Equations of fluid mechanics with random data: Analysis

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Equations of continuum mechanics

$\varrho = \varrho(t, x)$ .... mass density
$\mathbf{u} = \mathbf{u}(t, x)$ .... fluid velocity
$\mathbb{T}$ .... Cauchy stress
$\mathbf{f}$ .... external (driving) force
e .... internal energy
$\mathbf{q}$ .... internal energy flux

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum conservation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x \mathbb{T} + \varrho \mathbf{f}$$

Energy conservation

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) \mathbf{u} \right] - \mathrm{div}_x(\mathbb{T} \cdot \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \varrho \mathbf{f} \cdot \mathbf{u}$$

Fluid mechanics

$\mathbb{S}$ .... viscous stress

p .... pressure

Stokes law

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}$$

Thermodynamics, entropy

s .... entropy

ϑ .... (absolute) temperature

Gibbs' law

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

Entropy balance equation

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\vartheta}\mathbf{q} \cdot \nabla_x \vartheta\right)$$

Second law of thermodynamics, transport coefficients

Second law of thermodynamics

$$\Rightarrow$$

$$\text{entropy production rate } \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\vartheta}\mathbf{q}\cdot\nabla_x\vartheta\right) \geq 0$$

μ .... shear viscousity coefficient

η .... bulk viscosity coefficient

$$\mathbb{D}_x \mathbf{u} = \frac{1}{2}\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}\right)$$

Newton's rheological law

$$\mathbb{S}(\mathbb{D}_x \mathbf{u}) = \mu\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I},\ \mu \geq 0,\ \eta \geq 0$$

κ .... heat conductivity coefficient

Fourier's law

$$\mathbf{q} = -\kappa\nabla_x\vartheta,\ \kappa \geq 0$$

Navier–Stokes–Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u}) + \varrho \mathbf{f}$$

Entropy balance

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) - \mathrm{div}_x \left(\frac{\kappa \nabla_x \vartheta}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S} : \mathbb{D}_x \mathbf{u} + \frac{\kappa}{\vartheta} |\nabla_x \vartheta|^2 \right)$$

Equation(s) of state

$$p = p(\varrho, \vartheta)$$

$$s = s(\varrho, \vartheta)$$

Equation of state (EOS) examples

Monoatomic gas

$$p = \frac{2}{3}\varrho e$$

$$p(\varrho, \vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right)$$

$$P \text{ linear } \Rightarrow \text{ Boyle-Mariotte EOS } p = \varrho\vartheta$$

Radiation pressure (atrophysics)

$$p_R = a\vartheta^4$$

Initial/boundary conditions

Initial conditions

$$\varrho(0,\cdot) = \varrho_0,\ (\varrho \mathbf{u})(0,\cdot) = \mathbf{m}_0,\ \varrho s(0,\cdot) = S_0$$

$Q \subset R^d$ .... domain

Boundary conditions
In/out flow

$$\mathbf{u}|_{\partial Q} = \mathbf{u}_B$$

$$\varrho|_{\Gamma_{\rm in}} = \varrho_B,\ \Gamma_{\rm in} \left\{ x \in \partial Q \ \middle| \ \mathbf{u}_B \cdot \mathbf{n} < 0 \right\}$$

Prescribed boundary temperature

$$\vartheta|_{\partial Q} = \vartheta_B$$

Boundary heat flux

$$\nabla_x \vartheta \cdot \mathbf{n}|_{\partial Q} = F$$

Examples of boundary value problems

Rayleigh–Bénard problem [cf. Davidson]

$$Q \subset R^3 \text{ infinite slab } Q = R^2 \times [a, b]$$

$$\mathbf{u}|_{\partial Q} = 0$$

$$\vartheta = \vartheta_a \text{ if } x_3 = a$$

$$\vartheta = \vartheta_b \text{ if } x_3 = b$$

Gravitational force

$$\mathbf{f} = \nabla_x G, \ \ G = -x_3$$

Taylor–Couette flow [cf. Davidson]

$$Q = O \setminus \cup_i B_i \ \ B_i \text{ (rotating) balls}$$

$$\mathbf{u}|_{\partial B_i} = \mathbf{u}_{B,i}, \ \ \mathbf{u}_{B,i} \cdot \mathbf{n} = 0$$

$$\mathbf{u}|_{\partial O} = 0$$

Examples of “data”

Rheological/material properties

transport coefficients $\mu,\ \eta,\ \kappa$

parameters in EOS (equation of state)

External forcing

$$\mathbf{f} = \nabla_x G = g\mathbf{e}$$

Initial state

$$\varrho_0,\ \mathbf{m}_0,\ S_0\ \ldots$$

Boundary data

$$\mathbf{u}_B,\ \varrho_B,\ \vartheta_B \ldots$$

Uncertain data

$D \in \mathcal{D}$ – data space

$\mathcal{D}$ Polish – metrizable, separable, complete

Probability bases

$$\left\{\Omega, \mathfrak{B}, \mathcal{P}\right\}$$

Ω (topological) space, $\mathfrak{B}$ σ–field of measurable sets (containing Borel sets)

$\mathcal{P}$ complete (Borel) probability measure

Random data

$D = D(\omega) : \Omega \to \mathcal{D}$ Borel measurable mapping

Distribution (law) of D

$\mathcal{L}[D]$ – Borel probability measure on the data space $\mathcal{D}$

$$\mathcal{L}[D]\{\mathcal{B}\} = \mathcal{P}\{D^{-1}(\mathcal{B})\}$$

Basic problems

Strong formulation
Given $D : \Omega \to \mathcal{D}$ a family of random data, identify the corresponding solution of the problem as a random variable

Associated numerical methods are the stochastic Galerkin method, stochastic collocation method ...

Weak formulation
Given a law $\mathcal{L}[D]$ a family of random data, identify the law of the corresponding solution

Associated numerical methods are Monte Carlo and related methods ...

Principal difficulties

- *Solvability* of the problem for a given family of data
- *Uniqueness* of solutions for given data
- *Dependence* of solutions on the data

Basic tools of stochastic analysis, I

Prokhorov's theorem
Let $(\nu_N)_{N=1}^{\infty}$ be a family of probability measures on a Polish space X. The following is equivalent:

- $(\nu_N)_{N=1}^{\infty}$ is weakly precompact, meaning there is a subsequence such

$$\nu_{N_k} \to \nu \text{ weakly in } \mathfrak{P}(X).$$

- $(\nu_N)_{N=1}^{\infty}$ is tight, meaning for any $\varepsilon > 0$, there is a compact set $K(\varepsilon) \subset X$ such that

$$\mu_N(K) \geq 1 - \varepsilon \text{ for all } N = 1, 2, \ldots.$$

Basic tools of stochastic analysis, II

Skorokhod (representation) theorem
Let $(\mathbf{U}^M)_{M=1}^{\infty}$ be a sequence of random variables ranging in a Polish space X. Suppose that their laws are tight in X, meaning for any $\varepsilon > 0$, there exists a compact set $K(\varepsilon) \subset X$ such that

$$\mathbb{P}[\mathbf{U}^M \in X \setminus K(\varepsilon)] \leq \varepsilon \text{ for all } M = 1, 2, \dots .$$

Then there is a subsequence $M_n \to \infty$ and a sequence of random variables $(\widetilde{\mathbf{U}}^{M_n})_{n=1}^{\infty}$ defined on the standard probability space

$$\left(\widetilde{\Omega} = [0,1], \mathfrak{B}[0,1], \mathrm{d}y \right)$$

satisfying:

- $$\widetilde{\mathbf{U}}^{M_n} \approx_X \mathbf{U}^{M_n} \text{ (they are equally distributed random variables),}$$
- $$\widetilde{\mathbf{U}}^{M_n} \to \widetilde{\mathbf{U}} \text{ in } X \text{ for every } y \in [0,1].$$

Basic tools of stochastic analysis, III

Gyöngy–Krylov theorem
Let X be a Polish space and $(\mathbf{U}^M)_{M\geq 1}$ a sequence of $X-$valued random variables.
Then $(\mathbf{U}^M)_{M=1}^{\infty}$ converges in probability if and only if for any sequence of joint laws of

$$(\mathbf{U}^{M_k}, \mathbf{U}^{N_k})_{k=1}^{\infty}$$

there exists further subsequence that converge weakly to a probability measure μ on $X \times X$ such that

$$\mu\left[(x, y) \in X \times X,\ x = y\right] = 1.$$

Barotropic Navier–Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u})$$

No–slip boundary condition

$$Q \subset R^d,\ d = 2, 3 \text{ bounded, smooth}$$

$$\mathbf{u}|_{\partial Q} = 0$$

Initial data

$$\varrho(0, \cdot) = \varrho_0,\ \inf \varrho_0 > 0,\ (\varrho \mathbf{u})(0, \cdot) = \mathbf{m}_0 = \varrho_0 \mathbf{u}_0$$

$$p(\varrho) = a\varrho^\gamma,\ a > 0,\ \gamma > 1$$

Navier–Stokes system – weak solutions

Equation of continuity

$$\int_0^\infty \int_Q \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \, \mathrm{d}x\mathrm{d}t = - \int_Q \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x$$

for any $\varphi \in C^1_c([0, \infty) \times \overline{Q})$

Momentum equation

$$\int_0^\infty \int_Q \left[\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x\mathrm{d}t$$

$$= \int_0^\infty \int_Q \mathbb{S}(\mathbb{D}_x \mathbf{u} : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t - \int_Q \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x$$

for any $\boldsymbol{\varphi} \in C^1_c([0, \infty) \times Q; R^d)$

Energy inequality

$$- \int_0^\infty \partial_t \psi \int_Q \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \, \mathrm{d}x\mathrm{d}t + \int_0^\infty \psi \int_Q \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} \, \mathrm{d}x\mathrm{d}t$$

$$\leq \psi(0) \int_Q \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right) \, \mathrm{d}x$$

for any $\psi \in C^1_c[0, \infty)$, $\psi \geq 0$, $P'(\varrho)\varrho - P(\varrho) = p(\varrho)$

Solvability of the Navier–Stokes system

- **Local existence of smooth solutions** [Valli, Zajaczkowski [1986]]

$$\varrho_0 \in W^{k,2}(Q),\ \inf \varrho_0 > 0,\ \mathbf{u}_0 \in W^{k,2}(Q;R^d),\ k \geq 3$$

$$+$$

compatibility conditions

$$\Rightarrow$$

There exists a regular (classical) solution

$$\varrho \in C([0,T_{\rm max}); W^{k,2}(Q)),\ \mathbf{u} \in C([0,T_{\rm max}); W^{k,2}(Q;R^d)),\ T_{\rm max} > 0$$

- **Global existence of weak solutions** [Lions [1998], EF [2000]]

$$\varrho_0 \geq 0,\ \int_Q \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right] \ {\rm d}x < \infty,\ \gamma > \frac{d}{2}$$

$$\Rightarrow$$

There exists global in time weak solution

$$\varrho \in C([0,T]; L^1(Q)) \cap C_{\rm weak}([0,T]; L^\gamma(Q)),$$

$$\varrho \mathbf{u} \in C_{\rm weak}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(Q;R^d)),\ \mathbf{u} \in L^2(0,T; W^{1,2}_0(Q;R^d)) \text{ for any } T > 0$$

Conditional regularity, weak–strong uniqueness

A priori **bounds** [Sun, Wang, and Zhang [2011]]

$$\|\varrho(t,\cdot)\|_{W^{k,2}(Q)} + \|\mathbf{u}(t,\cdot)\|_{W^{k,2}(Q)}$$

$$\leq \Lambda\left(T, \|\varrho_0\|_{W^{k,2}(Q)},\ \inf \varrho_0, \|\mathbf{u}_0\|_{W^{k,2}(Q)}, \boxed{\|\varrho\|_{L^\infty(0,T)\times Q)}, \|\mathbf{u}\|_{L^\infty(0,T)\times Q)}}\right)$$

$$t \in [0, T]$$

Weak–strong uniqueness [EF, Jin, Novotný [2012]]
Any weak solutions emanating from sufficiently regular initial data coincides with the unique strong solutions as long as the latter exists

Corollary
Any weak solution emanating from sufficiently regular initial data that remain uniformly bounded is a classical solution

Consistent approximation

$$\int_0^\infty \int_Q \left[\varrho_\varepsilon \partial_t \varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi\right] \, \mathrm{d}x\mathrm{d}t = - \int_Q \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x + e_c(\varphi, \varepsilon)$$

for any $\varphi \in C^1_c([0,\infty) \times \overline{Q})$

$$\int_0^\infty \int_Q \left[\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \boldsymbol{\varphi} + p(\varrho_\varepsilon) \mathrm{div}_x \boldsymbol{\varphi}\right] \, \mathrm{d}x\mathrm{d}t$$

$$= \int_0^\infty \int_Q \mathbb{S}(\mathbb{D}_x \mathbf{u}_\varepsilon) : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t - \int_Q \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x + e_m(\boldsymbol{\varphi}, \varepsilon)$$

for any $\boldsymbol{\varphi} \in C^1_c([0,\infty) \times Q; R^d)$

$$\int_Q \left(\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + P(\varrho_\varepsilon)\right)(\tau, \cdot) \, \mathrm{d}x + \int_0^\tau \int_Q \mathbb{S}(\mathbb{D}_x \mathbf{u}_\varepsilon) : \mathbb{D}_x \mathbf{u}_\varepsilon \, \mathrm{d}x\mathrm{d}t$$

$$\leq \int_Q \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0)\right) \, \mathrm{d}x + e_e(\tau, \varepsilon)$$

Vanishing consistency error: e_c, e_m, $e_e \to 0$ as $\varepsilon \to 0$

Limit of consistent approximation

Weak convergence

$$\varrho_\varepsilon \to \varrho \text{ weak-(*) in } L^\infty(0,T;L^\gamma(Q))$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2((0,T;W_0^{1,2}(Q;R^d))$$

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u}} \text{ weak-(*) in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(Q;R^d))$$

Lions–Aubin argument (under some extra hypotheses)

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}$$

Limit system – dissipative solutions

$$\int_0^\infty \int_Q \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \, \mathrm{d}x\mathrm{d}t = - \int_Q \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x$$

for any $\varphi \in C^1_c([0, \infty) \times \overline{Q})$

$$\int_0^\infty \int_Q \left[\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x\mathrm{d}t$$

$$= \int_0^\infty \int_Q \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t - \int_Q \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x - \int_0^\infty \int_Q \mathfrak{R} : \nabla_x \boldsymbol{\varphi} \mathrm{d}t$$

for any $\boldsymbol{\varphi} \in C^1_c([0, \infty) \times Q; R^d)$

$$\int_Q \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) (\tau, \cdot) \, \mathrm{d}x + \int_Q \mathfrak{E}(\tau, \cdot) + \int_0^\tau \int_Q \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} \, \mathrm{d}x\mathrm{d}t$$

$$\leq \int_Q \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right) \, \mathrm{d}x$$

Reynolds stress and energy defect

Energy defect

$$\mathfrak{E} = \lim_{\varepsilon \to 0} \left[\frac{1}{2} \frac{|\mathbf{m}_\varepsilon|^2}{\varrho_\varepsilon} + P(\varrho_\varepsilon) \right] - \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right], \ \mathbf{m}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$$

Reynolds defect

$$\mathfrak{R} = \lim_{\varepsilon \to 0} \left[\frac{1}{2} \frac{\mathbf{m}_\varepsilon \otimes \mathbf{m}_\varepsilon}{\varrho_\varepsilon} + p(\varrho_\varepsilon) \mathbb{I} \right] - [\varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I}]$$

$$[\varrho, \mathbf{m}] \mapsto \left[\frac{1}{2} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho) \mathbb{I} \right] : \xi \otimes \xi = \left[\frac{1}{2} \frac{|\mathbf{m} \cdot \xi|^2}{\varrho} + p(\varrho) |\xi|^2 \mathbb{I} \right] \ \text{convex}$$

$$\Rightarrow$$

Compatibility

$$0 \leq \mathfrak{R}, \ 0 \leq \mathrm{trace}[\mathfrak{R}] \leq c \mathfrak{E}$$

Relative energy

Relative energy

$$E\left(\varrho, \mathbf{u}\middle| r, \mathbf{U}\right) = \frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r)$$

Relative energy as Bregman distance

$$E(\varrho, \mathbf{m}) = \begin{cases} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0, \\ 0 \text{ if } \varrho = 0, \mathbf{m} = 0, \\ \infty \text{ otherwise} \end{cases}$$

convex l.s.c. function

$$E\left(\varrho, \mathbf{m}\middle| r, \mathbf{M}\right) = E(\varrho, \mathbf{m}) - \partial_{\varrho, \mathbf{m}} E(r, \mathbf{M}) - E(r, \mathbf{M})$$

Decomposition

$$\int_Q E\left(\varrho, \mathbf{u}\middle| r, \mathbf{U}\right) \, \mathrm{d}x = \int_Q \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \, \mathrm{d}x$$

$$+ \int_Q \varrho \left(\frac{1}{2}|\mathbf{U}|^2 - P'(r)\right) \, \mathrm{d}x - \int_Q \varrho\mathbf{u}\cdot\mathbf{U} \, \mathrm{d}x + \int_Q p(r) \, \mathrm{d}x$$

Relative energy inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_Q E\left(\varrho, \mathbf{u} \;\middle|\; r, \mathbf{U}\right)\ \mathrm{d}x + \int_Q \mathfrak{E}\right)\\
&+ \int_Q \Big(\mathbb{S}(\mathbb{D}_x \mathbf{u}) - \mathbb{S}(\mathbb{D}_x \mathbf{U})\Big) : \Big(\mathbb{D}_x \mathbf{u} - \mathbb{D}_x \mathbf{U}\Big)\ \mathrm{d}x\\
\leq &- \int_Q \varrho(\mathbf{U} - \mathbf{u}) \cdot (\mathbf{U} - \mathbf{u}) \cdot \mathbb{D}_x \mathbf{U}\ \mathrm{d}x\\
&- \int_Q \Big[p(\varrho) - p'(r)(\varrho - r) - p(r)\Big] \mathrm{div}_x \mathbf{U}\ \mathrm{d}x\\
&+ \int_Q \left(\frac{\varrho}{r} - 1\right)(\mathbf{U} - \mathbf{u}) \cdot \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{U})\ \mathrm{d}x\\
&+ \int_Q \frac{\varrho}{r}(\mathbf{U} - \mathbf{u}) \cdot \Big[\partial_t (r\mathbf{U}) + \mathrm{div}_x (r\mathbf{U} \otimes \mathbf{U}) + \nabla_x p(r) - \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{U})\Big]\ \mathrm{d}x\\
&+ \int_Q \left(\frac{\varrho}{r}(\mathbf{u} - \mathbf{U}) \cdot \mathbf{U} + p'(r)\left(1 - \frac{\varrho}{r}\right)\right)\Big[\partial_t r + \mathrm{div}_x (r\mathbf{U})\Big]\ \mathrm{d}x\\
&- \int_Q \mathbb{D}_x \mathbf{U} : \mathfrak{R}
\end{aligned}
$$

Compatibility, weak strong uniqueness

Compatibility
If a dissipative solution $\varrho, \mathbf{u}$ belongs to the class C^2 and $\varrho_0 > 0$, then ϱ, $\mathbf{u}$ is a classical solutions

Weak–strong uniqueness
A dissipative solution coincides with the (unique) classical solution emanating from the same initial data as long as the latter solution exists

[EF, Lukáčová-Medviďová, Mizerová, She [2022]]

Data dependence, measurable selection

Background, weak solutions

- $p(\varrho) = a\varrho^\gamma$, $\gamma > \frac{d}{2}$
- ϱ_0, $\mathbf{m}_0 = \varrho_0 \mathbf{u}_0$ initial data
- $\mathcal{U}[\varrho_0, \mathbf{m}_0]$ – the set of all weak solutions to the Navier–Stokes system on the time interval $[0, \infty)$ emanating from the initial data $[\varrho_0, \mathbf{m}_0]$

Existence (E)
The set $\mathcal{U}[\varrho_0, \mathbf{m}_0]$ is non–empty for any

$$\varrho_0 \geq 0, \ \int_Q \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x < \infty$$

Compactness, closed graph (C)

$$\text{If } \varrho_{0,n} \to \varrho_0 \text{ weakly in } L^\gamma(Q), \ \mathbf{m}_{0,n} \to \mathbf{m}_0 \text{ weakly in } L^{\frac{2\gamma}{\gamma+1}}(Q; R^d)$$

$$\int_Q \left[\frac{1}{2} \frac{|\mathbf{m}_{0,n}|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) \right] \ \mathrm{d}x \to \int_Q \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$

$$\text{and } [\varrho_n, \mathbf{m}_n] \in \mathcal{U}[\varrho_{0,n}, \mathbf{m}_{0,n}]$$

Then, for a suitable subsequence,

$[\varrho_{n,k}, \mathbf{m}_{n,k}] \to [\varrho, \mathbf{m}]$ in $C_{\rm weak, loc}([0, \infty; L^\gamma \times L^{\frac{2\gamma}{\gamma+1}}(Q; R^d))$, where $[\varrho, \mathbf{m}] \in \mathcal{U}[\varrho_0, \mathbf{m}_0]$.

Semigroup (semiflow) selection

Set of data

$$\mathcal{D} = \left\{ \varrho, \mathbf{m}, E \ \Big| \ \int_\Omega \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \ \mathrm{d}x \leq E \right\}$$

Set of trajectories

$$\mathcal{T} = \left\{ \varrho(t, \cdot), \mathbf{m}(t, \cdot), E(t-, \cdot) \Big| t \in (0, \infty) \right\}$$

Solution set

$$\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] = \left\{ [\varrho, \mathbf{m}, E] \ \Big| [\varrho, \mathbf{m}, E] \text{ dissipative solution} \right.$$

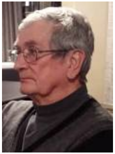
$$\left. \varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0, \ E(0+) \leq E_0 \right\}$$

Semiflow selection – semigroup

$$U[\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0], \ [\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{D}$$

$$U(t_1+t_2)[\varrho_0, \mathbf{m}_0, E_0] = U(t_1) \circ \Big[U(t_2)[\varrho_0, \mathbf{m}_0, E_0] \Big], \ t_1, t_2 > 0$$

Andrej Markov (1856–1933)



N. V. Krylov

Abstract setting

Phase space

$$(\varrho, \mathbf{m}, E) \in X = W^{-\ell,2}(Q) \times W^{-\ell,2}(Q; R^N) \times R$$

Data space

$$D = \left\{ [\varrho_0, \mathbf{m}_0, E_0] \in X \ \middle| \ \varrho_0 \geq 0, \ \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + \frac{a}{\gamma - 1} \varrho_0^\gamma \right] \ \mathrm{d}x \leq E_0 \right\}.$$

Trajectory space

$$\Omega = C_{\rm loc}([0,\infty); W^{-\ell,2}(Q)) \times C_{\rm loc}([0,\infty); W^{-\ell,2}(Q; R^N)) \times L^1_{\rm loc}(0,\infty)$$

Method by Krylov adapted by Cardona and Kapitanski

Multi–valued solution mapping

$$\mathcal{U} : [\varrho_0, \mathbf{m}_0, E_0] \mapsto [\varrho, \mathbf{m}, E] \in 2^{\Omega}$$

Time shift

$$S_T \circ \xi, \ S_T \circ \xi(t) = \xi(T + t), \ t \geq 0.$$

Continuation

$$\xi_1 \cup_T \xi_2(\tau) = \begin{cases} \xi_1(\tau) \text{ for } 0 \leq \tau \leq T, \\ \xi_2(\tau - T) \text{ for } \tau > T. \end{cases}$$

Basic axioms

(A1) Compactness: For any $[\varrho_0, \mathbf{m}_0, E_0] \in D$, the set $\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$ is a non–empty compact subset of Ω
(A2) The mapping

$$D \ni [\varrho_0, \mathbf{m}_0, E_0] \mapsto \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \in 2^{\Omega}$$

is **Borel measurable**, where the range of $\mathcal{U}$ is endowed with the Hausdorff metric on the subspace of compact sets in 2^{Ω}
(A3) Shift invariance: For any

$$[\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0],$$

we have

$$S_T \circ [\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho(T), \mathbf{m}(t), E(T-)] \text{ for any } T > 0.$$

(A4) Continuation: If $T > 0$, and

$$[\varrho^1, \mathbf{m}^1, E^1] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0],\ [\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho^1(T), \mathbf{m}^1(T), E^1(T-)],$$

then

$$[\varrho^1, \mathbf{m}^1, E^1] \cup_T [\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0].$$

Induction argument

System of functionals

$$I_{\lambda,F}[\varrho,\mathbf{m},E] = \int_0^\infty \exp(-\lambda t) F(\varrho,\mathbf{m},E)\,\mathrm{d}t,\ \lambda > 0$$

where

$$F : X = W^{-\ell,2}(Q) \times W^{-\ell,2}(Q;R^N) \times R \to R$$

is a bounded and continuous functional

Semiflow reduction

$$I_{\lambda,F} \circ \mathcal{U}[\varrho_0,\mathbf{m}_0,E_0]$$
$$= \Big\{ [\varrho,\mathbf{m},E] \in \mathcal{U}[\varrho_0,\mathbf{m}_0,E_0] \ \Big|$$
$$I_{\lambda,F}[\varrho,\mathbf{m},E] \leq I_{\lambda,F}[\widetilde{\varrho},\widetilde{\mathbf{m}},\widetilde{E}] \text{ for all } [\widetilde{\varrho},\widetilde{\mathbf{m}},\widetilde{E}] \in \mathcal{U}[\varrho_0,\mathbf{m}_0,E_0] \Big\}$$

Induction argument

$$\mathcal{U} \text{ satisfies (A1) - (A4) } \Rightarrow I_{\lambda,F} \circ \mathcal{U} \text{ satisfies (A1) - (A4)}$$

Maximal dissipation

Comparison of energy dissipation

$$[\varrho_1, \mathbf{m}_1, E_1] \prec [\varrho_2, \mathbf{m}_2, E_2] \ \Leftrightarrow\ E_1(t\pm) \leq E_2(t\pm) \text{ for any } t$$

Admissible solutions
Dissipative solution is admissible if it is minimal with respect to $\prec$

Admissibility of semigroup selection

The choice of the testinf functionals can be arranged in the way that the chosen solution is admissible

Semiflow selection (energy excluded)

Borel measurable mapping

$$\mathbf{U} : t \in [0, \infty) \times (\varrho_0, \mathbf{m}_0) \in L^\gamma \times L^{\frac{2\gamma}{\gamma+1}} \mapsto (\varrho, \mathbf{m}) \in C_{\rm weak,loc}([0, \infty); L^\gamma \times L^{\frac{2\gamma}{\gamma+1}})$$

$$U(\cdot; \varrho_0, \mathbf{m}_0) \in \mathcal{U}[\varrho_0, \mathbf{m}_0]$$

Semigroup property

$$\mathbf{U}(t + s; \varrho_0, \mathbf{m}_0) = \mathbf{U}(t; \mathbf{U}(s; \varrho_0, \mathbf{m}_0))$$

for any $t \geq 0$ and a.a. $s \geq 0$ including $s = 0$.

[Basarič [2021], Cardona and Kapitanski [2020]]

Statistical solutions – framework

Data (phase) space

$$\mathcal{D} = \left\{ [\varrho_0, \mathbf{m}_0] \ \middle| \varrho_0 \in L^\gamma(Q),\ \mathbf{m}_0 \in L^{\frac{2\gamma}{\gamma+1}}(Q; R^d)) \int_Q E(\varrho_0, \mathbf{m}_0) \ \mathrm{d}x < \infty \right\}$$

$$\subset X_{\mathcal{D}} = W^{-k,2}(Q) \times W^{-k,2}(Q; R^d) \ \text{ – Polish space}$$

Probability measures

$$\mathfrak{P}[\mathcal{D}] \text{ – the set of probability measures on } X_{\mathcal{D}} \text{ supported by } \mathcal{D}$$

Statistical solution

- Family of Markov operators

$$M_t : \mathfrak{P}[\mathcal{D}] \to \mathfrak{P}[\mathcal{D}]$$

- $$M_0(\nu) = \nu \text{ for any } \nu \in \mathfrak{P}[\mathcal{D}]$$

- $$M_t\left(\sum_{i=1}^N \alpha_i \nu_i\right) = \sum_{i=1}^N \alpha_i M_t(\nu_i),\ \alpha_i \geq 0,\ \sum_{i=1}^N \alpha_i = 1$$

- $$M_{t+s} = M_t \circ M_s \text{ for any } t \geq 0 \text{ and a.a. } s \geq 0$$

- $$t \mapsto M_t \text{ continuous with respect to the weak topology on } \mathfrak{P}[\mathcal{D}]$$

- $$M_t(\delta_{[\varrho_0, \mathbf{m}_0]}) = \delta_{(\varrho(t,\cdot), \mathbf{m}(t,\cdot))}$$
$$[\varrho(t,\cdot), \mathbf{m}(t,\cdot)] = \mathbf{U}(t; \varrho_0, \mathbf{m}_0)$$

Statistical solution – pushforward measure

Semiflow selection

$$\mathbf{U} : [0, \infty) \times \mathcal{D} \to \mathcal{D}$$

Pushforward measure

$$\nu_0 \in \mathfrak{P}[\mathcal{D}] \text{ given}$$

$$M_t(\nu_0)[B] = \nu_0[\mathbf{U}^{-1}(t, B)]$$

$$\int_{X_{\mathcal{D}}} F(\varrho, \mathbf{m}) \, \mathrm{d}M_t(\nu_0) = \int_{\mathcal{D}} F(\mathbf{U}(t; \varrho_0, \mathbf{m}_0)) \, \mathrm{d}\nu_0(\varrho_0, \mathbf{m}_0)$$

for any

$$F \in BC(X_{\mathcal{D}})$$

[Fanelli and EF [2020]]

Summary of the theoretical part, Navier–Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u})$$

No–slip boundary condition

$$Q \subset R^d,\ d = 2, 3 \text{ bounded, smooth}$$

$$\mathbf{u}|_{\partial Q} = 0$$

Initial data

$$\varrho(0, \cdot) = \varrho_0,\ \inf \varrho_0 > 0,\ (\varrho \mathbf{u})(0, \cdot) = \mathbf{m}_0 = \varrho_0 \mathbf{u}_0$$

$$p(\varrho) = a\varrho^\gamma,\ a > 0,\ \gamma > 1$$

Summary of the theoretical part, concepts of solutions

strong (classical) solutions $\subset$ **weak** solutions $\subset$ **dissipative** solutions

Strong solutions
Local in time existence for smooth data., global in time existence for the data close to equilibrium, uniqueness and continuous dependence on the data

Weak solutions
Global in time existence for $\gamma > \frac{d}{2}$, uniqueness – open problem, possibility to select a solution semigroup, measurable dependence of solutions on the data

Dissipative solutions
Limits of consistent approximations – numerical schemes.

Summary of the theoretical part, fundamenal results, I

Weak (dissipative) – strong uniqueness principles
A dissipative solution coincides with the strong solutions emanating from the same (smooth) initial data as long as the strong solution exists

Conditional regularity
A strong solution exists as long as the density and the momentum remain bounded (in the L^∞ norm)

Corollary
Any bounded dissipative solution emanating from smooth initial data is a strong (classical) solution. In addition, if the dissipative solution is a limit of a sequence of consistent approximations, then the convergnce is strong a.a. pointwise

Summary of the theoretical part, fundamenal results, II

Semiflow selection
The set of families of (global in time) weak solutions admits a measurable semiflow selection

Statistical (random) data – Markov semigroup
There exists a semigroup of Markov operators on the space of probability measures on the (initial) data space – a statistical solution to the barotropic Navier–Stokes system

Bibliography, I

D. Basarić.
Semiflow selection for the compressible Navier-Stokes system.
J. Evol. Equ., **21**(1):277–295, 2021.

J. E. Cardona and L. Kapitanski.
Semiflow selection and Markov selection theorems.
Topol. Methods Nonlinear Anal., **56**(1):197–227, 2020.

P. A. Davidson.
Turbulence:An introduction for scientists and engineers.
Oxford University Press, Oxford, 2004.

F. Fanelli and E. Feireisl.
Statistical solutions to the barotropic Navier-Stokes system.
J. Stat. Phys., **181**(1):212–245, 2020.

E. Feireisl.
Dynamics of viscous compressible fluids.
Oxford University Press, Oxford, 2004.

E. Feireisl, Bum Ja Jin, and A. Novotný.
Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system.
J. Math. Fluid Mech., **14**:712–730, 2012.

Bibliography, II

E. Feireisl, M. Lukáčová-Medviďová, H. Mizerová, and B. She.
Numerical analysis of compressible fluid flows.
Springer-Verlag, Cham, 2022.

P.-L. Lions.
Mathematical topics in fluid dynamics, Vol.2, Compressible models.
Oxford Science Publication, Oxford, 1998.

Y. Sun, C. Wang, and Z. Zhang.
A Beale-Kato-Majda criterion for three dimensional compressible viscous heat-conductive flows.
Arch. Ration. Mech. Anal., **201**(2):727–742, 2011.

A. Valli and M. Zajaczkowski.
Navier-Stokes equations for compressible fluids: Global existence and qualitative properties of the solutions in the general case.
Commun. Math. Phys., **103**:259–296, 1986.

Equations of fluid mechanics with random data: Numerics

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based on joint work with M. Lukáčová (Mainz), B. She (Prague), Y. Yuan (Mainz)

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GAČR CZECH SCIENCE FOUNDATION

Summary of the theoretical part, Navier–Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u})$$

Periodic boundary conditions

$$Q = \mathbb{T}^d = \left([-1,1]|_{\{-1,1\}} \right)^d, \ d = 2,3$$

Initial data

$$\varrho(0,\cdot) = \varrho_0, \ \inf \varrho_0 > 0, \ (\varrho \mathbf{u})(0,\cdot) = \mathbf{m}_0 = \varrho_0 \mathbf{u}_0$$

$$p(\varrho) = a\varrho^\gamma, \ a > 0, \ \gamma > 1$$

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Statistical (random) data – Markov semigroup
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Tools from probability theory I

Skorokhod (representation) theorem

Let $(\mathbf{U}^M)_{M=1}^{\infty}$ be a sequence of random variables ranging in a Polish space X. Suppose that their laws are tight in X, meaning for any $\varepsilon > 0$, there exists a compact set $K(\varepsilon) \subset X$ such that

$$\mathbb{P}[\mathbf{U}^M \in X \setminus K(\varepsilon)] \leq \varepsilon \text{ for all } M = 1, 2, \dots .$$

Then there is a subsequence $M_n \to \infty$ and a sequence of random variables $(\widetilde{\mathbf{U}}^{M_n})_{n=1}^{\infty}$ defined on the standard probability space

$$\left(\widetilde{\Omega} = [0,1], \mathfrak{B}[0,1], \mathrm{d}y\right)$$

satisfying:

- $$\widetilde{\mathbf{U}}^{M_n} \approx_X \mathbf{U}^{M_n} \text{ (they are equally distributed random variables)},$$
- $$\widetilde{\mathbf{U}}^{M_n} \to \widetilde{\mathbf{U}} \text{ in } X \text{ for every } y \in [0,1].$$

Tools from probability theory II

Gyöngy–Krylov theorem
Let X be a Polish space and $(\mathbf{U}^M)_{M\geq 1}$ a sequence of $X-$valued random variables.
Then $(\mathbf{U}^M)_{M=1}^{\infty}$ converges in probability if and only if for any sequence of joint laws of

$$(\mathbf{U}^{M_k}, \mathbf{U}^{N_k})_{k=1}^{\infty}$$

there exists further subsequence that converge weakly to a probability measure μ on $X \times X$ such that

$$\mu\left[(x, y) \in X \times X,\ x = y\right] = 1.$$

Regularity criterion for the Navier–Stokes system

Theorem (Regularity criterion)

Let $k \geq 5$*. Let* $(\varrho, \mathbf{u})$ *be a local solution of the Navier–Stokes system. Then*

$$\sup_{t \in [0,T]} \left(\|\varrho(t,\cdot)\|_{W^{k,2}(\mathbb{T}^d)} + \|\mathbf{u}(t,\cdot)\|_{W^{k,2}(\mathbb{T}^d;R^d)} \right) + \int_0^T \|\mathbf{u}\|^2_{W^{k+1}(\mathbb{T}^d;R^d)} \mathrm{d}t$$

$$\leq C \Big(T, \|(\varrho_0, \mathbf{u}_0)\|_{W^{k,2}}, \inf \varrho_0, \|(\varrho, \mathbf{u})\|_{L^\infty((0,T)\times\mathbb{T}^d;R^{d+1})} \Big)$$

for any $0 < T < T_{\max}$*, where* C *is a bounded function of bounded arguments. In particular,*

$$T_{\max} < \infty \ \Rightarrow \ \limsup_{t \to T_{\max}} \|(\varrho, \mathbf{u})(t,\cdot)\|_{L^\infty(\mathbb{T}^d;R^{d+1})} \to \infty.$$

Numerical approximation

(Initial) data

$$\varrho_0,\ \mathbf{m}_0 = \varrho_0 \mathbf{u}_0 \in \mathcal{D} \subset X_{\mathcal{D}}$$

Numerical approximation

$$\varrho^h,\ \mathbf{u}^h,\ h = h(\ell) \to 0 \text{ as } \ell \to \infty$$

Numerical scheme

$(\varrho^h, \mathbf{u}^h) \in V_h$, where $V_h \subset L^\infty((0,T) \times \mathbb{T}^d); R^{d+1})$ is a finite dimensional space,

$$\inf \varrho^h > 0 \text{ for any } h,$$

$$\mathcal{A}\left(h, [\varrho_0, \mathbf{u}_0,], \varrho^h, \mathbf{u}^h\right) = 0,$$

where

$$\mathcal{A} : (0,\infty) \times \mathcal{D} \times V_h \to R^m,\ m = m(h)$$

is a Borel measurable (typically continuous) mapping representing a finite system of algebraic equations called *numerical scheme*

Convergent numerical approximation

We say that a numerical approximation is *convergent* if for any sequence of data

$$[\varrho_0^N, \mathbf{u}_0^N] \in \mathcal{D} \to [\varrho_0, \mathbf{u}_0] \text{ in } X_{\mathcal{D}} \text{ as } N \to \infty,$$

the numerical approximation $(\varrho^{h,N}, \mathbf{u}^{h,N})$ satisfies:

- $$\varrho^{h,N} > 0;$$

- $$\varrho^{h,N} \to \varrho \text{ in } L^1((0,T) \times \mathbb{T}^d),$$
  $$\mathbf{u}^{h,N} \to \mathbf{u} \text{ in } L^1((0,T) \times \mathbb{T}^d; R^d) \text{ as } N \to \infty,\ h \to 0,$$

  for any $0 < T < T_{\max}$, where $(\varrho, \mathbf{u})$ is the unique classical solution of the problem with the data $[\varrho_0, \mathbf{u}_0]$ defined on the maximal time interval $[0, T_{\max})$.

Bounded graph property

If $N = N(\ell) \nearrow \infty$, $h = h(\ell) \searrow 0$,

$$\left[\varrho_0^N, \mathbf{u}_0^N\right] \in \mathcal{D} \to [\varrho_0, \mathbf{u}_0] \text{ in } X_{\mathcal{D}} \text{ as } N \to \infty,$$

and the associated numerical approximation satisfies

$$\sup_{h,N} \left\| (\varrho^{h,N}, \mathbf{u}^{h,N}) \right\|_{L^\infty((0,T)\times\mathbb{T}^d;R^{d+1})} < \infty,$$

then

$$\varrho^{h,N} \to \varrho \text{ in } L^1((0,T)\times\mathbb{T}^d),$$

$$\mathbf{u}^{h,N} \to \mathbf{u} \text{ in } L^1((0,T)\times\mathbb{T}^d;\mathbb{T}^d) \text{ as } h \to 0,\ N \to \infty,$$

where $(\varrho, \mathbf{u})$ is the unique classical solution of the Navier–Stokes system with the initial the data $[\varrho_0, \mathbf{u}_0]$.

Corollary
Any convergent numerical scheme possesses the bounded graph property

Random data, weak approach

$$\varrho_0, \mathbf{u}_0 \in \mathcal{D} \subset X_{\mathcal{D}}$$

weak approach $\Leftrightarrow$ determining distribution (law) of solutions

Generating sequences of random data

$$[\varrho_0^n, \mathbf{u}_0^n] \in \mathcal{D}$$

$$\frac{1}{N} \sum_{n=1}^{N} F\left[\varrho_0^n, \mathbf{u}_0^n\right] \to \mathbb{E}\left[F[\varrho_0, \mathbf{u}_0]\right] \text{ as } N \to \infty$$

for any $F \in BC(X_{\mathcal{D}})$

Expected value

$$\mathbb{E}\left[F[\varrho_0, \mathbf{u}_0]\right] = \int_{X_D} F\left(\hat{\varrho}, \hat{\mathbf{u}}\right) \, \mathrm{d}\mathcal{L}[\varrho_0, \mathbf{u}_0]$$

Distribution of the initial data

$$\mathcal{L}[\varrho_0, \mathbf{u}_0] \in \mathfrak{P}[\mathcal{D}] \text{ -- probability measure on the space of data}$$

Weak approach, main goal I

$[\varrho_0^n, \mathbf{u}_0^n] \in \mathcal{D} \ \to \ [\varrho^{h,n}, \mathbf{u}^{h,n}]$ numerical approximation

Sequence of empirical measures

$$\frac{1}{N} \sum_{n=1}^{N} \delta_{\varrho^{h,n}, \mathbf{u}^{h,n}}$$

Convergence in law

$$\frac{1}{N} \sum_{n=1}^{N} F[\varrho^{h,n}, \mathbf{u}^{h,n}] \to \mathbb{E}\left[F[\varrho, \mathbf{u}] \right] \text{ as } h \to 0, \ N \to \infty$$

for any $F \in BC\Big(W^{-m,2}((0,T) \times \mathbb{T}^d) \times W^{-m,2}((0,T) \times \mathbb{T}^d; R^d) \Big)$

Limit solution

$$\mathbb{E}\left[F[\varrho, \mathbf{u}] \right] = \int_{X_D} F\left[(\varrho, \mathbf{u})[\hat{\varrho}, \hat{\mathbf{u}}] \right] \mathrm{d}\mathcal{L}[\varrho_0, \mathbf{u}_0]$$

$(\varrho, \mathbf{u})$ - smooth (whence unique) statistical solution of the Navier-Stokes system

Weak approach, main goal II

Convergence of empirical means

$$\frac{1}{N}\sum_{n=1}^{N}(\varrho^{h,n}, \mathbf{u}^{h,n}) \to \mathbb{E}\left[\varrho, \mathbf{u}\right] \text{ as } N \to \infty,\ \ h \to 0$$

in $L^q((0,T) \times \mathbb{T}^d; R^{d+1})$, $q \geq 1$

Expected value

$$\mathbb{E}\left[\varrho, \mathbf{u}\right] = \int_{X_D} (\varrho, \mathbf{u})[\hat{\varrho}, \hat{\mathbf{u}}] \ \mathrm{d}\mathcal{L}[\varrho_0, \mathbf{u}_0]$$

Bochner integral in a suitable Banach space

Neither the approximate sequence $[\varrho_0^n, \mathbf{u}_0^n]$ nor the associated numerical solutions $(\varrho^{h,n}, \mathbf{u}^{h,n})$ are uniquely determined by the data $[\varrho_0, \mathbf{u}_0]$. Practical implementations deal with a large number of *samples* – sequences $[\varrho_0^n, \mathbf{u}_0^n]$ – generated independently mimicking the Strong law of large numbers

[Mishra, Schwab et al.]

Random data, strong approach

Data as random variable

$$[\varrho_0, \mathbf{u}_0] : \{\Omega, \mathcal{B}, \mathcal{P}\} \to X_{\mathcal{D}}.$$

Main goal

Identify the exact solution $(\varrho, \mathbf{u})$ as a random variable on the same probability space

Stochastic collocation method

$\Omega = \cup_{n=1}^{N} \Omega_n^N,\ \Omega_n^N\ \mathcal{P} -$ measurable, $\Omega_i^N \cap \Omega_j^N = \emptyset$ for $i \neq j,\ \cup_{n=1}^{N} \Omega_N^n = \Omega$

Approximate random data

$$[\varrho_{0,N}, \mathbf{u}_{0,N}] = \sum_{n=1}^{N} \mathbb{1}_{\Omega_N^n}(\omega)[\varrho_0, \mathbf{u}_0](\omega_n),\ \omega_n \in \Omega_N^n.$$

$$\sum_{n=1}^{N} \mathbb{1}_{\Omega_N^n}(\omega)[\varrho_0, \mathbf{u}_0](\omega_n) \to [\varrho_0, \mathbf{u}_0] \text{ in } X_{\mathcal{D}}\ \mathcal{P} - \text{a.s.}$$

Collocation method - convergence of data approximation

Probability space, class $\mathcal{R}$

$$\Omega - \text{compact metric space}$$

$$\mathcal{R}(\Omega,\mathbb{P}) = \left\{ f : \Omega \to R \ \middle| \ f \text{ bounded}, \ \mathbb{P}\{\omega \in \Omega \ \middle| \ f \text{ is not continuous at } \omega\} = 0 \right\}$$

Unconditional convergence of data approximation
Suppose the (initial data) belong to the class $\mathcal{R}$ (in a weak sense - Fourier modes).
Then

$$\sum_{n=1}^{N} \mathbb{1}_{\Omega_N^n}(\omega)[\varrho_0, \mathbf{u}_0](\omega_n) \to [\varrho_0, \mathbf{u}_0] \text{ in } X_{\mathcal{D}} \ \mathcal{P} - \text{a.s.}$$

independently of the choice of the collocation points provided diameters of the partition tend to zero

[EF, Lukáčová-Medviďová [2021]]

Boundedness in probability, weak approach

Approximate solutions

$$h = h(\ell),\ N = N(\ell),\ h(\ell) \searrow 0,\ N(\ell) \nearrow \infty \text{ as } \ell \to \infty.$$

$$\frac{1}{N}\sum_{n=1}^{N} \delta_{[\varrho^{h,n},\mathbf{u}^{h,n}]}$$

Boundedness in probability (weak)
For any $\varepsilon > 0$, there is $M = M(\varepsilon)$ such that

$$\frac{\#\left\{\|\varrho^{h,n},\mathbf{u}^{h,n}\|_{L^\infty((0,T)\times\mathbb{T}^d;R^{d+1})} > M,\ n \leq N\right\}}{N} < \varepsilon \text{ for any } \ell = 1, 2, \dots$$

Boundedness in probability, strong approach

Approximate solutions

$$h = h(\ell),\ N = N(\ell),\ h(\ell) \searrow 0,\ N(\ell) \nearrow \infty \text{ as } \ell \to \infty.$$

$$\sum_{n=1}^{N} \mathbb{1}_{\Omega_N^n}(\omega)[\varrho^{h,n}, \mathbf{u}^{h,n}]$$

Boundedness in probability (strong)
For any $\varepsilon > 0$, there is $M = M(\varepsilon)$ such that

$$\sum_{n \leq N, \left\{ \| \varrho^{n,h}, \mathbf{u}^{n,h} \|_{L^\infty((0,T)\times \mathbb{T}^d; R^{d+1})} > M \right\}} |\Omega_n^N| < \varepsilon \text{ for } \ell = 1, 2, \ldots$$

Weak to strong

Weak (statistical data)

$$\frac{1}{N}\sum_{n=1}^{N}\delta_{[\varrho_0^n,\mathbf{u}_0^n]}$$

Application of Skorokhod representation theorem

$$\mathcal{L}[\varrho_{0,N},\mathbf{u}_{0,N}] = \mathcal{L}\left[\frac{1}{N}\sum_{n=1}^{N}\delta_{[\varrho_0^n,\mathbf{u}_0^n]}\right]$$

$$[\varrho_{0,N},\mathbf{u}_{0,N}] \to [\widetilde{\varrho}_0,\widetilde{\mathbf{u}}_0] \text{ in } X_{\mathcal{D}} \ \mathrm{d}\mathcal{P} - \text{a.s.}$$

on a probability basis $\{\Omega,\mathcal{B},\mathcal{P}\}$

$$[\widetilde{\varrho}_0,\widetilde{\mathbf{u}}_0] \sim [\varrho_0,\mathbf{u}_0]$$

$\sim$ - equivalence in law

Convergence of approximate solutions, I

Approximate (numerical) solutions

$$(\varrho^{h,N}, \mathbf{u}^{h,N}),\ N = 1, 2, \dots,\ \mathcal{P}\left\{ \left\| \varrho^{h,N}, \mathbf{u}^{h,N} \right\|_{L^\infty((0,T)\times\mathbb{T}^d;R^{d+1}} \geq M \right\} \leq \varepsilon.$$

Application of Skorokhod theorem

$$Y_{h,N} = \left\{ [\varrho_{0,N}, \mathbf{u}_{0,N}]; (\varrho^{h,N}, \mathbf{u}^{h,N}); \Lambda_{h,N} \right\},\quad \text{with } \Lambda_{h,N} = \|\varrho^{h,N}, \mathbf{u}^{h,N}\|_{L^\infty},$$

a sequence of random variables ranging in the Polish space

$$X = X_{\mathcal{D}} \times W^{-m,2}((0,T) \times \mathbb{T}^d; R^{d+1}) \times R,\ m > d + 1.$$

Convergence of approximate solutions, II

$$\mathcal{L}[Y_{h,N}] \text{ tight in } X$$

$$\Rightarrow$$

$$\left\{ [\widetilde{\varrho}_{0,N_k}, \widetilde{\mathbf{u}}_{0,N_k}]; \left(\widetilde{\varrho}^{h_k,N_k}, \widetilde{\mathbf{u}}^{h_k,N_k} \right); \widetilde{\Lambda}_{h_k,N_k} \right\}$$

$$\sim \left\{ [\varrho_{0,N_k}, \mathbf{u}_{0,N_k}]; \left(\varrho^{h_k,N_k}, \mathbf{u}^{h_k,N_k} \right), \Lambda_{h_k,N_k} \right\},$$

$$[\widetilde{\varrho}_{0,N_k}, \widetilde{\mathbf{u}}_{0,N_k}] \to [\widetilde{\varrho}_0, \widetilde{\mathbf{u}}_0] \text{ in } X_{\mathcal{D}} \ \widetilde{\mathcal{P}} - \text{a.s.},$$

$$\text{where } [\widetilde{\varrho}_0, \widetilde{\mathbf{u}}_0] \sim [\varrho_0, \mathbf{u}_0]$$

$$\left(\widetilde{\varrho}^{h_k,N_k}, \widetilde{\mathbf{u}}^{h_k,N_k} \right) \to (\widetilde{\varrho}, \widetilde{\mathbf{u}}) \ \text{ in } W^{-m,2}((0,T) \times \mathbb{T}^d; R^{d+1}) \ \widetilde{\mathcal{P}} - \text{a.s.},$$

and

$$\widetilde{\Lambda}_{h_k,N_k} = \|(\widetilde{\varrho}^{h_k,N_k}, \widetilde{\mathbf{u}}^{h_k,N_k})\|_{L^\infty} \to \widetilde{\Lambda} \ \widetilde{\mathcal{P}} - \text{a.s.}$$

on a probability space $\{\widetilde{\Omega}; \widetilde{\mathcal{B}}; \widetilde{\mathcal{P}}\}$

Convergence of approximate solutions, conclusion

Bounded graph property

$$\left(\widetilde{\varrho}^{h_k,N_k},\widetilde{\mathbf{u}}^{h_k,N_k}\right) \to (\widetilde{\varrho},\widetilde{\mathbf{u}}) \text{ strongly in } L^q((0,T)\times\mathbb{T}^d;R^{d+1}) \ \ \widetilde{\mathcal{P}}-\text{a.s.}$$

$$\text{for any } 1 \leq q < \infty$$

where $(\widetilde{\varrho},\widetilde{\mathbf{u}})$ is the unique (statistical) solution of the Navier–Stokes system

Gyöngy–Krylov criterion

$$\left(\varrho^{h,N},\mathbf{u}^{h,N}\right) \to (\varrho,\mathbf{u}) \text{ in } L^q((0,T)\times\mathbb{T}^d;R^{d+1}) \text{ in } \mathcal{P}-\text{probability}$$

on the original probability basis

Convergence in expectations

Strong convergence in expectations [EF [2022]]
Suppose that the energy of the numerical solutions is bounded in expectations, meaning

$$\sum_{n=1}^{N} |\Omega_n^M| \int_{\mathbb{T}^d} \left[\frac{1}{2} \varrho^{h,n} |\mathbf{u}^{h,n}|^2 + P(\varrho^{h,n}) \right] (\tau, \cdot) \, \mathrm{d}x \lesssim 1 \text{ for } \tau \in (0,T),\ \ell = 1, 2, \dots$$

Then

$$\mathbb{E} \left[\left\| \sum_{n=1}^{N} \mathbb{1}_{\Omega_n^N} \varrho^{h,n} - \varrho \right\|^r_{L^\gamma((0,T)\times\mathbb{T}^d)} \right] \to 0 \text{ as } \ell \to \infty \text{ for any } 1 \leq r < \gamma,$$

$$\mathbb{E} \left[\left\| \sum_{n=1}^{N} \mathbb{1}_{\Omega_n^N} \varrho^{h,n} \mathbf{u}^{h,n} - \varrho \mathbf{u} \right\|^s_{L^{\frac{2\gamma}{\gamma+1}}((0,T)\times\mathbb{T}^d;R^d)} \right] \to 0 \text{ as } \ell \to \infty$$

$$\text{for any } 1 \leq s < \frac{2\gamma}{\gamma+1}$$

r-barycenter

r-barycenter

$\mathbb{E}_r[Y]$ of a random variable Y defined on a Polish space $(X; d_X)$:

$$\mathbb{E}_r[Y] \in X,\ \mathbb{E}\left[d_X\left(Y; \mathbb{E}_r[Y]\right)^r\right] = \min_{Z \in X} \mathbb{E}\left[d_X\left(Y; Z\right)^r\right],\ r \geq 1,$$

meaning

$$E_r(Y) = \arg\min_{Z \in X} \mathbb{E}\left[d_X\left(Y; Z\right)^r\right].$$

If $X = L^q((0, T) \times \mathbb{T}^d; R^d)$ and $1 < r < \infty$, then

- there exists a unique $r-$barycenter for any Y, $\mathbb{E}\left[\|Y\|_{L^q}^r\right] < \infty$,
- $\mathbb{E}_r[Y]$ depends only on the distribution (law) of Y

Convergence of barycenters

Strong convergence of barycenters [EF [2022]]

Suppose that the energy of the numerical solutions is bounded in expectations.

Then

- $$\frac{1}{N}\sum_{n=1}^{N} \varrho^{h,n} \to \mathbb{E}\left[\varrho\right] \ \text{in} \ L^{\gamma}((0,T)\times\mathbb{T}^d),$$

  $$\frac{1}{N}\sum_{n=1}^{N} \varrho^{h,n}\mathbf{u}^{h,n} \to \mathbb{E}\left[\varrho\mathbf{u}\right] \ \text{in} \ L^{\frac{2\gamma}{\gamma+1}}((0,T)\times\mathbb{T}^d;R^d)$$

  as $\ell \to \infty$

- $$\mathbb{E}_r\left[\frac{1}{N}\sum_{n=1}^{N}\delta_{\varrho^{h,n}}\right] \to \mathbb{E}_r[\varrho] \ \text{in} \ L^{\gamma}(\mathbb{T}^d), \ 1 < r < \gamma,$$

  $$\mathbb{E}_s\left[\frac{1}{N}\sum_{n=1}^{N}\delta_{\varrho^{h,n}\mathbf{u}^{h,n}}\right] \to \mathbb{E}_s[\varrho\mathbf{u}] \ \text{in} \ L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;R^d), \ 1 < s < \frac{2\gamma}{\gamma+1}$$

  as $\ell \to \infty$.

Bibliography

E. Feireisl.
Compressible fluid motion with uncertain data.
Archive Preprint Series, 2022.
arxiv preprint No. 2203.15049.

E. Feireisl and M. Lukáčová-Medviďová.
Convergence of a stochastic collocation finite volume method for the compressible Navier–Stokes system.
Archive Preprint Series, 2021.
arxiv preprint No.2111.07435.

U. Koley, N.H. Risebro, C. Schwab, and F. Weber.
A multilevel Monte Carlo finite difference method for random scalar degenerate convection-diffusion equations.
J. Hyperbolic Differ. Equ., **14**(3):415–454, 2017.

F. Y. Kuo, R. Scheichl, C. Schwab, I. H. Sloan, and E. Ullmann.
Multilevel quasi-Monte Carlo methods for lognormal diffusion problems.
Math. Comp., **86**(308):2827–2860, 2017.

Bibliography

F. Leonardi, S. Mishra, and C. Schwab.
Numerical approximation of statistical solutions of planar, incompressible flows.
Math. Models Methods Appl. Sci., **26**(13):2471–2523, 2016.

S. Mishra and Ch. Schwab.
Sparse tensor multi-level Monte Carlo finite volume methods for hyperbolic conservation laws with random initial data.
Math. Comp., **81**(280):1979–2018, 2012.