v > 0,\ s(\varrho,\vartheta) = c_v\log\vartheta - \log\varrho$$

Data

Physical space

$$Q \subset R^d,\ d = 1, 2, 3 \text{ (bounded) domain}$$

Impermeable boundary

$$\mathbf{u} \cdot \mathbf{n}|_{\partial Q} = 0$$

Kinematic boundary condition, complete slip

$$[\mathbb{S}(\mathbb{D}_x \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial Q} = 0$$

Kinematic boundary condition, tangential velocity

$$\mathbf{u} \times \mathbf{n}|_{\partial Q} = \mathbf{u}_B \times \mathbf{n}$$

Boundary temperature

$$\vartheta|_{\partial Q} = \vartheta_B$$

Thermal insulation – zero heat flux

$$\mathbf{q} \cdot \mathbf{n}|_{\partial Q} = 0$$

Initial/boundary value problem

Initial state of the system

$$\varrho(0,\cdot) = \varrho_0,\ \vartheta(0,\cdot) = \vartheta_0,\ \varrho_0 > 0, \vartheta_0 > 0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

$$+ \text{ compatibility conditions}$$

Existence of local–in–time strong solutions

- **Valli, Valli–Zajaczkowski [1986], Kawashima–Shizuta [1988]**

$$\varrho_0 \in W^{k,2}(Q),\ \vartheta_0 \in W^{k,2}(Q),\ \mathbf{u}_0 \in W^{k,2}(Q;R^d),\ k \geq 3$$

- **Cho–Kim [2006]**

$$\varrho_0 \in W^{1,p}(Q),\ \vartheta_0 \in W^{2,2}(Q),\ \mathbf{u}_0 \in W^{2,2}(Q;R^d),\ 3 < p \leq 6$$

$$\mathbf{u}_B = 0,\ \mathbf{q}\cdot\mathbf{n}|_{\partial Q} = 0$$

- **Kotschote [2015]**

$$\varrho_0 \in W^{1,p}(Q),\ \vartheta_0 \in W^{2-\frac{1}{p},p}(Q),\ \mathbf{u}_0 \in W^{2-\frac{1}{p},p}(Q;R^d),\ p > 3$$

Conditional regularity

John F. Nash [1928-2015]

Nash's conjecture: *Probably one should first try to prove a conditional existence and uniqueness theorem for flow equations. This should give existence, smoothness, and unique continuation (in time) of flows, conditional on the non-appearance of certain gross types of singularity, such as infinities of temperature or density.*

- **EF, Wen, Zhu [2022]**

$$\mathbf{u}_B = 0, \ \mathbf{q} \cdot \mathbf{n}|_{\partial Q} = 0$$

$$\sup_{t \in [0,T)} \left(\sup_Q \varrho(t,\cdot) + \sup_Q \vartheta(t,\cdot) \right) < \infty \ \Rightarrow \ T_{\max} > T$$

- **Basarić, EF, Mizerová [2023]**

$$\mathbf{u}_B \cdot \mathbf{n} = 0, \ \vartheta|_{\partial Q} = \vartheta_B$$

$$\sup_{t \in [0,T)} \left(\sup_Q \varrho(t,\cdot) + \sup_Q \vartheta(t,\cdot) + \sup_Q |\mathbf{u}(t,\cdot)| \right) < \infty \ \Rightarrow \ T_{\max} > T$$

Data space

$$\vartheta_D \in L^p(0,\infty; W^{2,p}(Q)),\ \partial_t \vartheta_D \in L^p(0,\infty; L^p(Q)),\ \vartheta_D > 0,$$

$$\vartheta_D(0,\cdot) = \vartheta_0,\ \vartheta_D|_{\partial Q} = \vartheta_B$$

$$\mathbf{u}_D \in L^p(0,\infty; W^{2,p}(Q;R^d)),\ \partial_t \mathbf{u}_D \in L^p(0,\infty; L^p(Q;R^d))$$

$$\mathbf{u}_D(0,\cdot) = \mathbf{u}_0,\ \mathbf{u}_D|_{\partial Q} = \mathbf{u}_B$$

Data space

$$X_D = \left\{ (\varrho_D, \vartheta_D, \mathbf{u}_D) \ \middle|\ \varrho_D = \varrho_0,\ \inf_Q \varrho_D > 0\ +\ \text{compatibility conditions} \right\}$$

Topology on the data space

$$\begin{aligned} \|D\|_{X_D} = & \|\varrho_D^{-1}\|_{W^{1,p}(Q)} + \|\vartheta_D^{-1}\|_{W^{1,p}(Q)} \\ & + \|\varrho_D\|_{W^{1,p}(Q)} + \|\vartheta_D\|_{L^p(0,\infty;W^{1,p}(Q))\cap W^{1,p}(0,\infty;L^p(Q))} \\ & + \|\mathbf{u}_D\|_{L^p(0,\infty;W^{1,p}(Q;R^d))\cap W^{1,p}(0,\infty;L^p(Q;R^d))},\ \ p > 3 \end{aligned} \tag{1}$$

Solution space (trajectory space)

Solutions (trajectories)

$$\mathbf{U} = (\varrho, \vartheta, \mathbf{u}) \in X_T, \quad T < T_{\rm max}, \ \ T_{\rm max} = T_{\rm max}[D]$$

Trajectory space

$$\varrho \in C^1([0,T]; W^{1,p}(Q))$$
$$\vartheta \in L^p(0,T; W^{2,p}(Q)) \cap W^{1,p}(0,T; L^p(Q))$$
$$\mathbf{u} \in L^p(0,T; W^{2,p}(Q; R^d)) \cap W^{1,p}(0,T; L^p(Q; R^d))$$

Stability with respect to the data

$$D_n = [\varrho_{D,n}, \vartheta_{D,n}, \mathbf{u}_{D,n}] \to D = [\varrho_D, \vartheta_D, \mathbf{u}_D] \text{ in } X_D$$
$$\Rightarrow$$
$$\liminf_{n \to \infty} T_{\rm max}[D_n] \geq T_{\rm max}[D] > 0$$
$$(\varrho, \vartheta, \mathbf{u})[D_n] \to (\varrho, \vartheta, \mathbf{u})[D] \text{ in } X_T \text{ for any } 0 < T < T_{\rm max}$$

Analytical results, summary

Existence and uniqueness
For any data $D = (\varrho_D, \vartheta_D, \mathbf{u}_D) \in X_D$, there exists a unique solution $(\varrho, \vartheta, \mathbf{u})$ on a maximal time interval $[0, T_{\max})$, $T_{\max} > 0$.
Stability
The mapping $D \in X_D \mapsto T_{\max}[D]$ is lower semi–continuous. If

$$D_n \to D \text{ in } X_D,$$

then

$$(\varrho, \vartheta, \mathbf{u})[D_n] \to (\varrho, \vartheta, \mathbf{u})[D] \text{ in } X_T \text{ for any } T < T_{\max}$$

Conditional regularity

$$\|\varrho(t,\cdot)\|_{W^{1,p}(\Omega)} + \|\vartheta(t,\cdot)\|_{W^{2-\frac{1}{p},p}(\Omega)} + \|\mathbf{u}(t,\cdot)\|_{W^{2-\frac{1}{p},p}(\Omega;R^d)}$$

$$\leq C\left(T, \|D\|_{X_D}, \sup_{t\in[0,T]}\left(\sup_Q \varrho(t,\cdot) + \sup_Q \vartheta(t,\cdot) + \sup_Q |\mathbf{u}(t,\cdot)|\right)\right.$$

for any $0 \leq t \leq T < T_{\max}$, C bounded for bounded arguments

Problems with uncertain data

Probability space

$$\{\Omega; \mathcal{B}, \mathbb{P}\},\ \Omega \text{ measurable space}$$

$$\mathcal{B}\ \sigma - \text{algebra of measurable sets},\ \mathbb{P} - \text{complete probability measure}$$

Random data

$$\omega \in \Omega \mapsto D \in X_D \text{ Borel measurable mapping}$$

Solutions as random variables

$$T_{\max} = T_{\max}[D] - \text{random variable}$$

$$D \mapsto (\varrho, \vartheta, \mathbf{u})[D] \text{ random variable}$$

Statistical solution

$$\text{strong sense: } \omega \in \Omega \mapsto (\varrho, \vartheta, \mathbf{u})(t, \cdot)[D]$$

$$\text{weak sense: } \mathcal{L}[(\varrho, \vartheta, \mathbf{u})(t, \cdot)[D]]$$

$$\mathcal{L} \text{ - law (distribution) of } (\varrho, \vartheta, \mathbf{u})(t, \cdot) \text{ in } W^{1,p} \times W^{2-\frac{1}{p},p} \times W^{2-\frac{1}{p},p}$$

Strong stability problem I

Data convergence

$$D_n = [\varrho_{D,n}, \vartheta_{D,n}, \mathbf{u}_{D,n}] \to D = [\varrho_D, \vartheta_D, \mathbf{u}_D] \text{ in } X_D$$

$$\mathbb{P}- \text{ a.s.}$$

Solution convergence

$$(\varrho, \vartheta, \mathbf{u})[D_n] \to (\varrho, \vartheta, \mathbf{u})[D] \text{ in } X_T$$

$$T < T_{\max}[D]$$

$$\mathbb{P}- \text{ a.s.}$$

Weak stability problem I

Data convergence in law (in distribution)

$$\mathcal{L}[D_n] = \mathcal{L}[\varrho_{D,n}, \vartheta_{D,n}, \mathbf{u}_{D,n}] \to \mathcal{L}$$

narrowly in $\mathfrak{P}[X_D]$

Tools from probability theory I

Skorokhod (representation) theorem
Let $(\mathcal{L}_n)_{n=1}^{\infty}$ be a sequence of probability measures on a Polish space X. Suppose that the sequence is tight in X, meaning for any $\varepsilon > 0$, there exists a compact set $K(\varepsilon) \subset X$ such that

$$\mathcal{L}_n[X \setminus K(\varepsilon)] \leq \varepsilon \text{ for all } n = 1, 2, \dots .$$

Then there is a subsequence $n_k \to \infty$ as $k \to \infty$ and a sequence of random variables $(\widetilde{D}_{n_k})_{k=1}^{\infty}$ defined on the standard probability space

$$\left(\widetilde{\Omega} = [0,1], \mathfrak{B}[0,1], \mathrm{d}y\right)$$

satisfying:

- $$\mathrm{law}[\widetilde{D}_{n_k}] = \mathcal{L}_{n_k},$$
- $$\widetilde{D}_k \to \widetilde{D} \text{ in } X \text{ for every } y \in [0,1].$$

Convergence in weak stability problem I

Skorokhod representation theorem

$$D_n \approx_{X_D} \widetilde{D}_{n_k}$$

Strong convergence in the new probability space

$$(\widetilde{\varrho}_k, \widetilde{\vartheta}_k, \widetilde{\mathbf{u}}_k) \equiv (\varrho, \vartheta, \mathbf{u})[\widetilde{D}_{n_k}] \to (\varrho, \vartheta, \mathbf{u})[\widetilde{D}]$$

in X_T surely $\mathrm{d}y$

Equivalence in law (Borel measurability of the solution mapping)

$$(\widetilde{\varrho}_n, \widetilde{\vartheta}_n, \widetilde{\mathbf{u}}_n) \approx (\varrho, \vartheta, \mathbf{u})[D_n]$$

Conclusion

$$\mathcal{L}[(\varrho, \vartheta, \mathbf{u})[D_n]] \to \mathcal{L}[(\varrho, \vartheta, \mathbf{u})[\widetilde{D}]]$$

narrowly

Strong stability problem II - global in time convergence

Data convergence

$$D_n = [\varrho_{D,n}, \vartheta_{D,n}, \mathbf{u}_{D,n}] \to D = [\varrho_D, \vartheta_D, \mathbf{u}_D] \text{ in } X_D$$

$$\mathbb{P}- \text{ a.s.}$$

Hypothesis of boundedness in probability

For any $\varepsilon > 0$, there exists $M > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}\left\{ \sup_{(0,T)\times Q} \varrho[D_n] + \sup_{(0,T)\times Q} \vartheta[D_n] + \sup_{(0,T)\times Q} |\mathbf{u}[D_n]| > M \right\} < \varepsilon$$

Conclusion (to be shown below)

$$(\varrho, \vartheta, \mathbf{u})[D_n] \to (\varrho, \vartheta, \mathbf{u})[D] \text{ in } X_T$$

in probability

Strong stability problem II - proof of convergence

Skorokhod representation theorem

augmented sequence of random variables $(D_n, (\varrho, \vartheta, \mathbf{u})[D_n],\ \Lambda_n)_{n=1}^{\infty}$

$$\Lambda_n = \sup_{(0,T)\times Q} \varrho[D_n] + \sup_{(0,T)\times Q} \vartheta[D_n] + \sup_{(0,T)\times Q} |\mathbf{u}[D_n]|$$

Skorokhod representation

$$(\widetilde{D}_n, (\varrho, \vartheta, \mathbf{u})[\widetilde{D}_n], \widetilde{\Lambda}_n)_{n=1}^{\infty}$$

$$\widetilde{\Lambda}_n = \sup_{(0,T)\times Q} \varrho[\widetilde{D}_n] + \sup_{(0,T)\times Q} \vartheta[\widetilde{D}_n] + \sup_{(0,T)\times Q} |\mathbf{u}[\widetilde{D}_n]| \to \widetilde{\Lambda}$$

$\mathrm{d}y$ surely

Conclusion by conditional regularity

$$\widetilde{D}_n \to \widetilde{D} \text{ in } X_D$$

$$(\varrho, \vartheta, \mathbf{u})[\widetilde{D}_n] \to (\varrho, \vartheta, \mathbf{u})[\widetilde{D}]$$

in X_T, $\mathrm{d}y$ surely

Tools from probability theory II

Gyöngy–Krylov theorem

Let X be a Polish space and $(\mathbf{U}_M)_{M\geq 1}$ a sequence of $X-$valued random variables.

Then $(\mathbf{U}_M)_{M=1}^{\infty}$ converges in probability if and only if for any sequence of joint laws of

$$(\mathbf{U}_{M_k}, \mathbf{U}_{N_k})_{k=1}^{\infty}$$

there exists further subsequence that converge weakly to a probability measure μ on $X \times X$ such that

$$\mu\left[(x, y) \in X \times X,\ x = y\right] = 1.$$

Approximate solutions

Approximate solutions

$$(\varrho, \mathbf{u}, \vartheta)_h[D], \ D \in X_D, \ h > 0 \text{ discretization parameter}$$

$$D \in X_D \mapsto (\varrho, \mathbf{u}, \vartheta)_h \in L^1((0,T) \times Q; R^{d+2}) \text{ Borel measurable for any } h > 0.$$

Consistent approximation

Conservative boundary conditions (for simplicity)

$$\mathbf{u}|_{\partial Q} = 0,\ \mathbf{q} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

Approximate field equations

$$\partial_t \varrho_n + \mathrm{div}_x(\varrho_n \mathbf{u}_n) = e_n^1 \text{ in } \mathcal{D}'((0,T) \times Q),$$

Consistent approximation

$$\varrho_n = \varrho_{h_n}[D],\ \vartheta_n = \vartheta_{h_n}[D],\ \mathbf{u}_n = \mathbf{u}_{h_n}[D]$$

$$\begin{aligned}\partial_t(\varrho_n \mathbf{u}_n) &+ \mathrm{div}_x(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) + \nabla_x p(\varrho_n, \vartheta_n) \\ &= \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u}_n) + \varrho_n \nabla_x G + e_n^2 \text{ in } \mathcal{D}'((0,T) \times Q; R^d)\end{aligned}$$

$$\begin{aligned}\partial_t(\varrho_n s(\varrho_n, \vartheta_n)) &+ \mathrm{div}_x(\varrho_n s(\varrho_n, \vartheta_n) \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q_n}}{\vartheta_n} \right) \\ &\geq \frac{1}{\vartheta_n} \left(\mathbb{S}(\mathbb{D}_x \mathbf{u}_n) : \mathbb{D}_x \mathbf{u}_n - \frac{\mathbf{q}_n \cdot \nabla_x \vartheta_n}{\vartheta_n} \right) + e_n^3 \text{ in } \mathcal{D}'((0,T) \times Q)\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_Q \left[\varrho_n |\mathbf{u}_n|^2 + \varrho_n e(\varrho_n, \vartheta_n) - \varrho_n G \right] \mathrm{d}x \leq e_n^4 \text{ in } \mathcal{D}'(0,T)$$

$$e_n^1, e_n^2, e_n^3, e_n^4 \to 0 \text{ as } n \to \infty \text{ in a ``weak'' sense}$$

Convergence of consistent approximations, I

Strong data convergence

$$D_n = [\varrho_{D,n}, \vartheta_{D,n}, \mathbf{u}_{D,n}] \to D = [\varrho_D, \vartheta_D, \mathbf{u}_D] \text{ in } X_D$$

$$\mathbb{P}- \text{ a.s.}$$

Hypothesis of boundedness in probability
For any $\varepsilon > 0$, there exists $M > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P} \left\{ \sup_{(0,T)\times Q} \varrho_n[D_n] + \sup_{(0,T)\times Q} \vartheta_n[D_n] + \sup_{(0,T)\times Q} |\mathbf{u}_n[D_n]| > M \right\} < \varepsilon$$

Consistent approximation

$[\varrho, \vartheta, \mathbf{u}]_n[D_n]$ a sequence of consistent approximations

Convergence of consistent approximations, II

1. Apply Skorokhod representation theorem to the sequence $(D_n, \varrho_n, \vartheta_n \mathbf{u}_n, \Lambda_n)_{n=1}^{\infty}$,

$$\Lambda_n = \sup_{(0,T)\times Q} \varrho_n[D_n] + \sup_{(0,T)\times Q} \vartheta_n[D_n] + \sup_{(0,T)\times Q} |\mathbf{u}_n[D_n]|$$

2. New sequence of data $\widetilde{D}_n$ with the same law on the standard probability space,

$$\widetilde{D}_n \to \widetilde{D} \text{ in } X_d, \ \mathrm{d}y \text{ surely.}$$

$$\widetilde{\Lambda}_n = \sup_{(0,T)\times Q} \varrho_n[\widetilde{D}_n] + \sup_{(0,T)\times Q} \vartheta_n[\widetilde{D}_n] + \sup_{(0,T)\times Q} |\mathbf{u}_n[\widetilde{D}_n]| \to \widetilde{\Lambda}$$

$$\mathrm{d}y \text{ surely}$$

$$\varrho_{n_k}[\widetilde{D}_{n_k}] \to \widetilde{\varrho} \text{ weakly-(*) in } L^\infty((0,T)\times Q)$$

$$\vartheta_{n_k}[\widetilde{D}_{n_k}] \to \widetilde{\vartheta} \text{ weakly-(*) in } L^\infty((0,T)\times Q)$$

$$\mathbf{u}_{n_k}[\widetilde{D}_{n_k}] \to \widetilde{\mathbf{u}} \text{ weakly-(*) in } L^\infty((0,T)\times Q; R^d)$$

$$\mathrm{dy} \ \text{ surely}$$

Convergence of consistent approximations, III

4. Show the limit is a measure–valued solution with the data $\widetilde{D}$ in the sense of Březina, EF, Novotný [2020], see also Chaudhuri [2022]
5. Apply the weak–strong uniqueness principle to conclude the $(\widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\mathbf{u}})$ is the unique strong solution associated to the data $\widetilde{D}$,

$$(\widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\mathbf{u}}) = (\varrho, \vartheta, \mathbf{u})[\widetilde{D}].$$

   Conclude there is no need of subsequence, $T_{\max}[\widetilde{D}] > T$, and convergence is strong for in L^q for any finite q.
6. Pass to the original space using Gyöngy–Krylov theorem

Conclusion

$$(\varrho_n, \vartheta_n, \mathbf{u}_n)[D_n] \to (\varrho, \vartheta, \mathbf{u})[D]$$

in $L^q((0,T) \times Q; R^{d+2})$ for any $1 \leq q < \infty$

in probability