

Euler system in fluid dynamics: Good and bad news

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Euler system of gas dynamics



Leonhard Paul
Euler
1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation – Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0, \quad p(\varrho) = a\varrho^\gamma$$

Impermeable boundary

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \Omega \subset \mathbb{R}^d, \quad d = 2, 3$$

Initial state (data)

$$\varrho(0, \cdot) = \varrho_0, \quad (\varrho \mathbf{u})(0, \cdot) = \varrho_0 \mathbf{u}_0$$

Admissibility

Energy

$$E(\varrho, \mathbf{u}) = \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho)$$

Pressure potential

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho), \quad P(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma$$

Dissipative (weak) solutions

$$\frac{d}{dt} \int_{\Omega} E(\varrho, \mathbf{u}) \, dx \leq 0, \quad \int_{\Omega} E(\varrho, \mathbf{u})(\tau, \cdot) \, dx \leq \int_{\Omega} E(\varrho_0, \mathbf{u}_0) \, dx$$

Admissible (weak) solutions

$$\partial_t E(\varrho, \mathbf{u}) + \operatorname{div}_x (E(\varrho, \mathbf{u})\mathbf{u} + p(\varrho)\mathbf{u}) \leq 0, \quad E(\varrho, \mathbf{u})(\tau, \cdot) \nearrow E(\varrho_0, \mathbf{u}_0), \quad \tau \rightarrow 0$$

Wild data

Initial state

$$\varrho(0, \cdot) = \varrho_0, (\varrho \mathbf{u})(0, \cdot) = \varrho_0 \mathbf{u}_0$$

The initial data are *wild* if there exists $T > 0$ such that the Euler system admits infinitely many (weak) *admissible* solutions on any time interval $[0, \tau]$, $0 < \tau < T$



Theorem (E. Chiodaroli, EF 2022) The set of wild data is dense in $L^2 \times L^2$

E. Chiodaroli (Pisa)

Related results for the incompressible Euler system by Székelyhidi–Wiedemann, Daneri–Székelyhidy

Related results for the barotropic Euler system by Ming, Vasseur, and You

$$\int_{\Omega} E(\varrho, \mathbf{u})(\tau) \, dx \leq \int_{\Omega} E(\varrho_0, \mathbf{u}_0) \, dx, \quad \tau \geq 0$$

Weak vs. strong continuity

$$\mathbf{U} = [\varrho, \mathbf{m}], \quad \mathbf{m} = \varrho \mathbf{u}$$

Weak continuity

$$\mathbf{U} \in C_{\text{weak}}([0, T]; L^p(\Omega; \mathbb{R}^d)), \quad t \mapsto \int_{\Omega} \mathbf{U} \cdot \varphi \, dx \in C[0, T]$$
$$\varphi \in L^{p'}(\Omega; \mathbb{R}^d)$$

Strong continuity

$$\tau \in [0, T], \quad \|\mathbf{U}(t, \cdot) - \mathbf{U}(\tau, \cdot)\|_{L^p(\Omega; \mathbb{R}^d)} \rightarrow 0 \text{ whenever } t \rightarrow \tau$$

Strong vs. weak

strong \Rightarrow weak, weak $\not\Rightarrow$ strong

Strong discontinuity

Theorem (A. Abbatiello, EF 2021)



Anna
Abbatiello
(Roma La
Sapienza)

Let $d = 2, 3$. Let \mathcal{R} denote the set of bounded Riemann integrable functions. Let ϱ_0, \mathbf{m}_0 be given such that

$$\varrho_0 \in \mathcal{R}, \quad 0 \leq \underline{\varrho} \leq \varrho_0 \leq \bar{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R}, \quad \operatorname{div}_x \mathbf{m}_0 \in \mathcal{R}, \quad \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^{\infty} \subset (0, T)$ be an arbitrary (countable dense) set of times.

Then the Euler problem admits infinitely many weak solutions ϱ, \mathbf{m} with a strictly decreasing total energy profile such that

$$\varrho \in C_{\text{weak}}([0, T]; L^\gamma(\Omega)), \quad \mathbf{m} \in C_{\text{weak}}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d))$$

but

$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)]$ is not strongly continuous at any τ_i

Consistent approximation

Approximate equation of continuity

$$\int_0^T \int_{\Omega} [\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi] dx dt = e_{1,n}[\varphi]$$

Approximate momentum equation

$$\int_0^T \int_{\Omega} \left[\mathbf{m}_n \cdot \partial_t \varphi + \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \varphi + p(\varrho_n) \operatorname{div}_x \varphi \right] dx dt = e_{2,n}[\varphi]$$

Stability - approximate energy inequality

$$\int_{\Omega} \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] dx \leq \int_{\Omega} \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] dx + e_{3,n}$$

Consistency

$$e_{1,n}[\varphi] \rightarrow 0, \quad e_{2,n}[\varphi] \rightarrow 0, \quad e_{3,n} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

Dissipative solutions – limits of consistent approximations



Dominic Breit
(Edinburgh)



Martina Hofmanová
(Bielefeld)

$$\partial_t \varrho + \operatorname{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \operatorname{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\operatorname{div}_x \mathfrak{R}$$

$$\frac{d}{dt} E(t) \leq 0, \quad E(t) \leq E_0, \quad E_0 = \int_{\Omega} \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] dx$$

$$E \equiv \int_{\Omega} \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] dx + c(\gamma) \int_{\bar{\Omega}} d \operatorname{trace}[\mathfrak{R}]$$

Reynolds stress

$$\mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(\bar{\Omega}; R_{\operatorname{sym}}^{d \times d}))$$

Basic properties of dissipative solutions

Well posedness, weak strong uniqueness

- **Existence.** Dissipative solutions exist globally in time for any finite energy initial data
- **Limits of consistent approximations** Limits of consistent approximations are dissipative solutions, in particular limits of consistent numerical schemes.
- **Compatibility.** Any C^1 dissipative solution $[\varrho, \mathbf{m}]$, $\varrho > 0$ is a classical solution of the Euler system
- **Weak–strong uniqueness.** If $[\tilde{\varrho}, \tilde{\mathbf{m}}]$ is a classical solution and $[\varrho, \mathbf{m}]$ a dissipative solution starting from the same initial data, then $\mathfrak{R} = 0$ and $\varrho = \tilde{\varrho}$, $\mathbf{m} = \tilde{\mathbf{m}}$.

Semiflow selection

Set of data

$$\mathcal{D} = \left\{ \varrho, \mathbf{m}, E \mid \int_{\Omega} \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \, dx \leq E \right\}$$

Set of trajectories

$$\mathcal{T} = \left\{ \varrho(t, \cdot), \mathbf{m}(t, \cdot), E(t-, \cdot) \mid t \in (0, \infty) \right\}$$

Solution set

$$\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] = \left\{ [\varrho, \mathbf{m}, E] \mid [\varrho, \mathbf{m}, E] \text{ dissipative solution} \right.$$

$$\left. \varrho(0, \cdot) = \varrho_0, \mathbf{m}(0, \cdot) = \mathbf{m}_0, E(0+) \leq E_0 \right\}$$

Semiflow selection – semigroup

$$U[\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0], [\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{D}$$

$$U(t_1+t_2)[\varrho_0, \mathbf{m}_0, E_0] = U(t_1) \circ [U(t_2)[\varrho_0, \mathbf{m}_0, E_0]], t_1, t_2 > 0$$



**Andrej Markov
(1856–1933)**



N. V. Krylov

Adaptation of Krylov's method by Cardona and Kapitanski



Jorge Cardona
(TU
Darmstadt)

Successive minimization of cost functionals

$$I_{\lambda, F}[\varrho, \mathbf{m}, E] = \int_0^{\infty} \exp(-\lambda t) F(\varrho, \mathbf{m}, E) dt, \quad \lambda > 0$$

where

$$F : X = W^{-\ell, 2}(\Omega) \times W^{-\ell, 2}(\Omega; \mathbb{R}^d) \times \mathbb{R} \rightarrow \mathbb{R}$$

is a bounded and continuous functional

Maximal dissipation principle

$$\tilde{E}(t) \leq E(t) \Rightarrow E(t) = \tilde{E}(t)$$

whenever, $\varrho_0 = \tilde{\varrho}_0$, $\mathbf{m}_0 = \tilde{\mathbf{m}}_0$, $E_0 = \tilde{E}_0$



Lev
Kapitanski
(Miami)

Euler system and turbulence

Fluid domain and obstacle

$$Q = R^d \setminus B, \quad d = 2, 3$$

B compact, convex

Navier–Stokes system

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$p(\varrho) \approx a\varrho^\gamma, \quad \gamma > 1, \quad \mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \operatorname{div}_x \mathbf{u} \mathbb{I},$$

Boundary and far field conditions

$$\mathbf{u}|_{\partial Q} = 0, \quad \varrho \rightarrow \varrho_\infty, \quad \mathbf{u} \rightarrow \mathbf{u}_\infty \quad \text{as } |x| \rightarrow \infty$$

High Reynolds number (vanishing viscosity) limit

Vanishing viscosity

$$\varepsilon_n \searrow 0, \mu_n = \varepsilon_n \mu, \mu > 0, \lambda_n = \varepsilon_n \lambda, \lambda \geq 0$$

Questions

- Identify the limit of the corresponding solutions $(\varrho_n, \mathbf{u}_n)$ as $n \rightarrow \infty$ in the fluid domain Q
- **Yakhot and Orszak [1986]:** *“The effect of the boundary in the turbulence regime can be modeled in a **statistically equivalent way** by fluid equations driven by stochastic forcing”*

Clarify the meaning of “statistically equivalent way”

Is the (compressible) Euler system driven by a general cylindrical white noise force adequate to describe the limit of $(\varrho_n, \mathbf{u}_n)$?

Statistical limit

Trajectory space

$$(\varrho_n, \mathbf{m}_n) \in \mathcal{T} \equiv C_{\text{weak}}([0, T]; L_{\text{loc}}^\gamma(\Omega) \times L_{\text{loc}}^{\frac{2\gamma}{\gamma+1}}(\Omega; \mathbb{R}^d))$$

$$\mathcal{V}_N = \frac{1}{N} \sum_{n=1}^N \delta_{(\varrho_n, \mathbf{m}_n)}, \quad \mathbf{m}_n = \varrho_n \mathbf{u}_n$$

Prokhorov theorem $\Rightarrow \mathcal{V}_N \rightarrow \mathcal{V}$ narrowly in $\mathfrak{P}[\mathcal{T}]$

$(\varrho, \mathbf{m}) \approx \mathcal{V}$ a random process with paths in \mathcal{T}

Hypothetical limit system

Euler system with stochastic forcing

$$\begin{aligned}d\tilde{\varrho} + \operatorname{div}_x \tilde{\mathbf{m}} dt &= 0 \\d\tilde{\mathbf{m}} + \operatorname{div}_x \left(\frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\varrho}} \right) dt + \nabla_x p(\tilde{\varrho}) dt &= \mathbf{F} dW\end{aligned}$$

$W = (W_k)_{k \geq 1}$ cylindrical Wiener process

$\mathbf{F} = (\mathbf{F}_k)_{k \geq 1}$ – diffusion coefficient

$$\mathbb{E} \left[\int_0^T \sum_{k \geq 1} \|\mathbf{F}_k\|_{W^{-\ell, 2}(Q; \mathbb{R}^d)}^2 dt \right] < \infty$$

we allow $\mathbf{F} = \mathbf{F}(\varrho, \mathbf{m})$

Statistical equivalence

statistical equivalence \Leftrightarrow identity in expectation of some quantities

(ϱ, \mathbf{m}) statistically equivalent to $(\tilde{\varrho}, \tilde{\mathbf{m}})$

\Leftrightarrow

■ density and momentum

$$\mathbb{E} \left[\int_D \varrho \right] = \mathbb{E} \left[\int_D \tilde{\varrho} \right], \quad \mathbb{E} \left[\int_D \mathbf{m} \right] = \mathbb{E} \left[\int_D \tilde{\mathbf{m}} \right]$$

■ kinetic and internal energy

$$\mathbb{E} \left[\int_D \frac{|\mathbf{m}|^2}{\varrho} \right] = \mathbb{E} \left[\int_D \frac{|\tilde{\mathbf{m}}|^2}{\tilde{\varrho}} \right], \quad \mathbb{E} \left[\int_D \rho(\varrho) \right] = \mathbb{E} \left[\int_D \rho(\tilde{\varrho}) \right]$$

■ angular energy

$$\mathbb{E} \left[\int_D \frac{1}{\varrho} (\mathbb{J}_{x_0} \cdot \mathbf{m}) \cdot \mathbf{m} \right] = \mathbb{E} \left[\int_D \frac{1}{\tilde{\varrho}} (\mathbb{J}_{x_0} \cdot \tilde{\mathbf{m}}) \cdot \tilde{\mathbf{m}} \right]$$

$$D \subset (0, T) \times Q, \quad x_0 \in R^d, \quad \mathbb{J}_{x_0}(x) \equiv |x - x_0|^2 \mathbb{I} - (x - x_0) \otimes (x - x_0)$$

Results (EF, M. Hofmanová 2022)

Hypothesis:

(ϱ, \mathbf{m}) statistically equivalent to a solution of the stochastic Euler system $(\tilde{\varrho}, \tilde{\mathbf{m}})$

Conclusion:

- **Noise inactive**

(ϱ, \mathbf{m}) is a statistical solution to a **deterministic** Euler system

- **S-convergence (up to a subsequence) to the limit system**

$$\frac{1}{N} \sum_{n=1}^N b(\varrho_n, \mathbf{m}_n) \rightarrow \mathbb{E} [b(\varrho, \mathbf{m})] \text{ strongly in } L^1_{\text{loc}}((0, T) \times Q)$$

for any $b \in C_c(R^{d+1})$, $\varphi \in C_c^\infty((0, T) \times Q)$

- **Conditional statistical convergence**

barycenter $(\bar{\varrho}, \bar{\mathbf{m}}) \equiv \mathbb{E}[(\varrho, \mathbf{m})]$ solves the Euler system

\Rightarrow

$$\frac{1}{N} \# \left\{ n \leq N \mid \|\varrho_n - \bar{\varrho}\|_{L^\gamma(K)} + \|\mathbf{m}_n - \bar{\mathbf{m}}\|_{L^{\frac{2\gamma}{\gamma+1}}(K; R^d)} > \varepsilon \right\} \rightarrow 0$$

as $N \rightarrow \infty$ for any $\varepsilon > 0$, and any compact $K \subset [0, T] \times Q$

Conclusion

- Stochastically driven Euler system **irrelevant** in the description of compressible turbulence (slightly extrapolated statement)

Possible scenarios:

- **Oscillatory limit.** The sequence $(\varrho_n, \mathbf{m}_n)$ generates a Young measure. Its barycenter (weak limit of $(\varrho_n, \mathbf{m}_n)$) **is not** a weak solution of the Euler system. Statistically, however, the limit is a single object. This scenario is **compatible** with the hypothesis that the limit is independent of the choice of $\varepsilon_n \searrow 0 \Rightarrow$ computable numerically.
- **Statistical limit.** The limit is a statistical solution of the Euler system. In agreement with Kolmogorov hypothesis concerning turbulent flow advocated in the compressible setting by Chen and Glimm. This scenario **is not compatible** with the hypothesis that the limit is independent of $\varepsilon_n \searrow 0$ (\Rightarrow numerically problematic) unless the limit is a monoatomic measure in which case the convergence must be strong.