Analysis of open fluid systems with uncertain data

Eduard Feireisl
based on joint work with D. Basarić (Praha), M. Lukáčová (Mainz), H. Mizerová (Bratislava)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

Kyushu University, Fukuoka
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Navier–Stokes–Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance (Newton' s second law)

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u}) + \varrho \nabla_x G$$

Internal energy balance (First law of thermodynamics)

$$\partial_t \varrho e(\varrho, \vartheta) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \mathbf{q}(\nabla_x \vartheta) = \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}$$

Newton's rheological law

$$\mathbb{S}(\mathbb{D}_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0$$

Fourier's law

$$\mathbf{q}(\nabla_x \vartheta) = -\kappa \nabla_x \vartheta, \ \kappa > 0$$

Thermodynamics

Gibbs' law, Second law of thermodynamics

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

Entropy balance equation (Second law of thermodynamics)

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S}(\mathbb{D}_x\mathbf{u}) : \mathbb{D}_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

Thermodynamic stability

$$(\varrho, S, \mathbf{m}) \mapsto \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + \varrho e(\varrho, S)\right] \text{ strictly convex, } S = \varrho s,\ \mathbf{m} = \varrho\mathbf{u}$$

Boyle-Mariotte equation of state

$$p(\varrho,\vartheta) = \varrho\vartheta,\ e(\varrho,\vartheta) = c_v\vartheta,\ c_v > 0,\ s(\varrho,\vartheta) = c_v\log\vartheta - \log\varrho$$

Data

Physical space

$$Q \subset R^d, \ d = 1, 2, 3 \text{ (bounded) domain}$$

Impermeable boundary

$$\mathbf{u} \cdot \mathbf{n}|_{\partial Q} = 0$$

Kinematic boundary condition, complete slip

$$[\mathbb{S}(\mathbb{D}_x \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial Q} = 0$$

Kinematic boundary condition, tangential velocity

$$\mathbf{u} \times \mathbf{n}|_{\partial Q} = \mathbf{u}_B \times \mathbf{n}$$

Boundary temperature

$$\vartheta|_{\partial Q} = \vartheta_B$$

Thermal insulation – zero heat flux

$$\mathbf{q} \cdot \mathbf{n}|_{\partial Q} = 0$$

Initial/boundary value problem

Initial state of the system

$$\varrho(0,\cdot) = \varrho_0,\ \vartheta(0,\cdot) = \vartheta_0,\ \varrho_0 > 0, \vartheta_0 > 0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

$$+ \text{ compatibility conditions}$$

Existence of local–in–time strong solutions

- **Valli, Valli–Zajaczkowski [1986], Kawashima–Shizuta [1988]**

$$\varrho_0 \in W^{k,2}(Q),\ \vartheta_0 \in W^{k,2}(Q),\ \mathbf{u}_0 \in W^{k,2}(Q; R^d),\ k \geq 3$$

- **Cho–Kim [2006]**

$$\varrho_0 \in W^{1,p}(Q),\ \vartheta_0 \in W^{2,2}(Q),\ \mathbf{u}_0 \in W^{2,2}(Q; R^d),\ 3 < p \leq 6$$

$$\mathbf{u}_B = 0,\ \mathbf{q} \cdot \mathbf{n}|_{\partial Q} = 0$$

- **Kotschote [2015]**

$$\varrho_0 \in W^{1,p}(Q),\ \vartheta_0 \in W^{2-\frac{1}{p},p}(Q),\ \mathbf{u}_0 \in W^{2-\frac{1}{p},p}(Q; R^d),\ p > 3$$

Conditional regularity

John F. Nash
[1928-2015]

Nash's conjecture: *Probably one should first try to prove a conditional existence and uniqueness theorem for flow equations. This should give existence, smoothness, and unique continuation (in time) of flows, conditional on the non-appearance of certain gross types of singularity, such as infinities of temperature or density.*

- **EF, Wen, Zhu [2022]**

$$\mathbf{u}_B = 0, \ \mathbf{q} \cdot \mathbf{n}|_{\partial Q} = 0$$

$$\sup_{t \in [0,T)} \left(\sup_Q \varrho(t,\cdot) + \sup_Q \vartheta(t,\cdot) \right) < \infty \ \Rightarrow \ T_{\rm max} > T$$

- **Basarić, EF, Mizerová [2023]**

$$\mathbf{u}_B \cdot \mathbf{n} = 0, \ \vartheta|_{\partial Q} = \vartheta_B$$

$$\sup_{t \in [0,T)} \left(\sup_Q \varrho(t,\cdot) + \sup_Q \vartheta(t,\cdot) + \sup_Q |\mathbf{u}(t,\cdot)| \right) < \infty \ \Rightarrow \ T_{\rm max} > T$$

Data space

Data

$$D = (\varrho_0, \vartheta_0, \mathbf{u}_0, \mathbf{u}_B, \vartheta_B)$$

Data space

$$X_D = \Big\{ (\varrho_0, \vartheta_0, \mathbf{u}_0, \mathbf{u}_B, \vartheta_B) \ \Big| \ \inf_Q \varrho_0 > 0,\ \inf_Q \vartheta_0 > 0,\ \inf_{\partial Q} \vartheta_B > 0, \mathbf{u}_B \cdot \mathbf{n} = 0 \\ + \text{ compatibility conditions} \Big\}$$

Topology on the data space

$$\vartheta_D \in W^{2-\frac{1}{p},p}(Q),\ \vartheta_D(0,\cdot) = \vartheta_0,\ \vartheta_D|_{\partial Q} = \vartheta_B$$

$$\mathbf{u}_D \in W^{2-\frac{1}{p},p}(Q; R^d),\ \vartheta_D(0,\cdot) = \mathbf{u}_0,\ \mathbf{u}_D|_{\partial Q} = \mathbf{u}_B$$

$$\|D\|_{X_D} = \|\varrho_0^{-1}\|_{W^{1,p}(Q)} + \|\vartheta_D^{-1}\|_{W^{2-\frac{1}{p},p}(Q)} \\ + \|\varrho_0\|_{W^{1,p}(Q)} + \|\vartheta_D\|_{W^{2-\frac{1}{p},p}(Q)} + \|\mathbf{u}_D\|_{W^{2-\frac{1}{p},p}(Q;R^d)},\ p > 3$$

Solution space (trajectory space)

Solutions (trajectories)

$$\mathbf{U} = (\varrho, \vartheta, \mathbf{u})$$

Trajectory space

$$\varrho \in C([0,T]; W^{1,p}(Q))$$

$$\vartheta \in C([0,T]; W^{2-\frac{1}{p},p}(Q))$$

$$\mathbf{u} \in C([0,T]; W^{2-\frac{1}{p},p}(Q;R^d)),\ \ T < T_{\max},\ \ T_{\max} = T_{\max}[D]$$

Stability with respect to the data

$$D_n = [\varrho_{0,n}, \vartheta_{0,n}, \mathbf{u}_{0,n}, \mathbf{u}_{B,n}, \vartheta_{B,n}] \to D = [\varrho_0, \vartheta_0, \mathbf{u}_0, \mathbf{u}_B, \vartheta_B] \text{ in } X_D$$

$$\Rightarrow$$

$$\liminf_{n\to\infty} T_{\max}[D_n] \geq T_{\max}[D] > 0,\ \ 0 < T < T_{\max}$$

$$(\varrho, \vartheta, \mathbf{u})[D_n] \to (\varrho, \vartheta, \mathbf{u})[D] \text{ weakly-* in } L^\infty(0,T; W^{1,p} \times W^{2-\frac{1}{p},p} \times W^{2-\frac{1}{p},p})$$

Analytical results, summary

Existence and uniqueness
For any data $D = (\varrho_0, \vartheta_0, \mathbf{u}_0, \vartheta_B, \mathbf{u}_B) \in X_D$, there exists a unique solution $(\varrho, \vartheta, \mathbf{u})$ on a maximal time interval $[0, T_{\max}) > 0$.
Stability
The mapping $D \in X_D \mapsto T_{\max}[D]$ is lower semi–continuous. If

$$D_n \to D \text{ in } X_D,$$

then

$$(\varrho, \vartheta, \mathbf{u})[D_n] \to (\varrho, \vartheta, \mathbf{u})[D] \text{ weakly-* in } L^\infty(0, T; W^{1,p} \times W^{2-\frac{1}{p},p} \times W^{2-\frac{1}{p},p}).$$

Conditional regularity

$$\|\varrho(t,\cdot)\|_{W^{1,p}(\Omega)} + \|\vartheta(t,\cdot)\|_{W^{2-\frac{1}{p},p}(\Omega)} + \|\mathbf{u}(t,\cdot)\|_{W^{2-\frac{1}{p},p}(\Omega;R^d)}$$
$$\leq C\left(T, \|D\|_{X_D}, \sup_{t\in[0,T]} \left(\sup_Q \varrho(t,\cdot) + \sup_Q \vartheta(t,\cdot) + \sup_Q |\mathbf{u}(t,\cdot)| \right)\right.$$

for any $0 \leq t < T < T_{\max}$, C bounded for bounded arguments

Problems with uncertain data

Probability space

$$\{\Omega; \mathcal{B}, \mathbb{P}\},\ \Omega \text{ measurable space}$$

$$\mathcal{B}\ \sigma - \text{algebra of measurable sets},\ \mathbb{P} - \text{complete probability measure}$$

Random data

$$\omega \in \Omega \mapsto D \in X_D \text{ Borel measurable mapping}$$

Solutions as random variables

$$T_{\max} = T_{\max}[D] - \text{random variable}$$

$$D \mapsto (\varrho, \vartheta, \mathbf{u})[D] \text{ random variable}$$

Statistical solution

$$\text{strong sense: } \omega \in \Omega \mapsto (\varrho, \vartheta, \mathbf{u})(t, \cdot)[D]$$

$$\text{weak sense: } \mathcal{L}[(\varrho, \vartheta, \mathbf{u})(t, \cdot)[D]]$$

$$\mathcal{L} \text{ - law (distribution) of } (\varrho, \vartheta, \mathbf{u})(t, \cdot) \text{ in } W^{1,p} \times W^{2-\frac{1}{p},p} \times W^{2-\frac{1}{p},p}$$

Strong stability problem I

Data convergence

$$D_n = [\varrho_{0,n}, \vartheta_{0,n}, \mathbf{u}_{0,n}, \mathbf{u}_{B,n}, \vartheta_{B,n}] \to D = [\varrho_0, \vartheta_0, \mathbf{u}_0, \mathbf{u}_B, \vartheta_B] \text{ in } X_D$$

$$\mathbb{P}- \text{ a.s.}$$

Solution convergence

$$(\varrho, \vartheta, \mathbf{u})[D_n] \to (\varrho, \vartheta, \mathbf{u})[D] \text{ weakly-* in } L^\infty(0, T; W^{1,p} \times W^{2-\frac{1}{p},p} \times W^{2-\frac{1}{p},p})$$

$$T < T_{\max}[D]$$

$$\mathbb{P}- \text{ a.s.}$$

Weak stability problem I

Data convergence in law (in distribution)

$$\mathcal{L}[D_n] = \mathcal{L}[\varrho_{0,n}, \vartheta_{0,n}, \mathbf{u}_{0,n}, \mathbf{u}_{B,n}, \vartheta_{B,n}] \to \mathcal{L}$$

narrowly in $\mathfrak{P}[X_D]$

Tools from probability theory I

Skorokhod (representation) theorem
Let $(\mathcal{L}_n)_{n=1}^{\infty}$ of probability measures on a Polish space X. Suppose that the sequence is tight in X, meaning for any $\varepsilon > 0$, there exists a compact set $K(\varepsilon) \subset X$ such that

$$\mathcal{L}_n[X \setminus K(\varepsilon)] \leq \varepsilon \text{ for all } n = 1, 2, \dots .$$

Then there is a subsequence $n_k \to \infty$ as $k \to \infty$ and a sequence of random variables $(\widetilde{D}_{n_k})_{k=1}^{\infty}$ defined on the standard probability space

$$\left(\widetilde{\Omega} = [0,1], \mathfrak{B}[0,1], \mathrm{d}y\right)$$

satisfying:

- $$\mathrm{law}[\widetilde{D}_{n_k}] = \mathcal{L}_{n_k},$$
- $$\widetilde{D}_k \to \widetilde{D} \text{ in } X \text{ for every } y \in [0,1].$$

Convergence in weak stability problem I

Skorokhod representation theorem

$$D_n \approx_{X_D} \widetilde{D}_{n_k}$$

Strong convergence in the new probability space

$$(\widetilde{\varrho}_k, \widetilde{\vartheta}_k, \widetilde{\mathbf{u}}_k) \equiv (\varrho, \vartheta, \mathbf{u})[\widetilde{D}_{n_k}] \to (\varrho, \vartheta, \mathbf{u})[\widetilde{D}]$$

weakly-* in $L^\infty(0, T; W^{1,p} \times W^{2-\frac{1}{p},p} \times W^{2-\frac{1}{p},p})$ surely $\mathrm{d}y$

Equivalence in law (Borel measurability of the solution mapping)

$$(\widetilde{\varrho}_n, \widetilde{\vartheta}_n, \widetilde{\mathbf{u}}_n) \approx (\varrho, \vartheta, \mathbf{u})[D_n]$$

Conclusion

$$\mathcal{L}[(\varrho, \vartheta, \mathbf{u})[D_n]] \to \mathcal{L}[(\varrho, \vartheta, \mathbf{u})[\widetilde{D}]]$$

narrowly

Strong stability problem II - global in time convergence

Data convergence

$$D_n = [\varrho_{0,n}, \vartheta_{0,n}, \mathbf{u}_{0,n}, \mathbf{u}_{B,n}, \vartheta_{B,n}] \to D = [\varrho_0, \vartheta_0, \mathbf{u}_0, \mathbf{u}_B, \vartheta_B] \text{ in } X_D$$

$$\mathbb{P} - \text{ a.s.}$$

Hypothesis of boundedness in probability

For any $\varepsilon > 0$, there exists $M > 0$ such that

$$\limsup_{n \to \infty} \mathbb{P} \left\{ \sup_{(0,T) \times Q} \varrho[D_n] + \sup_{(0,T) \times Q} \vartheta[D_n] + \sup_{(0,T) \times Q} |\mathbf{u}[D_n]| > M \right\} < \varepsilon$$

Conclusion (to be shown below)

$$(\varrho, \vartheta, \mathbf{u})[D_n] \to (\varrho, \vartheta, \mathbf{u})[D]$$

$$\text{weakly-* in } L^\infty(0, T; W^{1,p} \times W^{2-\frac{1}{p},p} \times W^{2-\frac{1}{p},p})$$

in probability

Strong stability problem II - proof of convergence

Skorokhod representation theorem

augmented sequence of random variables $(D_n, (\varrho, \vartheta, \mathbf{u})[D_n],\ \Lambda_n)_{n=1}^{\infty}$

$$\Lambda_n = \sup_{(0,T)\times Q} \varrho[D_n] + \sup_{(0,T)\times Q} \vartheta[D_n] + \sup_{(0,T)\times Q} |\mathbf{u}[D_n]|$$

Skorokhod representation

$$(\widetilde{D}_n, (\varrho, \vartheta, \mathbf{u})[\widetilde{D}_n], \widetilde{\Lambda}_n)_{n=1}^{\infty}$$

$$\widetilde{\Lambda}_n = \sup_{(0,T)\times Q} \varrho[\widetilde{D}_n] + \sup_{(0,T)\times Q} \vartheta[\widetilde{D}_n] + \sup_{(0,T)\times Q} |\mathbf{u}[\widetilde{D}_n]| \to \widetilde{\Lambda}$$

$\mathrm{d}y$ surely

Conclusion by conditional regularity

$$\widetilde{D}_n \to \widetilde{D} \text{ in } X_D$$

$$(\varrho, \vartheta, \mathbf{u})[\widetilde{D}_n] \to (\varrho, \vartheta, \mathbf{u})[\widetilde{D}]$$

weakly-* in $L^{\infty}(0, T; W^{1,p} \times W^{2-\frac{1}{p},p} \times W^{2-\frac{1}{p},p}),\ \mathrm{d}y$ surely

Tools from probability theory II

Gyöngy–Krylov theorem
Let X be a Polish space and $(\mathbf{U}_M)_{M\geq 1}$ a sequence of $X-$valued random variables.
Then $(\mathbf{U}_M)_{M=1}^{\infty}$ converges in probability if and only if for any sequence of joint laws of

$$(\mathbf{U}_{M_k}, \mathbf{U}_{N_k})_{k=1}^{\infty}$$

there exists further subsequence that converge weakly to a probability measure μ on $X \times X$ such that

$$\mu\left[(x, y) \in X \times X,\ x = y\right] = 1.$$

Consistent approximation

Conservative boundary conditions (for simplicity)

$$\mathbf{u}|_{\partial Q} = 0,\ \mathbf{q} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

Approximate field equations

$$\partial_t \varrho_n + \mathrm{div}_x(\varrho_n \mathbf{u}_n) = e_n^1 \text{ in } \mathcal{D}'((0,T) \times Q),$$

Consistent approximation

$$\varrho_n = [\varrho]_n[D],\ \vartheta_n = [\vartheta]_n[D],\ \mathbf{u}_n = [\mathbf{u}]_n[D]$$

$$\begin{aligned}\partial_t(\varrho_n \mathbf{u}_n) + \mathrm{div}_x(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) + \nabla_x p(\varrho_n, \vartheta_n) \\ = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u}_n) + \varrho_n \nabla_x G + e_n^2 \text{ in } \mathcal{D}'((0,T) \times Q; R^d)\end{aligned}$$

$$\begin{aligned}\partial_t(\varrho_n s(\varrho_n, \vartheta_n)) + \mathrm{div}_x(\varrho_n s(\varrho_n, \vartheta_n) \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q_n}}{\vartheta_n} \right) \\ \geq \frac{1}{\vartheta_n} \left(\mathbb{S}(\mathbb{D}_x \mathbf{u}_n) : \mathbb{D}_x \mathbf{u}_n - \frac{\mathbf{q}_n \cdot \nabla_x \vartheta_n}{\vartheta_n} \right) + e_n^3 \text{ in } \mathcal{D}'((0,T) \times Q)\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_Q \left[\varrho_n |\mathbf{u}_n|^2 + \varrho_n e(\varrho_n, \vartheta_n) - \varrho_n G \right] \mathrm{d}x \leq e_n^4 \text{ in } \mathcal{D}'(0,T)$$

$$e_n^1, e_n^2, e_n^3, e_n^4 \to 0 \text{ as } n \to \infty \text{ in a “weak” sense}$$

Convergence of consistent approximations

Strong data convergence

$$D_n = [\varrho_{0,n}, \vartheta_{0,n}, \mathbf{u}_{0,n}, \mathbf{u}_{B,n}, \vartheta_{B,n}] \to D = [\varrho_0, \vartheta_0, \mathbf{u}_0, \mathbf{u}_B, \vartheta_B] \text{ in } X_D$$

$$\mathbb{P}- \text{ a.s.}$$

Hypothesis of boundedness in probability
For any $\varepsilon > 0$, there exists $M > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}\left\{ \sup_{(0,T)\times Q} \varrho_n[D_n] + \sup_{(0,T)\times Q} \vartheta_n[D_n] + \sup_{(0,T)\times Q} |\mathbf{u}_n[D_n]| > M \right\} < \varepsilon$$

Consistent approximation

$[\varrho, \vartheta, \mathbf{u}]_n[D_n]$ a sequence of consistent approximations

Convergence of consistent approximations, I

1. Apply Skorokhod representation theorem to the sequence $(D_n, \varrho_n, \vartheta_n \mathbf{u}_n, \Lambda_n)_{n=1}^{\infty}$,

$$\Lambda_n = \sup_{(0,T)\times Q} \varrho_n[D_n] + \sup_{(0,T)\times Q} \vartheta_n[D_n] + \sup_{(0,T)\times Q} |\mathbf{u}_n[D_n]|$$

2. New sequence of data $\widetilde{D}_n$ with the same law on the standard probability space,

$$\widetilde{D}_n \to \widetilde{D} \text{ in } X_d,\ \mathrm{d}y \text{ surely.}$$

$$\widetilde{\Lambda}_n = \sup_{(0,T)\times Q} \varrho_n[\widetilde{D}_n] + \sup_{(0,T)\times Q} \vartheta_n[\widetilde{D}_n] + \sup_{(0,T)\times Q} |\mathbf{u}_n[\widetilde{D}_n]| \to \widetilde{\Lambda}$$

$$\mathrm{d}y \text{ surely}$$

$$\varrho_{n_k}[\widetilde{D}_{n_k}] \to \widetilde{\varrho} \text{ weakly-(*) in } L^{\infty}((0,T)\times Q)$$

$$\vartheta_{n_k}[\widetilde{D}_{n_k}] \to \widetilde{\vartheta} \text{ weakly-(*) in } L^{\infty}((0,T)\times Q)$$

$$\mathbf{u}_{n_k}[\widetilde{D}_{n_k}] \to \widetilde{\mathbf{u}} \text{ weakly-(*) in } L^{\infty}((0,T)\times Q; R^d)$$

$$\mathrm{dy} \ \text{ surely}$$

Convergence of consistent approximations, II

4. Show the limit is a measure–valued solution with the data $\widetilde{D}$ in the sense of Březina, EF, Novotný [2020].
5. Apply the weak–strong uniqueness principle to conclude the $(\widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\mathbf{u}})$ is the unique strong solution associated to the data $\widetilde{D}$,

$$(\widetilde{\varrho}, \widetilde{\vartheta}, \widetilde{\mathbf{u}}) = (\varrho, \vartheta, \mathbf{u})[\widetilde{D}].$$

   Conclude there is no need of subsequence, $T_{\max}[\widetilde{D}] > T$, and convergence is strong for in L^q for any finite q.
6. Pass to the original space using Gyöngy–Krylov theorem

Conclusion

$$(\varrho_n, \vartheta_n, \mathbf{u}_n)[D_n] \to (\varrho, \vartheta, \mathbf{u})[D]$$

in $L^q((0,T) \times Q; R^{d+2})$ for any $1 \leq q < \infty$

in probability