Euler system and turbulence: Computing oscillatory solutions in fluid dynamics

Eduard Feireisl
based on joint work with A. Abbatiello (Roma), E. Chiodaroli (Pisa), M. Hofmanová (Bielefeld), M. Lukáčová-Medvid'ová (Mainz), H. Mizerová (Bratislava), B.She (Beijing)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

Institute of Mathematics of CAS, Praha
March 17, 2023

Czech Academy of Sciences

GAČR
CZECH SCIENCE FOUNDATION

Euler system of gas dynamics

Leonhard Paul Euler 1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation – Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0, \ \ p(\varrho) = a\varrho^\gamma$$

Impermeable boundary

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \ \Omega \subset R^d, \ \ d = 2, 3$$

Initial state (data)

$$\varrho(0, \cdot) = \varrho_0, \ \ (\varrho \mathbf{u})(0, \cdot) = \varrho_0 \mathbf{u}_0$$

Admissibility

Energy

$$E(\varrho, \mathbf{u}) = \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)$$

Pressure potential

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho),\ P(\varrho) = \frac{a}{\gamma - 1}\varrho^\gamma$$

Dissipative (weak) solutions

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E(\varrho, \mathbf{u})\ \mathrm{d}x \leq 0,\ \int_\Omega E(\varrho, \mathbf{u})(\tau, \cdot)\ \mathrm{d}x \leq \int_\Omega E(\varrho_0, \mathbf{u}_0)\ \mathrm{d}x$$

Admissible (weak) solutions

$$\partial_t E(\varrho, \mathbf{u}) + \mathrm{div}_x \Big(E(\varrho, \mathbf{u})\mathbf{u} + p(\varrho)\mathbf{u} \Big) \leq 0,\ E(\varrho, \mathbf{u})(\tau, \cdot) \nearrow E(\varrho_0, \mathbf{u}_0),\ \tau \to 0$$

Wild data

Initial state

$$\varrho(0,\cdot) = \varrho_0,\ (\varrho\mathbf{u})(0,\cdot) = \varrho_0\mathbf{u}_0$$

The initial data are *wild* if there exists $T > 0$ such that the Euler system admits infinitely many (weak) *admissible* solutions on any time interval $[0,\tau]$, $0 < \tau < T$

Theorem (E. Chiodaroli, EF 2022) The set of wild data is dense in $L^2 \times L^2$

E. Chiodaroli (Pisa)

Related results for the incompressible Euler system by Székelyhidi–Wiedemann, Daneri–Székelyhidy

Related results for the barotropic Euler system by Ming, Vasseur, and You

$$\int_\Omega E(\varrho,\mathbf{u})(\tau)\ \mathrm{d}x \le \int_\Omega E(\varrho_0,\mathbf{u}_0)\ \mathrm{d}x,\ \tau \ge 0$$

Weak vs. strong continuity

$$\mathbf{U} = [\varrho, \mathbf{m}],\ \mathbf{m} = \varrho \mathbf{u}$$

Weak continuity

$$\mathbf{U} \in C_{\rm weak}([0,T]; L^p(\Omega; R^d)),\ t \mapsto \int_\Omega \mathbf{U} \cdot \boldsymbol{\varphi} \ {\rm d}x \in C[0,T]$$

$$\boldsymbol{\varphi} \in L^{p'}(\Omega; R^d)$$

Strong continuity

$$\tau \in [0,T],\ \|\mathbf{U}(t,\cdot) - \mathbf{U}(\tau,\cdot)\|_{L^p(\Omega;R^d)} \text{ whenever } t \to \tau$$

Strong vs. weak

$$\text{strong } \Rightarrow \text{weak},\ \text{weak} \not\Rightarrow \text{strong}$$

Strong discontinuity

Theorem (A.Abbatiello, EF 2021)

Anna Abbatiello (Roma La Sapienza)

Let $d = 2, 3$. Let $\mathcal{R}$ denote the set of bounded Riemann integrable functions. Let ϱ_0, $\mathbf{m}_0$ be given such that

$$\varrho_0 \in \mathcal{R},\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R},\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R},\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^{\infty} \subset (0, T)$ be an arbitrary (countable dense) set of times.

Then the Euler problem admits infinitely many weak solutions ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that

$$\varrho \in C_{\rm weak}([0, T]; L^{\gamma}(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

but

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)]\ \boxed{\text{is not}}\ \text{strongly continuous at any } \tau_i$$

FV numerical scheme

$$(\varrho_h^0, \mathbf{u}_h^0) = (\Pi_{\mathcal{T}} \varrho_0, \Pi_{\mathcal{T}} \mathbf{u}_0)$$

$$D_t \varrho_K^k + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} F_h(\varrho_h^k, \mathbf{u}_h^k) = 0$$

$$D_t(\varrho_h^k \mathbf{u}_h^k)_K + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \left(\mathbf{F}_h(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) + \overline{p(\rho_h^k)}\mathbf{n} - h^\beta \left[\left[\mathbf{u}_h^k \right]\right] \right) = 0.$$

Mária Lukáčová (Mainz)

Hana Mizerová (Bratislava)

Discrete time derivative

$$D_t r_K^k = \frac{r_K^k - r_K^{k-1}}{\Delta t}$$

Upwind, fluxes

$$\mathrm{Up}[r, \mathbf{v}] = \overline{r}\ \overline{\mathbf{v}} \cdot \mathbf{n} - \frac{1}{2} |\overline{\mathbf{v}} \cdot \mathbf{n}|\, [[r]]$$

$$F_h(r, \mathbf{v}) = \mathrm{Up}[r, \mathbf{v}] - h^\alpha\, [[r]]$$

Consistent approximation

$$\varrho \approx \varrho_n, \ \ \varrho \mathbf{u} \approx \mathbf{m}_n$$

Approximate equation of continuity

$$\int_0^T \int_\Omega \left[\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi \right] \mathrm{d}x \mathrm{d}t = - \int_\Omega \varrho_0 \varphi \ \mathrm{d}x + e_{1,n}[\varphi]$$

Approximate momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \varphi + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \mathrm{d}t$$

$$= - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi} \ \mathrm{d}x + e_{2,n}[\boldsymbol{\varphi}]$$

Stability - approximate energy inequality

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] \mathrm{d}x \leq \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \mathrm{d}x + e_{3,n}$$

Consistency

$$e_{1,n}[\varphi] \to 0, \ \ e_{2,n}[\boldsymbol{\varphi}] \to 0, \ \ e_{3,n} \to 0 \text{ as } n \to \infty$$

Lax equivalence principle

Formulation for LINEAR problems

Peter D. Lax

- **Stability** - uniform bounds of approximate solutions
- **Consistency** - vanishing approximation error

$$\Longrightarrow$$

- **Convergence** - approximate solutions converge to exact solution

Weak vs strong convergence

Weak convergence

$$\varrho_n \to \varrho \text{ weakly-(*) } L^\infty(0,T;L^\gamma(\Omega))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ weakly-(*) } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d))$$

Strong convergence (Theorem EF, M.Hofmanová)

- Suppose

$$\Omega \subset R^d \text{ bounded}$$

$$\varrho_n \to \varrho,\ \mathbf{m}_n \to \mathbf{m} \text{ a.a. pointwise in } \mathcal{U} \text{ open},\ \partial\Omega \subset \mathcal{U}$$

- Then the following is equivalent:

$$\varrho, \mathbf{m} \text{ weak solution to the Euler system}$$

$$\Leftrightarrow$$

$$\varrho_n \to \varrho,\ \mathbf{m}_n \to \mathbf{m} \text{ strongly (in } L^1) \text{ in } \Omega$$

Martina Hofmanová (Bielefeld)

Dissipative solutions – limits of consistent approximations

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \mathfrak{R}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq 0, \ E(t) \leq E_0, \ E_0 = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$

$$E \equiv \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x + c(\gamma) \int_{\overline{\Omega}} \mathrm{d} \ \mathrm{trace}[\mathfrak{R}]$$

Reynolds stress

$$\mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\mathrm{sym}}))$$

Convergence: Good and bad news

- **Weak–strong uniqueness.**

$$\text{stability} + \text{consistency} \;\Rightarrow\; \text{(strong) convergence}$$

as soon as the limit system admits a strong solution

- **Compatibility**

$$\text{stability} + \text{consistency} \;\Rightarrow\; \text{(strong) convergence}$$

as soon as the limit ϱ, $\mathbf{m}$ is smooth.

- **Limit is not Euler**
If consistent approximations converge strongly in a neighbourhood of $\partial\Omega$ but *weakly* otherwise, the the limit IS NOT a solution of the Euler system

Statistical description of oscillations – Young measures

Laurence Chisholm Young 1905–2000

Young measure

$$b(\varrho_n, \mathbf{m}_n) \to \overline{b(\varrho, \mathbf{m})} \text{ weakly-(*) in } L^\infty((0,T)\times\Omega)$$

(**up to a subsequence**) for any $b \in C_c(R^{d+1})$

Young measure $\mathcal{V}$ – a parametrized family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ on the phase space R^{d+1}:

$$\overline{b(\varrho, \mathbf{m})}(t,x) = \langle \mathcal{V}_{t,x}; b(\widetilde{\varrho}, \widetilde{\mathbf{m}})\rangle \text{ for a.a. } (t,x)$$

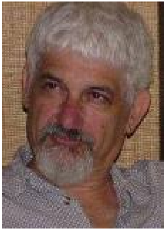
Visualizing Young measure

visualizing Young measure $\Leftrightarrow$ computing $\overline{b(\varrho, \mathbf{m})}$

Problems

- $b(\varrho_n, \mathbf{m}_n)$ converge only weakly
- extracting subsequences
- only statistical properties relevant $\Rightarrow$ knowledge of the "tail" of the sequence of approximate solutions absolutely necessary

Strong instead of weak (numerics)



Janos Komlós (Ruthers Univ.)

Komlós theorem (a variant of Strong Law of Large Numbers)

$$\{U_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q)$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} U_{n_k} \to \overline{U} \text{ a.a. in } Q \text{ as } N \to \infty$$

Elementary proof of Banach–Saks Theorem in L^2:

$$U_n \text{ an orthonormal basis, } U_n \to 0 \text{ weakly in } L^2$$

$$\int_Q \left(\sum_{n=1}^{N} U_n\right)^2 \mathrm{d}y = \sum_{n=1}^{N} \int_Q |U_n|^2 \, \mathrm{d}y = N$$

$$\Rightarrow$$

$$\left\| \frac{1}{N} \sum_{n=1}^{N} U_n \right\|_{L^2}^2 = \frac{N}{N^2} = \frac{1}{N}$$

Computing the limit distribution - measure

Alternatives to Young's definition via Komlós theorem

$$\mathbf{U}_n = [\varrho_n, \mathbf{m}_n] \in R^{d+1} \text{ phase space}$$

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q; R^d) \approx \nu^n_{t,x} = \delta_{\mathbf{U}_n(t,x)}$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} \nu^{n_k}_{t,x} \to \nu_{t,x} \text{ narrowly } \boxed{a.a.} \text{ in } Q \text{ as } N \to \infty$$

Erich J. Balder (Utrecht)

S–convergence

$$\frac{1}{N}\sum_{k=1}^{N} B(\varrho_{n_k}, \mathbf{m}_{n,k})(t,x) = \int B(\widetilde{\varrho}, \widetilde{\mathbf{m}}) \mathrm{d}\nu_{t,x}$$

for a.a. $(t,x) \in Q$

Convergence of numerical solutions - EF, M.Lukáčová, H.Mizerová 2018

$$\frac{1}{N}\sum_{k=1}^{N} \varrho_{n_k} \to \varrho, \ \frac{1}{N}\sum_{k=1}^{N} \mathbf{m}_{n_k} \to \mathbf{m} \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

Computing defect numerically -EF, M.Lukáčová, B.She

Monge–Kantorowich (Wasserstein) distance

$$\left\| \mathrm{dist} \left(\frac{1}{N} \sum_{k=1}^{N} \nu_{t,x}^{n_k} ; \nu_{t,x} \right) \right\|_{L^q((0,T)\times\Omega)} \to 0$$

for some $q > 1$

Convergence in the first variation

$$\frac{1}{N} \sum_{k=1}^{N} \left\langle \nu_{t,x}^{n_k} ; \left| \widetilde{\mathbf{U}} - \frac{1}{N} \sum_{k=1}^{N} \mathbf{U}_n \right| \right\rangle \to \left\langle \nu_{t,x} ; \left| \widetilde{\mathbf{U}} - \mathbf{U} \right| \right\rangle$$

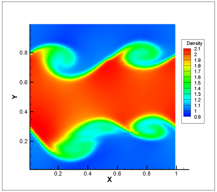
in $L^1(Q)$

Mária Lukáčová (Mainz)

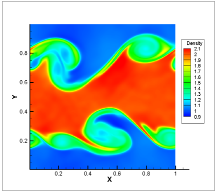
Bangwei She (CAS Praha)

Experiment I, density for Kelvin–Helmholtz problem (M. Lukáčová, Yue Wang)

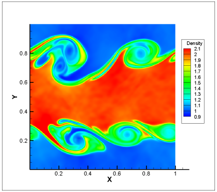
density ϱ
$n = 128, \; T = 2$



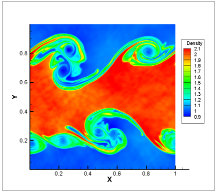
density ϱ
$n = 256, \; T = 2$



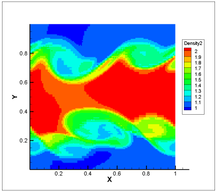
density ϱ
$n = 512, \; T = 2$



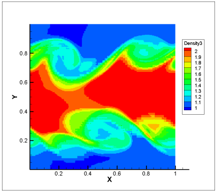
density ϱ
$n = 1024, \; T = 2$



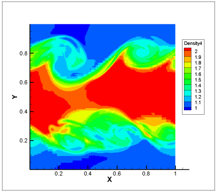
Cèsaro averages density ϱ
$n = 128, \; T = 2$



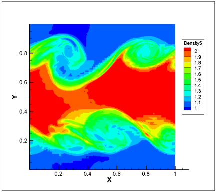
Cèsaro averages density ϱ
$n = 256, \; T = 2$



Cèsaro averages density ϱ
$n = 512, \; T = 2$

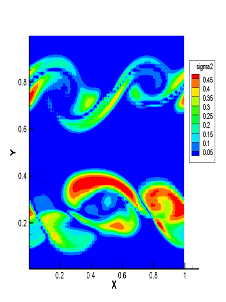


Cèsaro averages density ϱ
$n = 1024, \; T = 2$

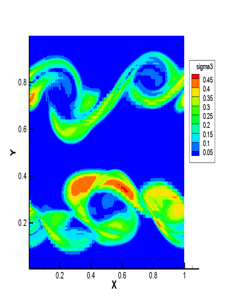


Experiment II, density variations for Kelvin–Helmholtz problem (M. Lukáčová, Yue Wang)

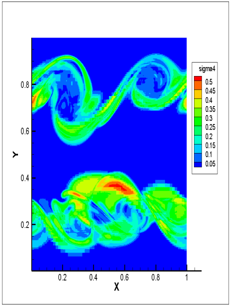
density variation $n = 128,\ T = 2$



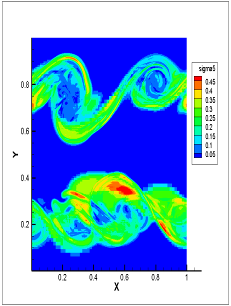
density variation $n = 256,\ T = 2$



density variation $n = 512,\ T = 2$



density variation $n = 1024,\ T = 2$



Yue Wang, Mainz

Mária Lukáčová, Mainz