Introduction to Sobolev spaces for PDEs

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This is a third iteration of a text, which is intended to be an introduction to the theory of Sobolev spaces, traces, embeddings, and interpolations which are useful in the theory of PDEs. The goal is to show the main ideas of the proofs, so that the reader can derive himself/herself particular formulas in cases that are not explicitly treated in textbooks. Hence, emphasis is put on methods rather than on a collection of formulas. Additional information can be found in [1, 2, 3, 11, 12].

Contents

1 Some analytical tools

Let X be a real Banach space. For a set $A \subset X$ we denote here and in the sequel by $\operatorname{Int} A$ the interior of A, by $\bar{A}$ the closure of A, and by $\partial A := \bar{A} \setminus \operatorname{Int} A$ the boundary of A. The symbol $B_\rho(x)$ denotes the open ball in X centered at $x \in X$ with radius r.

In this introductory section we recall for the reader's convenience some basic properties of the space $\mathbb{R}^N$ starting from the following classical lemma.

Lemma 1.1 *Let us consider two sets K and G, $K \subset G \subset \mathbb{R}^N$, K compact, G open. Then there exists a function $g \in C^\infty(\mathbb{R}^N)$ such that $0 \le g(x) \le 1$ for all $x \in \mathbb{R}^N$, $g(x) = 1$ for $x \in K$, $g(x) = 0$ for $x \in \mathbb{R}^N \setminus G$.*

Proof. The set K is compact, hence $\rho := \operatorname{dist}(K, \mathbb{R}^N \setminus G)$ is positive. Let $U := \bigcup_{x\in K} B_{\rho/2}(x)$. Then U is open, $\bar{U}$ is compact, and $\operatorname{dist}(\bar{U}, \mathbb{R}^N \setminus G) = \operatorname{dist}(K, \mathbb{R}^N \setminus U) = \rho/2$. Put $f(x) = 1$ for $x \in U$, $f(x) = 0$ for $x \in \mathbb{R}^N \setminus U$. We now choose a function $\varphi \in C^\infty(\mathbb{R}^N)$ called a *mollifier* such that

$$\varphi(x) \ge 0 \ \ \forall x \in \mathbb{R}^N, \quad \varphi(x) = 0 \ \text{ for } \ |x| \ge 1, \quad \int_{\mathbb{R}^N} \varphi(x)\,\mathrm{d}x = 1. \tag{1.1}$$

For a fixed value $\sigma \in (0, \rho/2)$ put

$$g(x) = \sigma^{-N} \int_{\mathbb{R}^N} \varphi\left(\frac{x-y}{\sigma}\right) f(y)\,\mathrm{d}y = \int_{\mathbb{R}^N} \varphi(\xi) f(x - \sigma\xi)\,\mathrm{d}\xi. \tag{1.2}$$

Then $g \in C^\infty(\mathbb{R}^N)$, $0 \le g(x) \le 1$ for all $x \in \mathbb{R}^N$. For $x \in K$ and $|\xi| < 1$ we have $x - \sigma\xi \in U$ and similarly, for $x \in \mathbb{R}^N \setminus G$ and $|\xi| < 1$ we have $x - \sigma\xi \in \mathbb{R}^N \setminus U$. Hence, $g(x) = 1$ on K, $g(x) = 0$ on $\mathbb{R}^N \setminus G$, which we wanted to prove. ■

Theorem 1.2 ***(Partition of unity)*** *Let $G_1, \dots, G_m$ be open subsets of $\mathbb{R}^N$, and let $K \subset \bigcup_{j=1}^m G_j$ be a compact set. Then there exist functions $\psi_1, \dots, \psi_m$ in $C^\infty(\mathbb{R}^N)$ such that for all $j = 1, \dots, m$ and all $x \in \mathbb{R}^N$ we have $0 \le \psi_j(x) \le 1$, $\psi_j(x) = 0$ for $x \in \mathbb{R}^N \setminus G_j$, and*

$$\sum_{j=1}^m \psi_j(x) = 1 \quad \forall x \in K.$$

Proof. For each $x \in K$ we denote $J(x) = \{j \in \{1, \dots, m\} : x \in G_j\}$, and find $r(x) > 0$ such that

$$B_{r(x)}(x) \subset \bigcap_{j \in J(x)} G_j.$$

We have indeed $K \subset \bigcup_{x \in K} B_{r(x)/2}(x)$. Using the compactness of K we select a finite subcovering

$$K \subset \bigcup_{i=1}^M B_{r(x_i)/2}(x_i).$$

For $j = 1, \dots, m$ put

$$\begin{aligned} R_j &:= \{i \in \{1, \dots, M\} : B_{r(x_i)}(x_i) \subset G_j\}, \\ V_j &:= \bigcup_{i \in R_j} B_{r(x_i)}(x_i), \\ W_j &:= \bigcup_{i \in R_j} B_{r(x_i)/2}(x_i). \end{aligned}$$

We have $\bar{W}_j \subset V_j \subset G_j$ for each $j = 1, \dots, m$, and

$$K \subset \bigcup_{j=1}^m W_j.$$

Using Lemma 1.1 we find functions $g_j \in C^\infty(\mathbb{R}^N)$ such that $g_j(x) = 1$ for $x \in \bar{W}_j$, $g_j(x) = 0$ for $x \in \mathbb{R}^N \setminus V_j$. We now construct a sequence for $x \in \mathbb{R}^N$ by the recurrent formula

$$\psi_0(x) = 0, \quad \psi_j(x) = g_j(x)\left(1 - \sum_{i=0}^{j-1} \psi_i(x)\right) \quad \text{for } j = 1, \dots, m.$$

By construction, we have $\psi_j(x) = 0$ for all $x \in \mathbb{R}^N \setminus G_j$ and for all $j = 1, \dots, m$. Furthermore,

$$\sum_{i=0}^{j} \psi_i(x) = g_j(x) + (1 - g_j(x)) \sum_{i=0}^{j-1} \psi_i(x),$$

We easily prove the implications

$$\sum_{i=0}^{j-1} \psi_i(x) \le 1 \implies \sum_{i=0}^{j} \psi_i(x) \le 1,$$
$$\sum_{i=0}^{j-1} \psi_i(x) = 1 \text{ for } x \in \bigcup_{i=1}^{j-1} W_i \implies \sum_{i=0}^{j} \psi_i(x) = 1 \text{ for } x \in \bigcup_{i=1}^{j} W_i,$$

and obtain in particular $\sum_{i=0}^m \psi_i(x) = 1$ on $\bigcup_{i=1}^m W_i \supset K$ which completes the proof. ■

2 Spaces $L^p(\Omega)$

Let $\Omega \subset \mathbb{R}^N$ be any open set, and let $u : \Omega \to \mathbb{R}$ be a Lebesgue measurable function. We say that the class of functions

$$\{\hat{u} : \Omega \to \mathbb{R} : \hat{u}(x) = u(x) \ \text{ a. e.}\}$$

belongs to $L^p(\Omega)$ for some $1 \le p \le \infty$ if the expression

$$|u|_{p,\Omega} = \begin{cases} \left(\int_\Omega |u(x)|^p \, \mathrm{d}x \right)^{1/p} & \text{if } 1 \le p < \infty\,, \\ \sup\operatorname{ess}_{x\in\Omega} |u(x)| & \text{if } p = \infty\,. \end{cases} \tag{2.1}$$

is finite. In the case $\Omega = \mathbb{R}^N$, we simply write $|\cdot|_p$ instead of $|\cdot|_{p,\mathbb{R}^N}$. It is easy to see that $|\cdot|_{\infty,\Omega}$ is a norm on $L^\infty(\Omega)$. We shall see below in (2.8) that $|\cdot|_{p,\Omega}$ is a norm on $L^p(\Omega)$ for all $p \ge 1$. Note that the spaces $L^p(\Omega)$ with the above norms are *Banach spaces*, see [1].

Let us begin with some auxiliary inequalities that are needed in the sequel.

Proposition 2.1 ***(Young's inequality)*** *Let $f : [0,\infty) \to [0,\infty)$ be an absolutely continuous increasing function, $f(0) = 0$. Then for every $x, y \ge 0$ we have (see Fig. 1)*

$$xy \;\le\; \int_0^x f(u)\,\mathrm{d}u + \int_0^y f^{-1}(v)\,\mathrm{d}v\,, \tag{2.2}$$

where f^{-1} is the inverse function to f.

Proof. Substituting $v = f(u)$ we have, with the convention $\int_a^b = -\int_b^a$ if $b < a$, that

$$\begin{aligned} \int_0^x f(u)\,\mathrm{d}u + \int_0^y f^{-1}(v)\,\mathrm{d}v \;&=\; \int_0^x (f(u) + uf'(u))\,\mathrm{d}u + \int_x^{f^{-1}(y)} uf'(u)\,\mathrm{d}u \\ &\ge\; xf(x) + x(y - f(x)) \;=\; xy\,. \end{aligned}$$

■

For $1 < p < \infty$, we denote by p' the *conjugate exponent*

$$p' \;=\; \frac{p}{p-1}\,. \tag{2.3}$$

Reciprocally, p is the conjugate of p' and we have

$$\frac{1}{p} + \frac{1}{p'} \;=\; 1\,, \quad p' - 1 \;=\; \frac{1}{p-1}\,. \tag{2.4}$$

As an immediate consequence of Proposition 2.1 we obtain, putting $f(x) = x^{p-1}$, that

$$xy \;\leq\; \frac{1}{p}x^p + \frac{1}{p'}y^{p'} \tag{2.5}$$

for every $x, y \geq 0$ and $1 < p < \infty$.

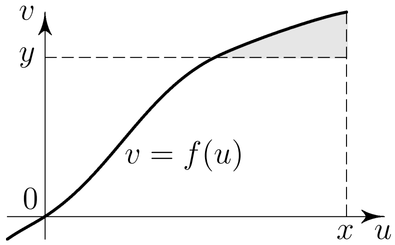

Figure 1: Young's inequality.

Proposition 2.2 ***(Hölder's inequality)*** *Let $\Omega \subset \mathbb{R}^N$ be any open set and let $1 \leq p \leq \infty$ be arbitrary. Then for every $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ we have*

$$\int_\Omega f(x)\,g(x)\,\mathrm{d}x \;\leq\; |f|_{p,\Omega}\,|g|_{p',\Omega}\,, \tag{2.6}$$

with the convention $1' = \infty$, $\infty' = 1$.

Proof. The cases $f = 0$, or $g = 0$, or $p = 1$, or $p = \infty$ are obvious. For $1 < p < \infty$ and $|f|_{p,\Omega} \neq 0$, $|g|_{p',\Omega} \neq 0$ we set

$$F(x) = \frac{f(x)}{|f|_{p,\Omega}}\,, \quad G(x) = \frac{g(x)}{|g|_{p',\Omega}}\,.$$

By (2.5) we have

$$|F(x)|\,|G(x)| \;\leq\; \frac{1}{p}|F(x)|^p + \frac{1}{p'}|G(x)|^{p'} \;=\; \frac{|f(x)|^p}{p\,|f|_{p,\Omega}^p} + \frac{|g(x)|^{p'}}{p'\,|g|_{p',\Omega}^{p'}}\,,$$

hence

$$\int_\Omega F(x)\,G(x)\,\mathrm{d}x \;\leq\; \int_\Omega |F(x)|\,|G(x)|\,\mathrm{d}x \;\leq\; \frac{1}{p} + \frac{1}{p'} \;=\; 1\,,$$

which we wanted to prove. ■

Proposition 2.3 ***(Minkowski's inequality)*** *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ be open sets, and let $f : X \times Y \to [0, \infty)$ be a measurable function. Then for every $1 \leq p < \infty$ we have*

$$\left(\int_Y \left(\int_X f(x,y)\,\mathrm{d}x\right)^p \mathrm{d}y\right)^{1/p} \leq \int_X \left(\int_Y f^p(x,y)\,\mathrm{d}y\right)^{1/p} \mathrm{d}x\,. \tag{2.7}$$

Proof. For $y \in Y$ and $R > 0$ set

$$F(y) = \int_X f_R(x,y)\,\mathrm{d}x\,, \quad g(y) = F^{p-1}(y)\,,$$

where

$$f_R(x,y) \;=\; \begin{cases} \min\{R, f(x,y)\} & \text{if } \max\{|x|,|y|\} < R\,, \\ 0 & \text{if } \max\{|x|,|y|\} \geq R\,. \end{cases}$$

Then

$$\begin{aligned} \int_Y F^p(y)\,\mathrm{d}y &= \int_Y F(y)\,g(y)\,\mathrm{d}y \;=\; \int_X \left(\int_Y f_R(x,y)\,g(y)\,\mathrm{d}y\right)\mathrm{d}x \\ &\overset{\text{Hölder}}{\leq} \int_X \left(\int_Y f_R^p(x,y)\,\mathrm{d}y\right)^{1/p} \left(\int_Y g^{p'}(y)\,\mathrm{d}y\right)^{1/p'} \mathrm{d}x \\ &= \int_X \left(\int_Y f_R^p(x,y)\,\mathrm{d}y\right)^{1/p} \mathrm{d}x \left(\int_Y F^p(y)\,\mathrm{d}y\right)^{1/p'}\,, \end{aligned}$$

hence

$$\left(\int_Y F^p(y)\,\mathrm{d}y\right)^{1/p} \leq \int_X \left(\int_Y f_R^p(x,y)\,\mathrm{d}y\right)^{1/p} \mathrm{d}x \leq \int_X \left(\int_Y f^p(x,y)\,\mathrm{d}y\right)^{1/p} \mathrm{d}x\,,$$

and we obtain the result from Fatou's Lemma letting R tend to $+\infty$. ■

Note that replacing X by a finite set $\{1,\dots,n\}$, $f(x,y)$ by $f_k(y)$, $k = 1,\dots,n$, and $\int_X \mathrm{d}x$ by $\sum_{k=1}^n$, the Minkowski inequality appears in the form

$$\left|\sum_{k=1}^n f_k\right|_{p,Y} \leq \sum_{k=1}^n |f_k|_{p,Y}\,, \tag{2.8}$$

which is nothing but the triangle inequality for the norm $|\cdot|_{p,Y}$.

Let us also mention the trivial "variant" of the Minkowski inequality for $p = \infty$, namely

$$\operatorname*{sup\,ess}_{y\in Y} \int_X f(x,y)\,\mathrm{d}x \leq \int_X \operatorname*{sup\,ess}_{y\in Y} f(x,y)\,\mathrm{d}x\,. \tag{2.9}$$

The following example shows that the Minkowski inequality cannot be reversed.

Example 2.4 Consider $X = Y = (0,1)$, and $f(x,y) = ((x-y)^+)^{-1/p}$ for some $p > 1$. Then

$$\begin{aligned} \left(\int_Y \left(\int_X f(x,y)\,\mathrm{d}x\right)^p \mathrm{d}y\right)^{1/p} &= \left(\int_0^1 \left(\int_y^1 (x-y)^{-1/p}\,\mathrm{d}x\right)^p \mathrm{d}y\right)^{1/p} &= \tfrac{1}{p-1}\,p^{1-1/p}, \\ \int_X \left(\int_Y f^p(x,y)\,\mathrm{d}y\right)^{1/p} \mathrm{d}x &= \int_0^1 \left(\int_0^x (x-y)^{-1}\,\mathrm{d}x\right)^{1/p} \mathrm{d}y &= +\infty\,. \end{aligned}$$

Remark 2.5 In the same way we prove that for every $1 \leq q < p < \infty$ we have

$$\left(\int_Y \left(\int_X f^q(x,y)\,\mathrm{d}x\right)^{p/q} \mathrm{d}y\right)^{1/p} \leq \left(\int_X \left(\int_Y f^p(x,y)\,\mathrm{d}y\right)^{q/p} \mathrm{d}x\right)^{1/q}. \tag{2.10}$$

We set in this case

$$F(y) = \int_X f_R^q(x,y)\,\mathrm{d}x\,, \quad g(y) = F^{(p/q)-1}(y)\,,$$

and estimate $\int_Y F^{p/q}(y)\,\mathrm{d}y$ similarly as in the proof of Proposition 2.3.

The proof of the Minkowski inequality is related to the so-called *reverse Hölder inequality*:

$$\int_\Omega f(x)\, g(x)\,\mathrm{d}x \;\le\; C\,|g|_{p',\Omega} \quad \forall g \in L^{p'}(\Omega) \;\Longrightarrow\; |f|_{p,\Omega} \;\le\; C\,. \tag{2.11}$$

To prove this statement, it suffices to choose $g(x) = \mathrm{sign}(f_R(x))\,|f_R(x)|^{p-1}$ with f_R defined analogously as in the proof of Proposition 2.3, use the fact that

$$\int_\Omega |f_R(x)|^p\,\mathrm{d}x \le \int_\Omega f(x)\,g(x)\,\mathrm{d}x \le \; C\,|g|_{p',\Omega} = C\,|f_R|_{p,\Omega}^{p/p'}\,,$$

and let R tend to ∞.

Proposition 2.6 ***(Young's inequality II for convolutions)*** *Let $1 \le p, q, r \le \infty$ be given such that*

$$\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}\,. \tag{2.12}$$

For $u \in L^p(\mathbb{R}^N)$, $v \in L^r(\mathbb{R}^N)$, and $x \in \mathbb{R}^N$ set

$$w(x) \;=\; \int_{\mathbb{R}^N} u(y)\,v(x-y)\,\mathrm{d}y\,.$$

Then $w \in L^q(\mathbb{R}^N)$ and

$$|w|_q \;\le\; |u|_p\,|v|_r\,. \tag{2.13}$$

Proof. The case $q = \infty$ follows immediately from Hölder's inequality. Hence, assume that $q < \infty$, and set $\alpha = r/q \in (0,1]$. To make the use of the Minkowski inequality more transparent, we write $\int_X \mathrm{d}x$, $\int_Y \mathrm{d}y$ instead of $\int_{\mathbb{R}^N} \mathrm{d}x$, $\int_{\mathbb{R}^N} \mathrm{d}y$. Then, using the fact that $1-\alpha = r/p'$ and that

$$\int_Y |v(x-y)|^r\,\mathrm{d}y \;=\; \int_Y |v(y)|^r\,\mathrm{d}y$$

for a.e. $x \in X$, we obtain

$$\begin{aligned}
|w|_q \;&=\; \left(\int_X \left|\int_Y u(y)\,v(x-y)\,\mathrm{d}y\right|^q \mathrm{d}x\right)^{1/q} \\
&\le\; \left(\int_X \left(\int_Y |u(y)|\,|v(x-y)|^\alpha\,|v(x-y)|^{1-\alpha}\,\mathrm{d}y\right)^q \mathrm{d}x\right)^{1/q} \\
&\overset{\text{Hölder}}{\le}\; \left(\int_X \left(\int_Y |u(y)|^p\,|v(x-y)|^{p\alpha}\,\mathrm{d}y\right)^{q/p} \mathrm{d}x\right)^{1/q} \left(\int_Y |v(y)|^{p'(1-\alpha)}\,\mathrm{d}y\right)^{1/p'} \\
&\overset{\text{Minkowski}}{\le}\; \left(\int_Y \left(\int_X |u(y)|^q\,|v(x-y)|^{q\alpha}\,\mathrm{d}x\right)^{p/q} \mathrm{d}y\right)^{1/p} |v|_r^{1-\alpha} \\
&=\; \left(\int_Y |u(y)|^p\,\mathrm{d}y\right)^{1/p} \left(\int_X |v(x)|^{q\alpha}\,\mathrm{d}x\right)^{1/q} |v|_r^{1-\alpha} \;=\; |u|_p\,|v|_r\,.
\end{aligned}$$

■

We devote the next section to the *Hardy-Littlewood inequality*, the proof of which is quite involved and requires a certain number of auxiliary steps. The proof we give here is a modification of the one from [2].

3 Hardy-Littlewood-Sobolev inequality

In this section we prove the following statement.

Proposition 3.1 ***(Hardy-Littlewood-Sobolev inequality)*** *Let* $1 < p, q, r < \infty$ *be such that* $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. *Then there exists a constant* $H_{pr} > 0$ *such that for every* $f \in L^p(\mathbb{R})$, $g \in L^q(\mathbb{R})$ *we have*

$$\iint_{\mathbb{R}^2} f(x)\, g(y)\, |x-y|^{-1/r}\, \mathrm{d}x\, \mathrm{d}y \;\leq\; H_{pr} |f|_p\, |g|_q\,. \tag{3.1}$$

An explicit estimate for H_{pr} will be given in (3.16) below. There exist also interesting counterparts in $\mathbb{R}^N$, see [6]. In $\mathbb{R}$, the most elementary approach consists in *rearrangements*. We recall here some basic facts about this technique.

We denote by $\mathcal{L}$ the space of functions $g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ such that $g(x) \geq 0$ a. e. For $g \in \mathcal{L}$ and $r \geq 0$ we define

$$\begin{aligned} A(r) &= \{x \in \mathbb{R} : g(x) > r\}, \\ \delta(r) &= \frac{1}{2} \mathrm{meas}\, A(r), \\ A^*(r) &= (-\delta(r), \delta(r)). \end{aligned}$$

We further put

$$\begin{aligned} A &= \{(x,r) \in \mathbb{R} \times [0,\infty) : x \in A(r)\}, \\ A^* &= \{(x,r) \in \mathbb{R} \times [0,\infty) : x \in A^*(r)\}, \\ g^*(x) &= \int_0^\infty \chi_{A^*}(x,r)\, \mathrm{d}r \qquad \text{for } \ x \in \mathbb{R}, \end{aligned}$$

where χ_{A^*} is the characteristic function of the set A^*.

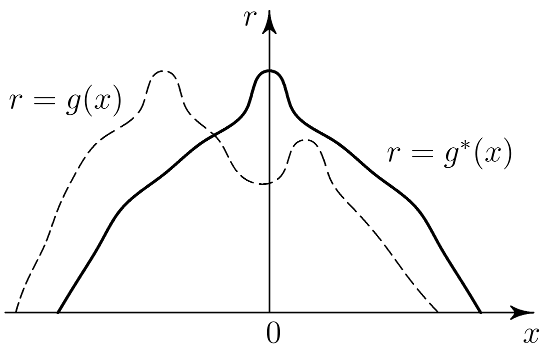

Figure 2: Rearrangement.

The function g^* is called the *rearrangement* of g, and admits the following characterization.

Lemma 3.2 *Let* $g \in \mathcal{L}$ *be given. Then for all* $x \in \mathbb{R}$ *we have*

$$g^*(x) = \sup\{r > 0 : \delta(r) > |x|\}$$

with the convention $\sup \emptyset = 0$.

Proof. Put

$$\rho(x) := \sup\{s > 0 : \delta(s) > |x|\}.$$

The function δ is nonincreasing in $[0, \infty)$. For $r > \rho(x)$ we thus have $\delta(r) \le |x|$, hence $x \notin A^*(r)$ and $\chi_{A^*}(x, r) = 0$. On the other hand, for $r < \rho(x)$ we have $\delta(r) > |x|$, hence $x \in A^*(r)$ and $\chi_{A^*}(x, r) = 1$. This yields

$$A^*(r) = \{x \in \mathbb{R} : g^*(x) > r\}, \quad g^*(x) = \int_0^{\rho(x)} \mathrm{d}r = \rho(x), \tag{3.2}$$

and the proof is complete. ∎

Let us list some further properties of the rearrangement.

Lemma 3.3 *Let $g \in \mathcal{L}$ be given. Then for all $x \in \mathbb{R}$ and $p \ge 1$ we have*

(i) $g^p(x) = \int_0^\infty \chi_A(x, r^{1/p})\, \mathrm{d}r$ *;*

(ii) $(g^*)^p(x) = (g^p)^*(x) = \sup\{r > 0 : \delta(r^{1/p}) > |x|\} = \int_0^\infty \chi_{A^*}(x, r^{1/p})\, \mathrm{d}r$ *;*

(iii) $\int_{\mathbb{R}} |g(x)|^p\, \mathrm{d}x = \int_{\mathbb{R}} |g^*(x)|^p\, \mathrm{d}x$ *.*

Proof.
(i) We have for all $x \in \mathbb{R}$, $p \ge 1$, and $r \ge 0$ that

$$\chi_A(x, r^{1/p}) = \begin{cases} 1 & \text{for } r < g^p(x), \\ 0 & \text{for } r \ge g^p(x), \end{cases}$$

and the statement follows easily.

(ii) Let $x \in \mathbb{R}$ and $p \ge 1$ be arbitrarily chosen. We use Lemma 3.2 to conclude that for $r > (g^*)^p(x)$ we have $\delta(r^{1/p}) \le |x|$, hence $x \notin A^*(r^{1/p})$ and $\chi_{A^*}(x, r^{1/p}) = 0$, for $r < (g^*)^p(x)$ we have $\delta(r^{1/p}) > |x|$, hence $x \in A^*(r^{1/p})$ and $\chi_{A^*}(x, r^{1/p}) = 1$. Consequently,

$$\int_0^\infty \chi_{A^*}(x, r^{1/p})\, \mathrm{d}r = \int_0^{(g^*)^p(x)} \mathrm{d}r = (g^*)^p(x).$$

We have indeed $A_p(r) := \{x \in \mathbb{R} : g^p(x) > r\} = A(r^{1/p})$, and $\delta_p(r) := \operatorname{meas} A_p(r) = \delta(r^{1/p})$. Hence,

$$(g^p)^*(x) = \int_0^\infty \chi_{A^*}(x, r^{1/p})\, \mathrm{d}r$$

and the assertion follows.

(iii) In view of (ii), it suffices to prove that

$$\int_{\mathbb{R}} g(x)\, \mathrm{d}x = \int_{\mathbb{R}} g^*(x)\, \mathrm{d}x. \tag{3.3}$$

This follows from the elementary computation based on (i) and on Fubini's Theorem

$$\begin{aligned}\int_{\mathbb{R}} g(x)\,\mathrm{d}x &= \int_{\mathbb{R}}\int_0^\infty \chi_A(x,r)\,\mathrm{d}r\,\mathrm{d}x = \int_0^\infty\int_{\mathbb{R}} \chi_A(x,r)\,\mathrm{d}x\,\mathrm{d}r = \int_0^\infty \operatorname{meas} A(r)\,\mathrm{d}r \\ &= \int_0^\infty \operatorname{meas} A^*(r)\,\mathrm{d}r = \int_{\mathbb{R}}\int_0^\infty \chi_{A^*}(x,r)\,\mathrm{d}r\,\mathrm{d}x = \int_{\mathbb{R}} g^*(x)\,\mathrm{d}x\end{aligned}$$

and the proof is complete. ■

We introduce the space

$$\mathcal{R} = \{f \in \mathcal{L} : f(x) = f(-x) \text{ a.e.}, f \text{ nondecreasing in } (-\infty,0),\ \text{nonincreasing in } (0,+\infty)\}. \tag{3.4}$$

By construction, the function g^* belongs to $\mathcal{R}$, and we call the mapping

$$R : \mathcal{L} \to \mathcal{R} : g \mapsto R[g] = g^* \tag{3.5}$$

the *rearrangement operator*. It is an easy exercise to check that the identity

$$R[\mu g] = \mu R[g] \tag{3.6}$$

holds for every $g \in \mathcal{L}$ and every $\mu \geq 0$.

Let us mention one classical application of rearrangements.

Proposition 3.4 ***(Hardy-Littlewood inequality)*** *For every $f, g \in \mathcal{L}$, $f^* = R[f]$, $g^* = R[g]$, we have*

$$\int_{\mathbb{R}} f(x)g(x)\,\mathrm{d}x \leq \int_{\mathbb{R}} f^*(x)g^*(x)\,\mathrm{d}x. \tag{3.7}$$

Proof. For $f, g \in \mathcal{L}$ put $A_f = \{(x,r) \in \mathbb{R}\times(0,\infty) : f(x) > r\}$, $A_g = \{(x,r) \in \mathbb{R}\times(0,\infty) : g(x) > r\}$, $A_f^* = \{(x,r) \in \mathbb{R}\times(0,\infty) : f^*(x) > r\}$, $A_g^* = \{(x,r) \in \mathbb{R}\times(0,\infty) : g^*(x) > r\}$. By Lemma 3.3, the Fubini Theorem, and by definition of the rearrangement we have

$$\begin{aligned}\int_{\mathbb{R}} f(x)g(x)\,\mathrm{d}x &= \int_0^\infty\int_0^\infty\int_{\mathbb{R}} \chi_{A_f}(x,r)\chi_{A_g}(x,s)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}s = \int_0^\infty\int_0^\infty \operatorname{meas}\left(A_f(r)\cap A_g(s)\right)\mathrm{d}r\,\mathrm{d}s \\ &\leq \int_0^\infty\int_0^\infty \min\{\operatorname{meas} A_f(r), \operatorname{meas} A_g(s)\}\,\mathrm{d}r\,\mathrm{d}s \\ &= \int_0^\infty\int_0^\infty \operatorname{meas}\left(A_f^*(r)\cap A_g^*(s)\right)\mathrm{d}r\,\mathrm{d}s \\ &= \int_0^\infty\int_0^\infty\int_{\mathbb{R}} \chi_{A_f^*}(x,r)\chi_{A_g^*}(x,s)\,\mathrm{d}x\,\mathrm{d}r\,\mathrm{d}s = \int_{\mathbb{R}} f^*(x)g^*(x)\,\mathrm{d}x,\end{aligned}$$

which we wanted to prove. ■

Rearrangements of Lipschitz continuous functions preserve the Lipschitz continuity in the following sense.

Lemma 3.5 *Let $g \in \mathcal{L}$ be such that for all $x, y \in \mathbb{R}$ we have $|g(x) - g(y)| \leq |x - y|$. Then for $g^* = R[g]$ we have $|g^*(x) - g^*(y)| \leq |x - y|$ for every $x, y \in \mathbb{R}$.*

Proof. Let $r > s > 0$ and $x \in A(r)$ be arbitrarily chosen. For $y \in (x - r + s, x + r - s)$ we have $g(x) - g(y) \le r - s$, hence $g(y) \ge g(x) - r + s > s$, so that $y \in A(s)$. We thus have $\delta(s) = \frac{1}{2}\text{meas}\, A(s) \ge \frac{1}{2}\text{meas}\, A(r) + r - s$, and we conclude that

$$\delta(s) - \delta(r) \ge r - s \quad \text{for } \ r > s. \tag{3.8}$$

Assume now that there exist $y > x \ge 0$ and $m > 0$ such that

$$g^*(x) - g^*(y) \ge y - x + m. \tag{3.9}$$

Put $a = g^*(x)$, $b = g^*(y)$. By Lemma 3.2 we have $a = \sup\{s > 0 : \delta(s) > x\}$, $b = \sup\{s > 0 : \delta(s) > y$. Hence, $\delta(a - \varepsilon) > x$ and $\delta(b + \varepsilon) \le y$ for every $\varepsilon > 0$. From (3.8)–(3.9) we obtain

$$a - b \ge m + \delta(b + \varepsilon) - \delta(a - \varepsilon) \ge m + a - b - 2\varepsilon,$$

which is a contradiction for $\varepsilon < m/2$. Lemma 3.5 is proved. ∎

Rearrangements are invariant with respect to *decompositions*. We prove here the following formula.

Lemma 3.6 *Consider a function* $g \in \mathcal{L}$ *and fix a number* $c > 0$. *Let us denote for* $x \in \mathbb{R}$

$$\begin{aligned} g_-(x) &= \min\{c, g(x)\}, \\ g_+(x) &= \max\{c, g(x)\} - c. \end{aligned}$$

Then $g = g_+ + g_-$ *and* $R[g] = R[g_+] + R[g_-]$.

Proof. The identity $g(x) = g_+(x) + g_-(x)$ for all $x \in \mathbb{R}$ is obvious. Put $g_+^* = R[g_+]$, $g_-^* = R[g_-]$, $A_+(r) = \{x \in \mathbb{R} : g_+(x) > r\}$, $A_-(r) = \{x \in \mathbb{R} : g_-(x) > r\}$, $\delta_\pm(r) = \frac{1}{2}\text{meas}\, A_\pm(r)$ for $r > 0$. By Lemma 3.2 we have

$$\begin{aligned} g_-^*(x) &= \sup\{s > 0 : \delta_-(s) > |x|\}, \\ g_+^*(x) &= \sup\{s > 0 : \delta_+(s) > |x|\} \end{aligned}$$

with the convention $\sup \emptyset = 0$. Note that

$$\begin{aligned} \delta_-(r) &= \begin{cases} \delta(r) & \text{for } \ r < c, \\ 0 & \text{for } \ r \ge c, \end{cases} \\ \delta_+(r) &= \delta(r + c) \ \ \text{for } \ r \ge 0. \end{aligned}$$

For $|x| \ge \delta(c)$ we thus have $g_-^*(x) = g^*(x)$, $g_+^*(x) = 0$. For $|x| < \delta(c)$ we have $g_-^*(x) = c$, $g_+^*(x) = g^*(x) - c$, which we wanted to prove. ∎

To illustrate the interaction between rearrangements and integral means, we consider a function $g \in \mathcal{L}$ and some $\gamma > 0$, denote $g^* = R[g]$ as in (3.5), and define for $x \in \mathbb{R}$

$$G(x) = \int_{x-\gamma}^{x+\gamma} g(y)\,\mathrm{d}y, \qquad \hat{G}(x) = \int_{x-\gamma}^{x+\gamma} g^*(y)\,\mathrm{d}y, \qquad G^*(x) = R[G](x).$$

We have the following result.

Proposition 3.7 *For every $\lambda > 0$ we have*

$$\int_0^\lambda G^*(x)\,\mathrm{d}x \le \int_0^\lambda \hat{G}(x)\,\mathrm{d}x. \tag{3.10}$$

Proof. It is easy to see that both G^* and $\hat{G}$ belong to the class $\mathcal{R}$ introduced in (3.4). Moreover, they are Lipschitz continuous by virtue of Lemma 3.5 and identity (3.6). We split he argument of the proof into four steps.

Step 1. Check that $G^*(0) \le \hat{G}(0)$.

To prove the assertion, we notice that $G^*(0) = \max_{x\in\mathbb{R}} G(x)$, and

$$\begin{aligned} G(x) &= \int_{x-\gamma}^{x+\gamma} g(y)\,\mathrm{d}y = \int_{x-\gamma}^{x+\gamma}\int_0^\infty \chi_A(y,r)\,\mathrm{d}r\,\mathrm{d}y = \int_0^\infty\int_{x-\gamma}^{x+\gamma}\chi_A(y,r)\,\mathrm{d}y\,\mathrm{d}r \\ &= \int_0^\infty \mathrm{meas}\,(A(r)\cap(x-\gamma,x+\gamma))\,\mathrm{d}r \le \int_0^\infty \min\{\mathrm{meas}\,A(r), 2\gamma\}\,\mathrm{d}r \\ &= \int_0^\infty \mathrm{meas}\,((-\delta(r),\delta(r)\cap(-\gamma,\gamma))\,\mathrm{d}r = \int_0^\infty\int_{-\gamma}^{\gamma}\chi_{A^*}(y,r)\,\mathrm{d}y\,\mathrm{d}r = \hat{G}(0). \end{aligned}$$

Step 2. Check that $\int_0^\infty G^*(x)\,\mathrm{d}x = \int_0^\infty \hat{G}(x)\,\mathrm{d}x$.

This easily follows from Lemma 3.3 (iii) for $p = 1$ and from the Fubini Theorem:

$$\int_0^\infty G^*(x)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} G^*(x)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} G(x)\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}}\int_{-\gamma}^{\gamma} g(x+z)\,\mathrm{d}z\,\mathrm{d}x = \int_0^\infty \hat{G}(x)\,\mathrm{d}x.$$

Step 3. Inequality (3.10) holds for characteristic functions $g = \chi_S$ of any bounded measurable set $S \subset \mathbb{R}$.

In this case we have $g^*(x) = \chi_{(-\sigma,\sigma)}(x)$, where $\sigma = \frac{1}{2}\mathrm{meas}\,S$. Then

$$\hat{G}(x) = \int_{x-\gamma}^{x+\gamma}\chi_{(-\sigma,\sigma)}(y)\,\mathrm{d}y = \begin{cases} 0 & \text{if } x \le -Q, \\ x+Q & \text{if } -Q < x \le -q, \\ Q-q & \text{if } -q < x \le q, \\ Q-x & \text{if } q < x \le Q, \\ 0 & \text{if } x > Q, \end{cases}$$

where $Q = \sigma + \gamma$, $q = |\sigma - \gamma|$. The function G is Lipschitz continuous with Lipschitz constant 1, and by Lemma 3.5 we have

$$|G^*(x) - G^*(y)| \le |x-y| \quad \forall x, y \in \mathbb{R}. \tag{3.11}$$

To check that (3.10) holds, we distinguish three cases:

(i) $\lambda \le q$: Then, by Step 1,

$$\int_0^\lambda \hat{G}(x)\,\mathrm{d}x = \lambda\hat{G}(0) \ge \lambda G^*(0) \ge \int_0^\lambda G^*(x)\,\mathrm{d}x.$$

(ii) $\lambda \geq Q$: Then, by Step 2,

$$\int_0^\lambda \hat{G}(x)\,\mathrm{d}x = \int_0^\infty \hat{G}(x)\,\mathrm{d}x = \int_0^\infty G^*(x)\,\mathrm{d}x \geq \int_0^\lambda G^*(x)\,\mathrm{d}x.$$

(iii) $q < \lambda < Q$: Put $H(\lambda) = \int_0^\lambda (\hat{G}(x) - G^*(x))\,\mathrm{d}x$. By (i), (ii), and (3.11) we have

$$H''(\lambda) = \hat{G}'(\lambda) - (G^*)'(\lambda) \leq 0 \quad \text{a. e.},$$

hence H is concave, $H(\lambda) \geq \min\{H(Q), H(q)\} \geq 0$, which completes Step 3.

Step 4. $g \in \mathcal{L}$ is arbitrary.

By Lemma 3.6, (3.6), and Step 3, inequality (3.10) holds for every function g with finite range. For a general function $g \in \mathcal{L}$, it suffices to find pointwise convergent approximations of g with finite range and pass to the limit using the Lebesgue Dominated Convergence Theorem. ■

The next step in the proof of the Hardy-Littlewood-Sobolev inequality is the following lemma.

Lemma 3.8 *Let $f, g \in \mathcal{L}$ and $h \in \mathcal{R}$ be given, and let $f^* = R[f]$, $g^* = R[g]$. Then*

$$\iint_{\mathbb{R}^2} f(x)\, g(y)\, h(x-y)\,\mathrm{d}y\,\mathrm{d}x \leq \iint_{\mathbb{R}^2} f^*(x)\, g^*(y)\, h(x-y)\,\mathrm{d}y\,\mathrm{d}x\,. \tag{3.12}$$

Proof of Lemma 3.8. The function h can be uniformly approximated by piecewise constant functions from $\mathcal{R}$, and every piecewise constant function from $\mathcal{R}$ can be represented as a sum of characteristic functions of symmetric intervals. Hence, it suffices to prove (3.12) for $h(z) = \chi_{(-\gamma,\gamma)}(z)$ with an arbitrary $\gamma > 0$.

Put

$$G(x) = \int_{\mathbb{R}} g(y)\chi_{(-\gamma,\gamma)}(x-y)\,\mathrm{d}y = \int_{x-\gamma}^{x+\gamma} g(y)\,\mathrm{d}y.$$

Then by Lemma 3.4

$$\iint_{\mathbb{R}^2} f(x)\, g(y)\chi_{(-\gamma,\gamma)}(x-y)\,\mathrm{d}y\,\mathrm{d}x = \int_{\mathbb{R}} f(x)G(x)\,\mathrm{d}x \leq \int_{\mathbb{R}} f^*(x)G^*(x)\,\mathrm{d}x, \tag{3.13}$$

where $G^* = R[G]$. The function f^* belongs to $\mathcal{R}$ and can also be uniformly approximated by piecewise constant functions from $\mathcal{R}$, and every piecewise constant function from $\mathcal{R}$ can be represented by a sum of characteristic functions of symmetric intervals. For $f^*(x) = \chi_{(-\lambda,\lambda)}(x)$ we have by Proposition 3.7

$$\int_{\mathbb{R}} f^*(x)G^*(x)\,\mathrm{d}x = \int_{-\lambda}^{\lambda} G^*(x)\,\mathrm{d}x \leq \int_{-\lambda}^{\lambda} \hat{G}(x)\,\mathrm{d}x = \iint_{\mathbb{R}^2} \chi_{(-\lambda,\lambda)}(x)\, g^*(y)\, \chi_{(-\gamma,\gamma)}(x-y)\,\mathrm{d}y\,\mathrm{d}x, \tag{3.14}$$

and (3.12) follows by approximating h and f^* by piecewise constant functions. ■

We are now ready to pass to the proof of Proposition 3.1.

Proof of Proposition 3.1. We restrict ourselves to the case that $f, g \in \mathcal{L}$ and replace h by a truncation of the form $h(z) = \min\{|z|^{-1/r}, K\}$ with an arbitrary K which we let tend to ∞.

The general case then follows from the density argument. We estimate the right hand side of (3.12) by introducing an auxiliary function F by the formula

$$F(y) = \int_{\mathbb{R}} f^*(x)h(x-y)\,\mathrm{d}x \qquad \text{for} \quad y \in \mathbb{R}.$$

The function f^* belongs to $\mathcal{R}$, hence, by Lemma 3.3 (iii),

$$|f|_p^p = |f^*|_p^p \;\geq\; \int_{-|x|}^{|x|} (f^*(\xi))^p\,\mathrm{d}\xi \;\geq\; 2|x|(f^*(x))^p \qquad \forall x \in \mathbb{R}\,. \tag{3.15}$$

Choosing $\alpha = p/q'$ and using again Lemma 3.3, we obtain for every $y \in \mathbb{R}$ that

$$\begin{aligned}
F(y) \;&\leq\; \int_{\mathbb{R}} (f^*(x))^\alpha |2x|^{-(1-\alpha)/p} |f|_p^{1-\alpha}\,|x-y|^{-1/r}\,\mathrm{d}x \\
&=\; 2^{-1+1/r}\,|f|_p^{1-p/q'} \int_{\mathbb{R}} (f^*(x))^{p/q'} |x|^{-1+1/r}\,|x-y|^{-1/r}\,\mathrm{d}x \\
&=\; 2^{-1+1/r}\,|f|_p^{1-p/q'} \int_{\mathbb{R}} (f^*(yt))^{p/q'} |t|^{-1+1/r}\,|t-1|^{-1/r}\,\mathrm{d}t\,.
\end{aligned}$$

We now use the Minkowski inequality (2.7) to estimate the $L^{q'}$ norm of F. We have

$$\begin{aligned}
|F|_{q'} \quad &\leq \quad 2^{-1+1/r}\,|f|_p^{1-p/q'} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} (f^*(yt))^{p/q'} |t|^{-1+1/r}\,|t-1|^{-1/r}\,\mathrm{d}t \right)^{q'} \mathrm{d}y \right)^{1/q'} \\
&\overset{\text{Minkowski}}{\leq} \quad 2^{-1+1/r}\,|f|_p^{1-p/q'} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (f^*(yt))^p |t|^{-q'+q'/r}\,|t-1|^{-q'/r}\,\mathrm{d}y \right)^{1/q'} \mathrm{d}t \\
&= \quad 2^{-1+1/r}\,|f|_p^{1-p/q'} \int_{\mathbb{R}} |t|^{-1+1/r}\,|t-1|^{-1/r} \left(\int_{\mathbb{R}} (f^*(yt))^p\,\mathrm{d}y \right)^{1/q'} \mathrm{d}t \\
&= \quad 2^{-1+1/r}\,|f|_p \int_{\mathbb{R}} |t|^{-1/p}\,|t-1|^{-1/r}\,\mathrm{d}t\,.
\end{aligned}$$

By Hölder's inequality, Lemma 3.8, and inequality (3.12), the left-hand side of (3.1) is estimated from above by $|g|_q|F|_{q'}$. Hence, (3.1) holds with

$$H_{pr} \;=\; 2^{-1+1/r} \int_{\mathbb{R}} |t|^{-1/p}\,|t-1|^{-1/r}\,\mathrm{d}t\,. \tag{3.16}$$

■

4 Approximation of L^p-functions

Let us start with the following classical result, see also [5, Corollary 4.8.5].

Theorem 4.1 ***(Lusin)*** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, and let $u : \Omega \to \mathbb{R}$ be a bounded measurable function. Then for every $\delta > 0$ there exists a measurable set A_δ, $\operatorname{meas} A_\delta < \delta$, and a continuous function $g : \Omega \to \mathbb{R}$ such that $\sup_{x\in\Omega} |g(x)| \leq \sup_{x\in\Omega} |u(x)|$ and $g(x) = u(x)$ on $\Omega \setminus A_\delta$.*

Proof. We can assume that $0 \le u(x) < 1$ for all $x \in \Omega$, and define

$$E_i^{(n)} = \left\{ x \in \Omega : \frac{i-1}{2^n} \le u(x) < \frac{i}{2^n} \right\} \quad \text{for} \quad i = 1, \dots, 2^n, \ n \in \mathbb{N} \cap \{0\}.$$

Then $\Omega = \bigcup_{i=1}^{2^n} E_i^{(n)}$ for each n, and $E_i^{(n)} \cap E_j^{(n)} = \emptyset$ for $i \ne j$. We now put

$$u^{(n)}(x) = \sum_{i=1}^{2^n} \frac{i-1}{2^n} \chi_{E_i^{(n)}}(x) \quad \text{for} \quad x \in \Omega,$$

where χ_A denotes the characteristic function of a set $A \subset \Omega$, that is, $\chi_A(x) = 1$ if $x \in A$, $\chi_A(x) = 0$ if $x \notin A$. We see that for all $x \in \Omega$ and $n \in \mathbb{N} \cup \{0\}$ we have

$$0 \le u(x) - u^{(n)}(x) \le 2^{-n}, \tag{4.1}$$

hence $u^{(n)}$ converge uniformly to u. We now define an auxiliary sequence $\{w^{(n)}\}$ by

$$w^{(0)}(x) = u^{(0)}(x) = 0, \quad w^{(n)}(x) = u^{(n)}(x) - u^{(n-1)}(x) \quad \text{for} \quad n \in \mathbb{N}, \ x \in \Omega.$$

We have

$$E_i^{(n-1)} = E_{2i-1}^{(n)} \cup E_{2i}^{(n)},$$

which implies the representation

$$u^{(n-1)}(x) = \sum_{i=1}^{2^{n-1}} \frac{2i-2}{2^n} \left(\chi_{E_{2i-1}^{(n)}}(x) + \chi_{E_{2i}^{(n)}}(x) \right),$$

$$u^{(n)}(x) = \sum_{i=1}^{2^{n-1}} \left(\frac{2i-2}{2^n} \chi_{E_{2i-1}^{(n)}}(x) + \frac{2i-1}{2^n} \chi_{E_{2i}^{(n)}}(x) \right).$$

This yields in particular that

$$w^{(n)}(x) = 2^{-n} \chi_{\hat{E}^{(n)}}(x), \quad \hat{E}^{(n)} = \bigcup_{i=1}^{2^{n-1}} E_{2i}^{(n)}.$$

We conclude from (4.1) that

$$u(x) = \sum_{n=1}^{\infty} w_n(x)$$

and the series converges uniformly.

Let now $\delta > 0$ be given. We find for each $n \in \mathbb{N}$ a compact set K_n and an open set G_n such that $K_n \subset \hat{E}^{(n)} \subset G_n \subset \Omega$ and $\text{meas}\,(G_n \setminus K_n) < \delta 2^{-n}$. Put

$$A_\delta = \bigcup_{n=1}^{\infty} (G_n \setminus K_n).$$

Then $\text{meas}\, A_\delta < \delta$. By Lemma 1.1 there exist continuous functions g_n on $\mathbb{R}^N$ such that $g_n(x) = 1$ for $x \in K_n$, $g_n(x) = 0$ for $x \in \mathbb{R}^N \setminus G_n$. Put

$$g(x) = \sum_{n=1}^{\infty} 2^{-n} g_n(x).$$

This is a uniformly convergent series of continuous functions, hence g is continuous, and

$$g_n(x) = h_n(x) \quad \text{for} \quad x \in (\Omega \setminus G_n) \cup K_n.$$

We thus have $g(x) = u(x)$ for $x \in \Omega \setminus A_\delta$, and the proof is complete. ∎

As a first consequence of the Lusin Theorem we prove that continuous functions form a dense subset of $L^p(\Omega)$ for $1 \le p < \infty$.

Corollary 4.2 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, and let $u \in L^p(\Omega)$ with $1 \le p < \infty$ be arbitrary. Then*

$$\forall \varepsilon > 0 \ \exists g \in C(\bar{\Omega}) : \left(\int_\Omega |u(x) - g(x)|^p \, \mathrm{d}x \right)^{1/p} < \varepsilon.$$

Proof. We can assume that $u(x) \ge 0$ a. e., otherwise we prove the statement separately for the positive and negative parts u^+ and u^-. For $n \in \mathbb{N}$ and $x \in \Omega$ put

$$u_n(x) = \min\{u(x), n\}.$$

By virtue of Theorem 4.1 there exists for each $n \in \mathbb{N}$ a set A_n, $\operatorname{meas} A_n < n^{-2p}$ and a function $f_n \in C(\Omega)$ such that $f_n(x) = u_n(x)$ on $\Omega \setminus A_n$. For $x \in \Omega$ and $n \in \mathbb{N}$ put

$$f_n^*(x) = \min\{f_n(x), n\}^+.$$

Then $f_n^*(x) = u_n(x)$ on $\Omega \setminus A_n$ and we have

$$\int_\Omega |u_n(x) - f_n^*(x)|^p \, \mathrm{d}x = \int_{A_n} |u_n(x) - f_n^*(x)|^p \, \mathrm{d}x \le \left(\frac{2}{n} \right)^p. \tag{4.2}$$

Put $M_n = \{x \in \Omega : u(x) > n\}$. Then

$$U := \left(\int_\Omega |u(x)|^p \, \mathrm{d}x \right)^{1/p} \ge (\operatorname{meas} M_n)^{1/p},$$

hence $\operatorname{meas} M_n \le (U/n)^p$.

Let now $\varepsilon > 0$ be given. The absolute continuity of the Lebesgue integral (see, e. g., [7, Chapter III, §2, Theorem 3]) yields

$$\exists \delta > 0 \ \ \forall B \subset \Omega, \ \operatorname{meas} B < \delta : \left(\int_\Omega |u(x)|^p \, \mathrm{d}x \right)^{1/p} < \frac{\varepsilon}{2}.$$

We choose n sufficiently large such that

$$\frac{2}{n} < \frac{\varepsilon}{2}, \quad \left(\frac{U}{n} \right)^p < \delta.$$

Then

$$\begin{aligned} \left(\int_\Omega |u(x) - f_n^*(x)|^p \, \mathrm{d}x \right)^{1/p} &\le \left(\int_\Omega |u_n(x) - f_n^*(x)|^p \, \mathrm{d}x \right)^{1/p} + \left(\int_\Omega |u(x) - u_n(x)|^p \, \mathrm{d}x \right)^{1/p} \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and it suffices to put $g = f_n^*$. ■

The next consequence of Theorem 4.1 describes an important property of L^p-functions which will be often referred to in the sequel.

Theorem 4.3 ***(Mean Continuity)*** *Let Ω, Ω_∞ be bounded open sets in $\mathbb{R}^N$ such that $\Omega \subset \bar{\Omega} \subset \Omega_\infty$. Let $u \in L^p(\Omega_\infty)$ be given, $1 \le p < \infty$. Then*

$$\forall \varepsilon > 0 \ \exists \delta \in (0,1) \ \forall \xi \in \mathbb{R}^N, |\xi| < \delta : \left(\int_\Omega |u(x) - u(x+\xi)|^p \, \mathrm{d}x \right)^{1/p} < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. Using Corollary 4.2 we find $g \in C(\bar{\Omega}_\infty)$ such that

$$\left(\int_{\Omega_\infty} |u(x) - g(x)|^p \, \mathrm{d}x \right)^{1/p} < \frac{\varepsilon}{4}.$$

Let $\delta \in (0,1)$ be chosen such that $\Omega + B_\delta(0) \subset \Omega_\infty$ and

$$|g(x) - g(y)| < \frac{\varepsilon}{2 \,\mathrm{meas}\,(\bar{\Omega})^{1/p}} \quad \text{for} \quad x, y \in (\bar{\Omega}_\infty), \ |x - y| < \delta.$$

Then for $|\xi| < \delta$ we have

$$\begin{aligned} \left(\int_\Omega |u(x) - u(x+\xi)|^p \, \mathrm{d}x \right)^{1/p} &\le \left(\int_\Omega |u(x) - g(x)|^p \, \mathrm{d}x \right)^{1/p} + \left(\int_\Omega |u(x{+}\xi) - g(x{+}\xi)|^p \, \mathrm{d}x \right)^{1/p} \\ &\quad + \left(\int_\Omega |g(x) - g(x+\xi)|^p \, \mathrm{d}x \right)^{1/p} \\ &\le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and the proof is complete. ■

The Mean Continuity Theorem 4.3 can be easily extended to the whole $\mathbb{R}^N$ as follows.

Corollary 4.4 *Let $u \in L^p(\mathbb{R}^N)$ be given, $1 \le p < \infty$. Then*

$$\forall \varepsilon > 0 \ \exists \delta \in (0,1) \ \forall \xi \in \mathbb{R}^N, |\xi| < \delta : \left(\int_{\mathbb{R}^N} |u(x) - u(x+\xi)|^p \, \mathrm{d}x \right)^{1/p} < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. We find $r > 0$ sufficiently large such that $|u|_{p,\mathbb{R}^N \setminus B_{r-1}(0)} < \varepsilon/4$. Using Theorem 4.3 we find $\delta \in (0,1)$ such that for $|\xi| < \delta$ we have $|u - u(\cdot + \xi)|_{p,B_r(0)} < \varepsilon/2$. We have

$$\left(\int_{\mathbb{R}^N} |u(x) - u(x+\xi)|^p \, \mathrm{d}x \right)^{1/p} \le |u - u(\cdot + \xi)|_{p,B_r(0)} + 2|u|_{p,\mathbb{R}^N \setminus B_{r-1}(0)} < \varepsilon$$

which we wanted to prove. ■

We can now strengthen the result of Corollary 4.2 in the following sense.

Corollary 4.5 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, and let $u \in L^p(\Omega)$ with $1 \le p < \infty$ be arbitrary. Then*

$$\forall \varepsilon > 0 \ \ \exists g \in C^\infty(\mathbb{R}^N) : \left(\int_\Omega |u(x) - g(x)|^p \, \mathrm{d}x\right)^{1/p} < \varepsilon.$$

Proof. We choose a mollifier φ as in (1.1). For $u \in L^p(\Omega)$ we consider its extension

$$\tilde{u}(x) = \begin{cases} u(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

to the whole $\mathbb{R}^N$, and for a given $x \in \mathbb{R}^N$ and a parameter $\sigma \in (0,1]$ we set

$$u^\sigma(x) \ = \ \sigma^{-N} \int_{\mathbb{R}^N} \varphi\left(\frac{x-y}{\sigma}\right) \tilde{u}(y) \, \mathrm{d}y \, . \tag{4.3}$$

For all $\sigma \in (0,1]$, the function u^σ is of class C^∞ and we have

$$\begin{aligned} \left(\int_\Omega |u^\sigma - u|^p(x) \, \mathrm{d}x\right)^{1/p} \quad & \le \quad \left(\int_{\mathbb{R}^N} |u^\sigma - \tilde{u}|^p(x) \, \mathrm{d}x\right)^{1/p} \\ & = \quad \left(\int_{\mathbb{R}^N} \left|\int_{B_1(0)} \varphi(z)(\tilde{u}(x - \sigma\xi) - \tilde{u}(x)) \, \mathrm{d}\xi\right|^p \mathrm{d}x\right)^{1/p} \\ & \overset{\text{Minkowski}}{\le} \quad \int_{B_1(0)} \varphi(\xi) \left(\int_{\mathbb{R}^N} |\tilde{u}(x - \sigma\xi) - \tilde{u}(x)|^p \, \mathrm{d}x\right)^{1/p} \mathrm{d}\xi \, , \end{aligned}$$

hence $u^\sigma \to u$ strongly in $L^p(\Omega)$ as $\sigma \to 0+$ as a consequence of the Mean Continuity Theorem 4.3, and it suffices to put $g = u^\sigma$ for σ sufficiently small. ■

The regularization formula (4.3) will often be used in the sequel. Note that Proposition 2.6 yields for $u \in L^p(\mathbb{R}^N)$ the following relation between the L^q norm of u^σ and L^p norm of u:

$$|u^\sigma|_q \ \le \ \sigma^{-N(1/p - 1/q)} |\varphi|_r \, |u|_p \quad \forall q \ge p \, , \tag{4.4}$$

where r is as in (2.12).

5 Spaces $W^{1,p}(\Omega)$

We say that $v \in L^p(\Omega)$ is a *generalized partial derivative* of $u \in L^p(\Omega)$ with respect to x_i, $i \in \{1, \dots, N\}$, if for every smooth function $\varphi : \Omega \to \mathbb{R}$ with *compact support in* Ω, that is,

$$\exists K = \bar{K} \subset \Omega \ \ \forall x \in \Omega \setminus K : \ \varphi(x) = 0 \, , \tag{5.1}$$

we have

$$\int_\Omega u(x) \frac{\partial \varphi}{\partial x_i}(x) \, \mathrm{d}x \ = \ - \int_\Omega v(x) \, \varphi(x) \, \mathrm{d}x \, . \tag{5.2}$$

By [11, Chap. 2, Sect. 2.2], condition (5.2) is fulfilled if and only if u is absolutely continuous along almost all lines parallel to the x_i-axis and v coincides with $\partial u / \partial x_i$ almost everywhere.

The Sobolev space $W^{1,p}(\Omega)$ is defined as the subspace of $L^p(\Omega)$ of all functions u, which together with all generalized partial derivatives $\partial u/\partial x_i$ belong to $L^p(\Omega)$. With the norm

$$\|u\|_{1;p,\Omega} = |u|_{p,\Omega} + \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|_{p,\Omega}, \tag{5.3}$$

$W^{1,p}(\Omega)$ is also a Banach space.

Extensions and continuously differentiable approximations of functions from Sobolev spaces outside their domain of definition are more delicate. We cannot simply extend the function by zero as we did, for instance, in Theorem 4.3 for L^p-functions, since we have to preserve the absolute continuity along the lines parallel to the coordinate axes across the boundary. We therefore have to assume some regularity of the boundary. This can be done in the following way (see Fig. 3).

Definition 5.1 *Let $\Omega \subset \mathbb{R}^N$ be an open connected set. We say that Ω has Lipschitzian boundary if the following condition holds*

(L) *There exist $\delta > 0$ and $m \in \mathbb{N}$, and for each $k = 1, \ldots, m$ there exists an open convex set $\Delta_k \subset \mathbb{R}^{N-1}$, a Lipschitz continuous function $a_k : \Delta_k \to \mathbb{R}$, and a rotation A_k in $\mathbb{R}^N$ (represented by an $N \times N$ matrix, still denoted by A_k, such that $A_k^{-1} = A_k^{\mathsf{T}}$ and $\det A_k = 1$), such that*

(i) $\partial\Omega \subset \bigcup_{k=1}^m A_k(G_k)$,
(ii) $G_k = \{y \in \mathbb{R}^N \,;\, y = (y', y_N), y' = (y_1, \ldots, y_{N-1}) \in \Delta_k, y_N \in (a_k(y') - \delta, a_k(y') + \delta)\}$,
(iii) $G_k^- = \{y \in G_k \,;\, y_N \in (a_k(y') - \delta, a_k(y'))\}$,
(iv) $G_k^0 = \{y \in G_k \,;\, y_N = a_k(y')\}$,
(v) $\Omega \cap A_k(G_k) = A_k(G_k^-)$,
(vi) $\partial\Omega \cap A_k(G_k) = A_k(G_k^0)$.

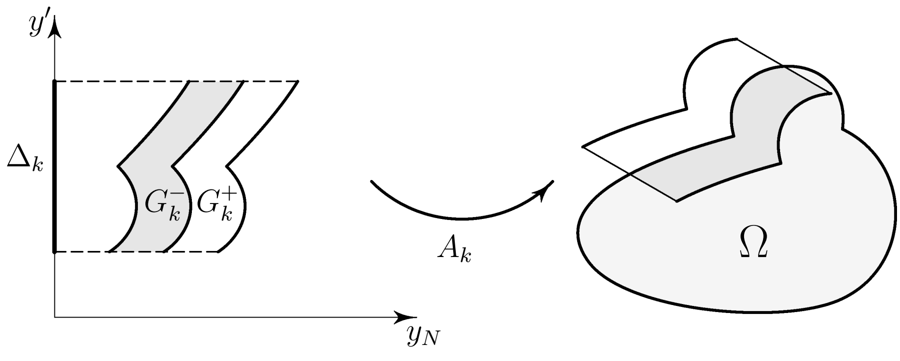

Figure 3: A domain with Lipschitzian boundary.

The prolongation result reads as follows.

Theorem 5.2 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, and let $R > 0$ be such that $\bar{\Omega} \subset B_R(0)$. Let $W_R^{1,p}$ be the set of all functions from $W^{1,p}(\mathbb{R}^N)$ which vanish outside $B_R(0)$, and let $\|\cdot\|_{1;p}$ denote the norm in $W^{1,p}(\mathbb{R}^N)$. Then there exists a linear prolongation operator $E_p : W^{1,p}(\Omega) \to W_R^{1,p}$ such that for every $u \in W^{1,p}(\Omega)$ we have*

(i) $E_p u(x) = u(x)$ *for a. e.* $x \in \Omega$ *;*

(ii) *There exists a constant* $c_p > 0$ *such that for every* $u \in W^{1,p}(\Omega)$ *we have*

$$|E_p u|_p \le c_p |u|_{p,\Omega}, \quad \|E_p u\|_{1;p} \le c_p \|u\|_{1;p,\Omega} \,.$$

The proof of Theorem 5.2 requires several intermediate steps and we start with an easy lemma.

Lemma 5.3 *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded set, and let* $\Omega_0 \subset \bar{\Omega}_0 \subset \Omega$ *and* $u \in W^{1,p}(\Omega)$ *be given. For* $\sigma \in (0,1]$ *we define*

$$u^\sigma(x) := \sigma^{-N} \int_\Omega \varphi\left(\frac{x-y}{\sigma}\right) u(y)\,\mathrm{d}y\,, \tag{5.4}$$

where φ *is as in* (1.1)*. Then we have*

$$\lim_{\sigma\to 0} \|u^\sigma - u\|_{1;p,\Omega_0} = 0\,.$$

Two things are worth mentioning. First, formula (5.4) is not the same as (4.3). Here, we integrate only over Ω. Second, the convergence takes place only on the subdomain $\Omega_0 \subset \Omega$.

Proof of Lemma 5.3. Put $\rho = \operatorname{dist}(\bar{\Omega}_0, \mathbb{R}^N \setminus \Omega)$. For $\sigma < \rho$ and $x \in \Omega_0$ has the function $y \mapsto \varphi((x-y)/\sigma)$ compact support in Ω. Hence, for $i = 1, \ldots, N$ the partial derivatives $\partial_i := \partial/\partial x_i$ of u^σ satisfy the identities

$$\begin{aligned} \partial_i u^\sigma(x) &= -\sigma^{-N-1} \int_\Omega \partial_i\varphi\left(\frac{x-y}{\sigma}\right) u(y)\,\mathrm{d}y = \sigma^{-N} \int_\Omega \varphi\left(\frac{x-y}{\sigma}\right) \partial_i u(y)\,\mathrm{d}y \\ &= \int_{B_1(0)} \varphi(\xi)\,\partial_i u(x-\sigma\xi)\,\mathrm{d}\xi. \end{aligned}$$

This yields

$$\partial_i u^\sigma(x) - \partial_i u(x) = \int_{B_1(0)} \varphi(\xi)\left(\partial_i u(x-\sigma\xi) - \partial_i u(x)\right)\mathrm{d}\xi. \tag{5.5}$$

From the Minkowski inequality (2.7) it follows that

$$|\partial_i u^\sigma - \partial_i u|_{p,\Omega_0} \le \int_{B_1(0)} \varphi(\xi)\,|\partial_i u(\cdot - \sigma\xi) - \partial_i u|_{p,\Omega_0}\,\mathrm{d}\xi.$$

In the same way we obtain a similar estimate for $|u^\sigma - u|_{p,\Omega_0}$, and the assertion follows. ■

Proposition 5.4 *Let* Ω *be as in Theorem 5.2 and let* $\Omega_\infty \supset \bar{\Omega}$ *be an open set. Then for every* $\varepsilon > 0$ *there exists a function* $g \in C^\infty(\mathbb{R}^N)$ *with compact support in* Ω_∞ *such that*

$$\|g - u\|_{1;p,\Omega} < \varepsilon.$$

Proof. We proceed in three steps.

Step 1. Let $\partial\Omega \subset \bigcup_{k=1}^m A_k(G_k) \subset \bigcup_{k=1}^m \overline{A_k(G_k)} \subset \Omega_\infty$ be the covering of $\partial\Omega$ from Definition 5.1, and let $\Omega_0 \subset \bar{\Omega}_0 \subset \Omega$ be such that

$$\bar{\Omega} \subset \bigcup_{k=0}^m \Omega_k, \quad \Omega_k = A_k(G_k) \ \text{ for } \ k = 1, \dots, m. \tag{5.6}$$

Let $\psi_0, \dots, \psi_m$ be a partition of unity associated with this covering according to Theorem 1.2, and for $x \in \Omega_k$ put $u_k(x) = \psi_k(x)u(x)$, $k = 0, \dots, m$. It suffices to prove that for every $\varepsilon > 0$ and every $k = 0, \dots, m$ there exists $g_k \in C^\infty(\mathbb{R}^N)$ with support in Ω_k such that

$$\|g_k - u_k\|_{1;p,\Omega_k} < \frac{\varepsilon}{m} \tag{5.7}$$

and put $g = \sum_{k=0}^m \psi_k g_k$.

Step 2. The case $k = 0$ of (5.7) is treated in Lemma 5.3. For $k \geq 1$ we denote by e_k the unit vector $e_k := A_k(0, \dots, 0, 1)^\mathsf{T}$, and for $t \in (0, \delta)$ and $x \in \Omega_k - te_k$ put $u_k^t(x) = u_k(x + te_k)$. We have indeed

$$\partial_i u_k^t(x) = \partial_i u_k(x + te_k)$$

for a. e. $x \in \Omega_k - te_k$ and all $i = 1, \dots, N$. Since the support $S_k := \operatorname{supp}(\psi_k)$ of ψ_k is compact in Ω_k, we have $\operatorname{dist}(S_k, \mathbb{R}^N \setminus \Omega_k) =: \rho_k > 0$, hence, the function u_k^t is defined on Ω_k for all $t < \rho_k$. Using Theorem 4.3 we can choose $t_k \in (0, \min\{\rho_k, \delta\})$ such that

$$\|u_k^t - u_k\|_{1;p,\Omega_k} < \frac{\varepsilon}{2m} \quad \text{ for } \ t \in (0, t_k). \tag{5.8}$$

Step 3. For $\delta_k \in (0, \rho_k - t_k)$ we have $B_{\delta_k}(x) \subset \Omega_k - t_k e_k$ for all $x \in \bar{\Omega} \cap S_k$. We now set for $x \in \Omega_k$, $t \in (0, t_k)$, and $\sigma \in (0, \delta_k)$

$$u_k^{t\sigma}(x) = \sigma^{-N} \int_{\Omega_k} \varphi\left(\frac{x - y}{\sigma}\right) u_k^t(y) \, \mathrm{d}y$$

with φ as in (1.1). Then we have for $i = 1, \dots, N$ similarly as in (5.5) that

$$\partial_i u_k^{t\sigma}(x) - \partial_i u_k^t(x) = \int_{B_1(0)} \varphi(\xi) \left(\partial_i u_k^t(x - \sigma\xi) - \partial_i u_k^t(x)\right) \, \mathrm{d}y$$

and similarly as in the proof of Lemma 5.3 we obtain $\lim_{\sigma \to 0} \|u_k^{t\sigma} - u_k^t\|_{1;p,\Omega_k} = 0$. Combining this result for σ sufficiently small with (5.8) we thus have

$$\|u_k^{t\sigma} - u_k\|_{1;p,\Omega_k} < \frac{\varepsilon}{m} \quad \text{ for } \ k = 1, \dots, m.$$

This is precisely (5.7) with $g_k = u_k^{t\sigma}$, which completes the proof. ∎

We are now ready to prove Theorem 5.2.

Proof. Let us consider the covering of Ω as in (5.6). Similarly as in the proof of Proposition 5.4, we consider the partition of unity $\{\psi_k : k = 0, \dots, m\}$ associated with the covering (5.6), and set $u_k(x) := \psi_k(x)u(x)$ for $k = 0, \dots, m$ and $x \in \Omega_k \cap \Omega$. It is enough to prove that each u_k admits a bounded linear extension from $W^{1,p}(\Omega_k \cap \Omega)$ to $W^{1,p}(\Omega_k)$.

There is nothing to prove for $k=0$. Let us consider now some $k\in\{1,\dots,m\}$ and for $y=(y',y_N)\in G_k^-$ put

$$v_k(y)=u_k(A_k y).$$

Then $v_k\in W^{1,p}(G_k^-)$, and we define

$$w_k(y',t):=v_k(y',a_k(y')+t) \tag{5.9}$$

for $y'\in\Delta_k$ and $t\in(-\delta,0)$. Let us check that $w_k\in W^{1,p}(\Delta_k\times(-\delta,0))$. Indeed, by Proposition 5.4 we find a sequence $\{v_k^{(n)}:n\in\mathbb{N}\}$ of smooth functions with support in G_k and such that $v_k^{(n)}\to v_k$ in $W^{1,p}(G_k^-)$, and set $w_k^{(n)}(y',t):=v_k^{(n)}(y',a_k(y')+t)$. We have

$$\begin{aligned}\partial_i w_k^{(n)}&=\partial_i v_k^{(n)}+\partial_N v_k^{(n)}\,\partial_i a_k \quad \text{for}\ \ i=1,\dots,N-1,\\ \partial_N w_k^{(n)}&=\partial_N v_k^{(n)}\end{aligned}$$

almost everywhere in $\Delta_k\times(-\delta,0)$, and there exists a constant $C>0$ such that for all $n,l\in\mathbb{N}$

$$\begin{aligned}|\partial_i w_k^{(n)}-\partial_i w_k^{(l)}|_{p,\Delta_k\times(-\delta,0)}&\le C\|v_k^{(n)}-v_k^{(l)}\|_{1;p,G_k^-}\quad \text{for}\ \ i=1,\dots,N,\\ |w_k^{(n)}-w_k^{(l)}|_{p,\Delta_k\times(-\delta,0)}&=|v_k^{(n)}-v_k^{(l)}|_{p,G_k^-}.\end{aligned}$$

It follows that $\{w_k^{(n)}:n\in\mathbb{N}\}$ is a Cauchy sequence in $W^{1,p}(\Delta_k\times(-\delta,0))$ and $w_k^{(n)}$ converge to w_k almost everywhere in $\Delta_k\times(-\delta,0)$, hence $w_k=\lim_{n\to\infty}w_k^{(n)}\in W^{1,p}(\Delta_k\times(-\delta,0))$.

We now extend the functions $w_k^{(n)}$ to $\Delta_k\times(-\delta,\delta)$ by the formula

$$\tilde w_k^{(n)}(y',t)=\begin{cases} w_k^{(n)}(y',-|t|) & \text{for}\ \ y'\in\Delta_k, t\in(-\delta,0)\cup(0,\delta),\\ w_k^{(n)}(y',0-) & \text{for}\ \ y'\in\Delta_k, t=0.\end{cases}$$

This extension preserves the absolute continuity along the coordinate axes and

$$|\partial_i\tilde w_k^{(n)}-\partial_i\tilde w_k^{(l)}|_{p,\Delta_k\times(-\delta,\delta)}\le 2^{1/p}|\partial_i w_k^{(n)}-\partial_i w_k^{(l)}|_{p,\Delta_k\times(-\delta,0)}\quad \text{for}\ \ i=1,\dots,N,$$

so that $\tilde w_k^{(n)}$ converge in $W^{1,p}(\Delta_k\times(-\delta,\delta))$ to a limit $\tilde w_k$ which coincides with w_k on $\Delta_k\times(-\delta,0)$. We now put

$$\tilde v_k(y)=\tilde w_k(y',y_N-a_k(y'))$$

for $y=(y',y_N)\in G_k$. We argue as above using Proposition 5.4 to check that $\tilde v_k\in W^{1,p}(G_k)$, $\tilde v_k=v_k$ on G_k^-, and that the function

$$\tilde u_k(x):=\tilde v_k(A_k^{\mathsf T}x)\quad \text{for}\ \ x\in\Omega_k$$

belongs to $W^{1,p}(\Omega_k)$ and $\tilde u_k=u_k$ on $\Omega_k\cap\Omega$. To complete the proof, it is enough to put

$$E_p u(x)=\sum_{k=0}^m \psi_k(x)\,\tilde u_k(x).$$

By construction, we have $E_pu(x)=u(x)$ on Ω and $E_pu(x)=0$ on $\mathbb{R}^N\setminus\bigcup_{k=0}^m\Omega_k\subset\mathbb{R}^N\setminus B_R(0)$ for $R>0$ sufficiently large. Since the mapping $E_p:W^{1,p}(\Omega)\to W_R^{1,p}$ is linear and bounded, it is continuous, which we wanted to prove. ■

6 Traces on the boundary

One important property of functions $u \in W^{1,p}(\Omega)$ for domains Ω with Lipschitzian boundary is that the *trace* of u on $\partial\Omega$ is well defined. In the situation of Definition 5.1, consider a partition of unity $\{\psi_k\}$ as in the proof of Theorem 5.2, and a function $f : \partial\Omega \to [0,\infty)$, and put $f_k(x) := \psi_k(x) f(x)$ for $x \in \partial\Omega \cap \Omega_k$, $\hat{f}_k(y) = f_k(A_k y)$ for $y \in G_k^0$, and $f_k^*(y') = \hat{f}_k(y', a_k(y'))$ for $y' \in \Delta_k$, $k = 1, \dots, m$. Assuming that f_k^* is integrable on Δ_k, we define the surface integral

$$\begin{aligned} \int_{\partial\Omega\cap\Omega_k} f_k(x)\,\mathrm{d}S(x) &:= \int_{\Delta_k} f_k^*(y') \left(1 + \sum_{i=1}^N (\partial_i a_k(y'))^2\right)^{1/2} \mathrm{d}y', \\ \int_{\partial\Omega} f(x)\,\mathrm{d}S(x) &:= \sum_{k=1}^m \int_{\partial\Omega\cap\Omega_k} f_k(x)\,\mathrm{d}S(x). \end{aligned} \tag{6.1}$$

We say that a function $\bar{u}$ belongs to the space $L^p(\partial\Omega)$ for $1 \le p < \infty$ if

$$|\bar{u}|_{p,\partial\Omega} := \left(\int_{\partial\Omega} |\bar{u}(x)|^p\,\mathrm{d}S(x)\right)^{1/p} < \infty,$$

with an obvious modification for $p = \infty$ as in (2.1). For functions in $W^{1,p}(\Omega)$ we have the following result.

Theorem 6.1 ***(The Trace Theorem)*** *Let $\Omega \subset \mathbb{R}^N$ be as in Theorem 5.2 and let $1 \le p < \infty$. Then there exists a linear continuous mapping $T_p : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that for every $u \in C^1(\bar{\Omega})$ and every $x \in \partial\Omega$ we have $T_p u(x) = u(x)$.*

Proof. Let $u \in W^{1,p}(\Omega)$ be given, and let $\tilde{v}_k^{(n)}, \tilde{w}_k^{(n)}$ be as in the proof of Theorem 5.2. By construction, the functions $\tilde{v}_k^{(n)}$ have compact support in G_k, and we have for $y' \in \Delta_k$ and $n, l \in \mathbb{N}$ that

$$\begin{aligned} |\tilde{w}_k^{(n)}(y') - \tilde{w}_k^{(l)}(y')|^p = |\tilde{v}_k^{(n)}(y', a_k(y')) - \tilde{v}_k^{(l)}(y', a_k(y'))|^p &= \int_{a_k(y')-\delta}^{a_k(y')} \partial_N |\tilde{v}_k^{(n)} - \tilde{v}_k^{(l)}|^p(y', y_N)\,\mathrm{d}y_N \\ &\le p \int_{a_k(y')-\delta}^{a_k(y')} |\tilde{v}_k^{(n)} - \tilde{v}_k^{(l)}|^{p-1}(y', y_N) |\partial_N(\tilde{v}_k^{(n)} - \tilde{v}_k^{(l)})|(y', y_N)\,\mathrm{d}y_N, \end{aligned}$$

hence, we have either

$$\int_{\Delta_k} |\tilde{w}_k^{(n)} - \tilde{w}_k^{(l)}|(y') \left(1 + \sum_{i=1}^N (\partial_i a_k(y'))^2\right)^{1/2} \mathrm{d}y' \le C \int_{G_k^-} |\partial_N \tilde{w}_k^{(n)} - \partial_N \tilde{w}_k^{(l)}|\,\mathrm{d}y$$

if $p = 1$ or, by Hölder's inequality if $p > 1$,

$$\begin{aligned} &\int_{\Delta_k} |\tilde{w}_k^{(n)} - \tilde{w}_k^{(l)}|^p(y') \left(1 + \sum_{i=1}^N (\partial_i a_k(y'))^2\right)^{1/2} \mathrm{d}y' \\ &\qquad \le C \left(\int_{G_k^-} |\tilde{w}_k^{(n)} - \tilde{w}_k^{(l)}|^p\,\mathrm{d}y\right)^{1/p'} \left(\int_{G_k^-} |\partial_N \tilde{w}_k^{(n)} - \partial_N \tilde{w}_k^{(l)}|^p\,\mathrm{d}y\right)^{1/p'} \end{aligned}$$

with a constant $C > 0$ independent of n, l. By definition of the surface integral, the functions $\tilde{v}_k^{(n)}$ form a Cauchy sequence on $L^p(G_k^0)$. There exists therefore a limit $\bar{v}_k = \lim_{n\to\infty} \tilde{v}_k^{(n)} \in L^p(G_k^0)$, and we have

$$|\bar{v}_k|_{p,G_k^0} \le C\|v_k\|_{1;p,G_k^-}.$$

It suffices to put $\bar{u}_k(x) = \bar{v}_k(A_k^\mathsf{T} x)$ for $x \in \partial\Omega \cap \Omega_k$, and $\bar{u}(x) = \sum_{k=1}^m \psi_k(x)\bar{u}_k(x)$ for $x \in \partial\Omega$, and the assertion follows. For continuously differentiable functions u the convergence $\tilde{v}_k^{(n)} \to v_k$ is uniform, and we consequently have in this case that $\bar{u} = u$ on $\partial\Omega$. ■

Let us mention the following interesting relation between the values of a function in Ω and its trace.

Proposition 6.2 ***(Gauss-Ostrogradsky)*** *Let Ω be as in Theorem 5.2 and let a function $u \in W^{1,1}(\Omega;\mathbb{R}^N)$ be given, $u = (u^1, \dots, u^N)$. Let $n(x)$ denote the unit outward normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$, and let us denote $\operatorname{div} u = \sum_{i=1}^N \partial_i u^i$. Then we have*

$$\int_\Omega \operatorname{div} u(x)\,\mathrm{d}x = \int_{\partial\Omega} \langle T_1 u(x), n(x)\rangle\ \mathrm{d}S(x), \tag{6.2}$$

where we denote by $\langle\cdot,\cdot\rangle$ the canonical scalar product in $\mathbb{R}^N$.

Proof. Let $A_k = (A_k)_{ij}$, $i, j = 1, \dots, N$, G_k, G_k^0, ψ_k, u_k be as in Definition 5.1 and in the proof of Proposition 5.4. It is enough to prove that for each $k = 1, \dots, m$ we have

$$\int_{A_k(G_k^-)} \operatorname{div} u_k(x)\,\mathrm{d}x = \int_{A_k(G_k^0)} \langle T_1 u_k(x), n(x)\rangle\ \mathrm{d}S(x) \tag{6.3}$$

and that

$$\int_{\Omega_0} \operatorname{div} u_0(x)\,\mathrm{d}x = 0. \tag{6.4}$$

Indeed, the general statement then follows from the formula $u = \sum_{k=0}^m u_k$. The identity (6.4) is easy. If necessary, we cover Ω_0 with finitely many cubes $Q_1, \dots, Q_r$ whose edges are parallel to the coordinate axes and construct a partition of unity $\zeta_1, \dots, \zeta_r$ associated with this covering. For each function $\hat{u}_j = \zeta_j u_0$, $j = 1, \dots, r$ we obtain (6.4) by a straightforward integration, the result for u_0 follows from the formula $u_0 = \sum_{j=1}^r \hat{u}_j$.

To prove (6.3) for $k \ge 1$, we put $v_k(y) := u_k(A_k y)$ for $y \in G_k^-$. Then

$$\frac{\partial u_k^i}{\partial x_i}(x) = \sum_{j=1}^N (A_k)_{ij} \frac{\partial v_k^i}{\partial y_j}(A_k^\mathsf{T} x), \tag{6.5}$$

and

$$\begin{aligned}\int_{A_k(G_k^-)} \operatorname{div} u_k(x)\,\mathrm{d}x &= \sum_{i,j=1}^N (A_k)_{ij} \int_{G_k^-} \frac{\partial v_k^i}{\partial y_j}(y)\,\mathrm{d}y \\ &= \sum_{i,j=1}^N (A_k)_{ij} \int_{\Delta_k} \int_{a_k(y')-\delta}^{a_k(y')} \frac{\partial v_k^i}{\partial y_j}(y', y_N)\,\mathrm{d}y_N\,\mathrm{d}y'.\end{aligned}$$

As in (5.9), we define for $y' \in \Delta_k$ and $t \in (-\delta, 0)$

$$w_k^i(y', t) := v_k^i(y', a_k(y') + t). \tag{6.6}$$

Then

$$\frac{\partial w_k^i}{\partial y_j}(y', t) = \frac{\partial v_k^i}{\partial y_j}(y', a_k(y') + t) + \frac{\partial v_k^i}{\partial y_N}(y', a_k(y') + t)\frac{\partial a_k}{\partial y_j}(y') \quad \text{for} \quad j = 1, \dots, N-1,$$
$$\frac{\partial w_k^i}{\partial y_N}(y', t) = \frac{\partial v_k^i}{\partial y_N}(y', a_k(y') + t).$$

We have for all i, k, t and $j = 1, \dots, N-1$ that

$$\int_{\Delta_k} \frac{\partial w_k^i}{\partial y_j}(y', t)\, \mathrm{d}y' = 0,$$

hence,

$$\int_{A_k(G_k^-)} \operatorname{div} u_k(x)\, \mathrm{d}x = \sum_{i,j=1}^N (A_k)_{ij} \int_{\Delta_k} \int_{a_k(y')-\delta}^{a_k(y')} \frac{\partial v_k^i}{\partial y_N}(y', y_N)\, n_j^*(y')\, \mathrm{d}y_N\, \mathrm{d}y'$$
$$= \sum_{i,j=1}^N (A_k)_{ij} \int_{\Delta_k} v_k^i(y', a_k(y'))\, n_j^*(y')\, \mathrm{d}y',$$

where we denote $n_j^*(y') = -\partial_j a_k(y')$ for $j = 1, \dots N-1$ and $n_N^*(y') = 1$.
We now introduce a unit vector $\hat{n}(y) = \hat{n}_1(y), \dots \hat{n}_N(y)$ for $y \in G_k^-$ by the formula

$$\hat{n}_j(y', y_N) = n_j^*(y') \left(1 + \sum_{i=1}^N (\partial_i a_k(y'))^2\right)^{-1/2}.$$

By definition (6.1) of the surface integral we obtain (6.3) with $n(x) = A_k \hat{n}(A_k^\mathsf{T} x)$. ■

The subspace of $W^{1,p}(\Omega)$ of functions with zero trace

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : T_p u = 0\},$$

that is, the null-space of the operator T_p, plays a particular role in the theory. It admits the following characterization.

Proposition 6.3 *Let Ω be as in Theorem 5.2 and let $1 \le p < \infty$. Then the space $C_C^\infty(\Omega)$ of infinitely differentiable functions with compact support in Ω is dense in $W_0^{1,p}(\Omega)$ with respect to the norm $\| \cdot \|_{1;p,\Omega}$.*

Proof. Let $u \in W_0^{1,p}(\Omega)$ and $\varepsilon > 0$ be given. The problem consists in finding $g \in C_C^\infty(\Omega)$ such that

$$\|u - g\|_{1;p,\Omega} < \varepsilon. \tag{6.7}$$

We keep below the notation from the proof of Theorem 5.2 and define the functions $w_k : \Delta_k \times (-\delta, 0) \to \mathbb{R}$ as in (5.9). Since $T_p u = 0$, we have $w_k(y', 0-) = 0$ for a. e. $y' \in \Delta_k$, and the function

$$\tilde{w}_k(y', t) = \begin{cases} w_k(y', t) & \text{for} \quad (y', t) \in \Delta_k \times (-\delta, 0), \\ 0 & \text{for} \quad (y', t) \in \Delta_k \times (0, \delta) \end{cases}$$

is an extension to $W^{1,p}(\Delta_k \times (-\delta, \delta)$ of w_k. For $\eta < \delta$ and $(y', t) \in \Delta_k \times (-\delta, 0)$ put

$$w_k^\eta(y', t) = \tilde{w}_k(y', t + \eta).$$

Then $w_k^\eta \in W^{1,p}(\Delta_k \times (-\delta, 0)$ and for η sufficiently small, the functions w_k^η have compact support in $\Delta_k \times (-\delta, 0)$. Let now $\kappa > 0$ be arbitrary. By the Mean Continuity Theorem 4.3 we can choose $\eta > 0$ such that

$$\|w_k^\eta - w_k\|_{1;p,\Delta_k \times (-\delta,0)} < \kappa.$$

We now put $v_k^\eta(y', y_N) = w_k^\eta(y', y_N - a_k(y'))$ and $u_k^\eta(x) = v_k^\eta(A^\mathsf{T} x)$. There exists a constant $C_1 > 0$ independent of κ such that

$$\|u_k^\eta - u_k\|_{1;p,A_k(G_k)} < C_1 \kappa.$$

We now put $u^\eta = \sum_{k=0}^m \psi_k u_k^\eta$ and find a constant $C_2 > 0$ such that

$$\|u^\eta - u\|_{1;p,\Omega} < C_2 \kappa.$$

The functions u^η vanish outside a set $\Omega^\eta \subset \bar{\Omega}^\eta \subset \Omega$, and it suffices to choose $\kappa > 0$ sufficiently small and apply Proposition 5.4 to obtain (6.7). ∎

7 Regularity

We start with an example motivated by mechanics, see [4, 10]. The state of a deformable body $\Omega \subset \mathbb{R}^N$ is described by the *displacement vector* $u : \Omega \to \mathbb{R}^N$ which indicates where is the point x located with respect to its referential position. The local state of the body is characterized by the *strain tensor* $\{\varepsilon_{ij} : i, j = 1, \ldots, N\}$ defined by the formula

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \tag{7.1}$$

This is indeed a symmetric tensor with $N(N+1)/2$ independent components. It can, however, control all the N^2 components of ∇u in the following sense.

Proposition 7.1 ***(Korn inequality)*** *Let Ω be as in Theorem 5.2. Then there exists a constant $C > 0$ such that for all $u \in W_0^{1,2}(\Omega; \mathbb{R}^N)$ we have*

$$|\nabla u|_{2,\Omega} \le C \max_{i,j \in \{1,\ldots,N\}} |\varepsilon_{ij}|_{2,\Omega}.$$

Proof. By virtue of of Proposition 6.3, it is enough to prove (7.1) for $u \in C_C^\infty(\Omega; \mathbb{R}^N)$. For such u and for all $i, j, k = 1, \ldots, N$ we have the identity

$$\partial_i \partial_k u_j = \partial_i \varepsilon_{jk} + \partial_k \varepsilon_{ij} - \partial_j \varepsilon_{ik}. \tag{7.2}$$

Hence, for every $j = 1, \dots, N$ and every $w \in C_C^\infty(\Omega)$,

$$\int_\Omega \langle \nabla u_j, \nabla w \rangle \, \mathrm{d}x = \int_\Omega \langle f_j, \nabla w \rangle \, \mathrm{d}x, \tag{7.3}$$

where f_j is the vector $(f_{1j}, \dots, f_{Nj})$ with components $f_{ij} = 2\varepsilon_{ij} - \delta_{ij} \sum_{k=1}^N \varepsilon_{kk}$ and δ_{ij} is the Kronecker tensor $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ij} = 1$ for $i = j$. Putting $w = u_j$ in (7.3) we obtain the assertion. ■

The concept of Sobolev spaces can be extended to higher derivatives. By induction, we can define on $\Omega \subset \mathbb{R}^N$ the spaces

$$W^{k,p}(\Omega) = \{u \in W^{k-1,p}(\Omega) : \partial_i u \in W^{k-1,p}(\Omega) \ \forall i \in \{1, \dots, N\}\}$$

for $k \in \mathbb{N}$ with the convention $W^{0,p}(\Omega) = L^p(\Omega)$. These are Banach spaces with the norm

$$\|u\|_{k;p,\Omega} = \|u\|_{k-1;p,\Omega} + \sum_{|\alpha|=k} |\partial_\alpha u|_{p,\Omega},$$

where the sum is taken over all multiindices $\alpha = (\alpha_1, \dots, \alpha_N)$, $\alpha_i \in \mathbb{N} \cup \{0\}$, such that $|\alpha| := \sum_{i=1}^N \alpha_i = k$, and ∂_α denotes the partial derivative of the k-th order

$$\partial_\alpha u := \partial_1^{\alpha_1} \dots \partial_N^{\alpha_N} u.$$

There are important particular cases in which the $W^{k,p}$-regularity of a function is obtained even if we do not control all partial derivatives of u of the k-th order. We show here one typical example which frequently arises in the theory of PDEs.

We say that a bounded open domain $\Omega \subset \mathbb{R}^N$ is of class $C^{1,1}$ if there exists a covering of $\partial\Omega$ as in Definition 5.1 and such that all functions a_k, $k = 1, \dots, m$, admit Lipschitz continuous partial derivatives of the first order with respect to y_i, $i = 1, \dots, N-1$.

Proposition 7.2 *Let Ω be a domain of class $C^{1,1}$, let B be a symmetric positive definite $N \times N$ matrix, let $f \in L^2(\Omega)$ be a given function, and let $u \in W_0^{1,2}(\Omega)$ be a solution of the problem*

$$\int_\Omega (\langle B\nabla u, \nabla w \rangle - gw) \, \mathrm{d}x = 0 \quad \forall w \in C_C^\infty(\Omega). \tag{7.4}$$

Then $u \in W^{2,2}(\Omega)$.

The existence and uniqueness of a solution to (7.4) is easy to prove, see, e. g., [11], and the regularity stated in Proposition 7.2 is also proved in [11] in a more general context. The case of non-homogeneous and nonlinear boundary conditions for parabolic problems is discussed in [8]. This result is, however, of interest in particular if B is the identity matrix. Then one scalar quantity, namely the sum of derivatives $\sum_{i=1}^N \partial_i^2 u$ is enough to control all second order derivatives of u.

Here, the assumption of $C^{1,1}$-regularity of the boundary $\partial\Omega$ is substantial. Before proving Proposition 7.2, we give an example showing that the statement does not hold if the domain Ω is only Lipschitzian.

Example 7.3 Consider the set in polar coordinates

$$\hat{\Omega}_\omega := \Big\{(r,\theta) : r \in (0,1), \theta \in \Big(-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\Big)\Big\}$$

for some $\omega > 1/2$, and put

$$\Omega_\omega := \{(x,y) \in \mathbb{R}^2 : x = r\cos\theta, y = r\sin\theta, (r,\theta) \in \hat{\Omega}_\omega\},$$

see Fig. 4. We define the functions $\hat{f}, \hat{v} : \hat{\Omega}_\omega \to \mathbb{R}$

$$\hat{f}(r,\theta) = r^\omega \cos\omega\theta, \quad \hat{v}(r,\theta) = r^4 \cos\omega\theta,$$

and the corresponding functions $f, v, g : \Omega_\omega \to \mathbb{R}$ given by the formula

$$\hat{f}(r,\theta) = f(r\cos\theta, r\sin\theta), \quad \hat{v}(r,\theta) = v(r\cos\theta, r\sin\theta), \quad g(x,y) = v_{xx}(x,y) + v_{yy}(x,y).$$

Then g is continuously differentiable on $\bar{\Omega}_\omega$, the function

$$u(x,y) := f(x,y) - v(x,y)$$

satisfies the equation

$$u_{xx} + u_{yy} = -g \quad \text{in} \quad \Omega, \quad v|_{\partial\Omega} = 0, \tag{7.5}$$

and $u \notin W^{2,2}(\Omega)$ provided $\omega < 1$.

The fact that (7.5) holds follows from the identity

$$f_{xx} + f_{yy} = \hat{f}_{rr} + \frac{1}{r}\hat{f}_r + \frac{1}{r^2}\hat{f}_{\theta\theta} = 0.$$

To check that $u \notin W^{2,2}(\Omega)$ for $\omega < 1$, we compute

$$f_{xx} = \hat{f}_{rr}\cos^2\theta + \frac{1}{r}\hat{f}_r \sin^2\theta + \frac{1}{r^2}\hat{f}_{\theta\theta}\sin^2\theta + \Big(\frac{2}{r^2}\hat{f}_\theta - \frac{2}{r}\hat{f}_{r\theta}\Big)\cos\theta\sin\theta = \omega(\omega-1)r^{\omega-2}\cos((2-\omega)\theta),$$

hence,

$$\int_\Omega |f_{xx}|^2 \,\mathrm{d}x\,\mathrm{d}y = \omega^2(\omega-1)^2 \int_0^1 r^{2\omega-3}\,\mathrm{d}r \int_{-\pi/(2\omega)}^{\pi/(2\omega)} \cos^2((2-\omega)\theta)\,\mathrm{d}\theta = \infty.$$

Proof of Proposition 7.2. Consider the covering $\bar{\Omega} \subset \bigcup_{k=1}^m \Omega_k$ as in Definition 5.1, and the partition of unity $\{\psi_k : k = 0,1,\ldots,m\}$ as in the proof of Proposition 5.4, and set $u_k = u\psi_k$. It is enough to prove that $u_k \in W^{2,2}(A_k(G_k^-))$ for each $k = 0,\ldots,m$, and the general statement follows from the formula $u = \sum_{k=0}^m u_k$ on Ω.

For a fixed k and each fixed $w \in C_C^\infty(\Omega)$ we have

$$\begin{aligned} \int_\Omega \langle B\nabla u_k, \nabla w\rangle \,\mathrm{d}x &= \int_\Omega \langle B\nabla u, \nabla(w\psi_k)\rangle + \langle B(u\nabla w - w\nabla u), \nabla\psi_k\rangle \,\mathrm{d}x \\ &= \int_\Omega (gw\psi_k + \langle B(u\nabla w - w\nabla u), \nabla\psi_k\rangle)\,\mathrm{d}x \\ &= \int_\Omega (g_k w + \langle h_k, \nabla w\rangle)\,\mathrm{d}x, \end{aligned} \tag{7.6}$$

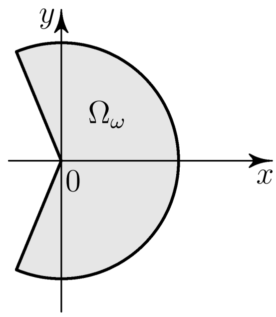

Figure 4: Counterexample to Proposition 7.2.

with functions $g_k := g\psi_k - \langle B\nabla u, \nabla\psi_k\rangle \in L^2(\Omega)$ and $h_k := uB\nabla\psi_k \in W_0^{1,2}(\Omega;\mathbb{R}^N)$, both with a compact support in $A_k(G_k^-)$.

Let now $w \in C_C^\infty(A_k(G_k^-))$ be arbitrarily chosen. We pass to the substitution $y = A_k x$ in (7.6), and by (6.5) we obtain for the new quantities $v_k(y) = u_k(A_k^\mathsf{T} y)$, $z(y) = w(A_k^\mathsf{T} y)$ the identity

$$\int_{G_k^-} \left\langle A_k^\mathsf{T} B A_k \nabla v_k, \nabla z\right\rangle \,\mathrm{d}y = \int_{G_k^-} \left(\tilde{g}_k z + \left\langle \tilde{h}_k, \nabla z\right\rangle\right) \mathrm{d}y, \tag{7.7}$$

where $\tilde{g}_k(y) = g_k(A_k y)$ and $\tilde{h}_k(y) = A_k^\mathsf{T} h_k(A_k y)$, and it is enough to prove that $v_k \in W^{2,2}(G_k^-)$.

As in the proof of Theorem 5.2, we pass from coordinates $y = (y', y_N)$, $y' = (y_1, \dots, y_{N-1})^\mathsf{T} \in \Delta_k$, $y_N \in (a_k(y') - \delta, a_k(y'))$ to new coordinates (y', s), $y' \in \Delta_k$, $s = y_N - a_k(y')$ and put $\hat{v}_k(y', s) = v_k(y', s + a_k(y'))$, $\hat{z}_k(y', s) = z_k(y', s + a_k(y'))$. In terms of the new variables, the identity (7.7) has the form

$$\begin{aligned} &\int_{-\delta}^0 \int_{\Delta_k} \langle B_k(y')\nabla\hat{v}_k(y', s), \nabla\hat{z}(y', s)\rangle \,\mathrm{d}y' \,\mathrm{d}s \\ &\qquad = \int_{-\delta}^0 \int_{\Delta_k} \left(\hat{g}_k(y', s)\hat{z}(y', s) + \left\langle \hat{h}_k(y', s), \nabla\hat{z}(y', s)\right\rangle\right) \mathrm{d}y' \,\mathrm{d}s \end{aligned} \tag{7.8}$$

for every $\hat{z} \in C_C^\infty(\Delta_k \times (-\delta, 0))$, where $B_k(y')$ is the symmetric positive definite matrix

$$B_k(y') := M_k^\mathsf{T}(y') A_k^\mathsf{T} B A_k M_k(y')$$

with

$$M_k(y') = \begin{pmatrix} 1 & 0 & \dots & 0 & -\partial_1 a_k(y') \\ 0 & 1 & \dots & 0 & -\partial_2 a_k(y') \\ & & \dots & & \\ 0 & 0 & \dots & 1 & -\partial_{N-1} a_k(y') \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

and $\hat{g}_k(y', s) = \tilde{g}(y', s + a_k(y'))$ belongs to $L^2(\Delta_k \times (-\delta, 0))$, $\hat{h}_k(y', s) = \tilde{h}(y', s + a_k(y'))M_k(y')$ belongs to $W_0^{1,2}(\Delta_k \times (-\delta, 0); \mathbb{R}^N)$. We can use in (7.8) the Gauss-Ostrogradsky formula (6.2) and write

$$\int_{-\delta}^0 \int_{\Delta_k} \langle B_k(y')\nabla\hat{v}_k(y', s), \nabla\hat{z}(y', s)\rangle \,\mathrm{d}y' \,\mathrm{d}s = \int_{-\delta}^0 \int_{\Delta_k} g_k^*(y', s)\hat{z}(y', s) \,\mathrm{d}y' \,\mathrm{d}s \tag{7.9}$$

with $g_k^* = \hat{g}_k - \operatorname{div} \hat{h}_k \in L^2(\Delta_k \times (-\delta, 0))$. Again, it will be enough to prove that

$$\hat{v}_k \in W^{2,2}(\Delta_k \times (-\delta, 0)). \tag{7.10}$$

To this end, we denote by e_i the unit vector in the direction of the i-th coordinate, $i \in \{1, \ldots, N-1\}$. All functions in (7.9) have compact support in $\Delta_k \times (-\delta, 0)$, so that for $|t| > 0$ sufficiently small we have

$$\int_{-\delta}^{0} \int_{\Delta_k} \langle B_k(y' + te_i) \nabla \hat{v}_k(y' + te_i, s), \nabla \hat{z}(y', s) \rangle \, \mathrm{d}y' \, \mathrm{d}s = \int_{-\delta}^{0} \int_{\Delta_k} g_k^*(y' + te_i, s) \hat{z}(y', s) \, \mathrm{d}y' \, \mathrm{d}s \tag{7.11}$$

for every $\hat{z} \in C_C^\infty(\Delta_k \times (-\delta, 0))$. We now subtract (7.9) from (7.11) and put $\hat{z}(y', s) = (\hat{v}_k(y' + te_i, s) - \hat{v}_k(y', s))/t^2$. This is indeed an admissible choice by virtue of Proposition 6.3. By hypothesis, the matrix $B_k(y')$ is symmetric positive definite and depends Lipschitz-continuously on y'. Furthermore, we know that $\nabla \hat{v}_k$ and g_k^* belong to $L^2(\Delta_k \times (-\delta, 0))$. We thus find a constant $C > 0$ independent of t such that

$$\begin{aligned} &\int_{-\delta}^{0} \int_{\Delta_k} \frac{|\nabla \hat{v}_k(y' + te_i, s) - \nabla \hat{v}_k(y', s)|^2}{t^2} \, \mathrm{d}y' \, \mathrm{d}s \\ &\quad \le C \left(1 + \left(\int_{-\delta}^{0} \int_{\Delta_k} \frac{|\hat{v}_k(y' + te_i, s) - 2\hat{v}_k(y', s) + \hat{v}_k(y' - te_i, s)|^2}{t^4} \, \mathrm{d}y' \, \mathrm{d}s \right)^{1/2} \right). \end{aligned} \tag{7.12}$$

We split the fraction on the right hand side into three terms and estimate its L^2-norm from above by the sum

$$\begin{aligned} &\left(\int_{-\delta}^{0} \int_{\Delta_k} \frac{|\partial_i \hat{v}_k(y' + te_i, s) - \partial_i \hat{v}_k(y', s)|^2}{t^2} \, \mathrm{d}y' \, \mathrm{d}s \right)^{1/2} \\ &+ \frac{2}{t^2} \left(\int_{-\delta}^{0} \int_{\Delta_k} \left| \hat{v}_k(y', s) - \hat{v}_k(y' - te_i, s) - t \partial_i \hat{v}_k(y', s) \right|^2 \, \mathrm{d}y' \, \mathrm{d}s \right)^{1/2}. \end{aligned}$$

The first term is dominated by the left-hand side of (7.12), while the second term can be estimated by

$$\frac{2}{t^2} \left(\int_{-\delta}^{0} \int_{\Delta_k} \left(\int_0^{|t|} |\partial_i \hat{v}_k(y' + \tau e_i, s) - \partial_i \hat{v}_k(y', s)| \, \mathrm{d}\tau \right)^2 \mathrm{d}y' \, \mathrm{d}s \right)^{1/2}$$

which can in turn be estimated using the Minkowski inequality (2.7) by

$$\frac{2}{t^2} \int_0^{|t|} \left(\int_{-\delta}^{0} \int_{\Delta_k} \left| \partial_i \hat{v}_k(y' + \tau e_i, s) - \partial_i \hat{v}_k(y', s) \right|^2 \mathrm{d}y' \, \mathrm{d}s \right)^{1/2} \mathrm{d}\tau.$$

For $t \ge 0$ put

$$V(t) := \int_0^t \left(\int_{-\delta}^{0} \int_{\Delta_k} \left| \nabla \hat{v}_k(y' + \tau e_i, s) - \nabla \hat{v}_k(y', s) \right|^2 \mathrm{d}y' \, \mathrm{d}s \right)^{1/2} \mathrm{d}\tau.$$

Then (7.12) yields that

$$\frac{1}{t^2} (\dot{V}(t))^2 \le C \left(1 + \frac{1}{t^2} V(t) \right)$$

for all $t > 0$, that is,

$$\frac{1}{t}\dot{V}(t) \le C\left(1 + \frac{1}{t}V^{1/2}(t)\right).$$

Put $U(t) := V(t)/t$ for $t > 0$. Then $\dot{U}(t) = \dot{V}(t)/t - V(t)/t^2 \le C(1 + U(t)/t^{1/2}) - U(t)/t$, hence $\dot{U}(t) \le C + C^2/4$. From the Mean Continuity Theorem it follows that $U(0-) = 0$, hence $U(t) \le (C + C^2/4)t$, and $V(t) \le (C + C^2/4)t^2$. We conclude from (7.12) that there exists a constant C^* independent of t such that for all $|t| > 0$ sufficiently small we have

$$\int_{-\delta}^{0}\int_{\Delta_k} \frac{|\nabla\hat{v}_k(y' + te_i, s) - \nabla\hat{v}_k(y', s)|^2}{t^2}\,\mathrm{d}y'\,\mathrm{d}s \le C^*. \tag{7.13}$$

Consequently, we can find a sequence $t_j \to 0$ and a function $\eta_{k,i} \in L^2(\Delta_k \times (-\delta, 0))$ such that

$$\frac{\nabla\hat{v}_k(y' + t_j e_i, s) - \nabla\hat{v}_k(y', s)}{t_j} \to \eta_{k,i}(y', s) \quad \text{weakly in } \ L^2(\Delta_k \times (-\delta, 0)).$$

On the other hand, for every $z \in C_C^\infty(\Delta_k \times (-\delta, 0); \mathbb{R}^N)$ we have

$$\begin{aligned} &\int_{-\delta}^{0}\int_{\Delta_k} \left\langle \frac{\nabla\hat{v}_k(y' + t_j e_i, s) - \nabla\hat{v}_k(y', s)}{t_j}, z(y', s)\right\rangle \mathrm{d}y'\,\mathrm{d}s \\ &\quad = -\int_{-\delta}^{0}\int_{\Delta_k} \left\langle \nabla\hat{v}_k(y', s), \frac{z(y', s) - z(y' - t_j e_i)}{t_j}\right\rangle \mathrm{d}y'\,\mathrm{d}s \\ &\quad \to -\int_{-\delta}^{0}\int_{\Delta_k} \langle \nabla\hat{v}_k(y', s), \partial_i z(y', s)\rangle\,\mathrm{d}y'\,\mathrm{d}s, \end{aligned}$$

and we conclude that $\partial_i\nabla\hat{v}_k = \eta_{k,i} \in L^2(\Delta_k \times (-\delta, 0))$ for $i = 1, \ldots, N-1$. It remains to prove that $\partial_N^2\hat{v}_k \in L^2(\Delta_k \times (-\delta, 0))$. This follows from (7.9), where we can integrate by parts in all terms of the left-hand side except for the term $(B_k)_{NN}\partial_N\hat{v}_k\partial_N\hat{z}_k$, and we obtain

$$\int_{-\delta}^{0}\int_{\Delta_k} (B_k)_{NN}(y')\partial_N\hat{v}_k(y', s)\partial_N\hat{z}(y', s)\,\mathrm{d}y'\,\mathrm{d}s = \int_{-\delta}^{0}\int_{\Delta_k} g_k^{**}(y', s)\hat{z}(y', s)\,\mathrm{d}y'\,\mathrm{d}s \tag{7.14}$$

with some function $g_k^{**} \in L^2(\Delta_k \times (-\delta, 0))$. Since B_k is symmetric and uniformly positive definite, we necessarily have $(B_k)_{NN}(y') \ge c$ for some constant $c > 0$, and we conclude that all second derivatives $\partial_i\partial_j\hat{v}_k$ belong to $L^2(\Delta_k \times (-\delta, 0))$, which we wanted to prove. ■

In the theory of elasticity, it is assumed that the *stress tensor* $\sigma = \sigma_{ij}$ is related to the strain tensor $\varepsilon = \varepsilon_{ij}$ given by (7.1) by a linear relation

$$\sigma_{ij} = \sum_{k,l=1}^{N} A_{ijkl}\varepsilon_{kl} \tag{7.15}$$

with a symmetric positive definite fourth order tensor $A = A_{ijkl}$, see [10]. Let $g = (g_1, \ldots g_N)$ be the vector of volume force density acting on the body. Then the equilibrium condition has the form $\operatorname{div} A\sigma = g$, that is,

$$\sum_{j,l,k=1}^{N} \partial_j A_{ijkl}\varepsilon_{kl} = g_i \quad \text{for } \ i = 1, \ldots, N. \tag{7.16}$$

We have the following counterpart of Proposition 7.2

Corollary 7.4 *Let Ω be as in Proposition 7.2 and let $g_i \in L^2(\Omega)$ for $i = 1,\dots,N$. Then the solution u of* (7.16) *with boundary condition $u = 0$ on $\partial\Omega$ belongs to $W^{2,2}(\Omega;\mathbb{R}^N) \cap W_0^{1,2}(\Omega;\mathbb{R}^N)$.*

We omit the proof which is similar to the proof of Proposition 7.2 except that it makes use of the Korn inequality in Proposition 7.1.

8 Embeddings

The word *embedding* is used in the situation of two Banach spaces U and V, endowed with respective norms $\|\cdot\|_U$ and $\|\cdot\|_V$, and such that

$$\left.\begin{array}{l} V \subset U\,, \\ \exists C > 0 \ \ \forall v \in V: \ \|v\|_U \le C\|v\|_V\,. \end{array}\right\} \tag{8.1}$$

If (8.1) holds, then we say that V *is embedded in* U.

The embedding is said to be *compact*, if every bounded set $A \subset V$ is *precompact* in U, that is,

$$\forall \varepsilon > 0 \ \ \exists a_1,\dots,a_n \in A \ \ \forall a \in A \ \ \exists k \in \{1,\dots,n\}: \ \ \|a - a_k\|_U < \varepsilon\,. \tag{8.2}$$

The following theorem represents a basic tool in the theory of compact embeddings in function spaces.

Theorem 8.1 ***(Arzelà-Ascoli)*** *Let X, Y be Banach spaces and let $A \subset X, B \subset Y$ be compact sets. Let $C(A;B)$ be the Banach space of all continuous mappings from A into B. Let $K \subset C(A;B)$ be an* equicontinuous *set, that is,*

$$\forall \varepsilon > 0 \ \ \exists \delta > 0 \ \ \forall f \in K \ \ \forall x, y \in A: \|x-y\|_X < \delta \Rightarrow \|f(x) - f(y)\|_Y < \varepsilon\,.$$

Then K is compact in $C(A;B)$. Conversely, every relatively compact set in $C(A;B)$ is equicontinuous.

Proof. Let $K \subset C(A;B)$ be equicontinuous, and let $\varepsilon > 0$ be given. We find $\delta > 0$ such that for all $f \in K$ we have $\|f(x) - f(y)\|_Y < \varepsilon/4$ whenever $\|x - y\|_X < \delta$. Since A is compact, there exist $x_1,\dots,x_p \in A$ such that for every $x \in A$ there exists $i \in \mathcal{I} := \{1,\dots,p\}$ such that $\|x - x_i\|_X < \delta$. Furthermore, B is compact, hence there exist $y_1,\dots,y_q \in B$ such that for every $y \in B$ there exists $j \in \mathcal{J} := \{1,\dots,q\}$ such that $\|y - y_j\|_Y < \varepsilon/4$.

For $z \in \mathcal{J}^p$, $z = \{z_1,\dots,z_p\}$, we now denote

$$K_z \ = \ \left\{ f \in K\,;\, \forall i \in \mathcal{I}: \ \|f(x_i) - y_{z_i}\|_Y < \frac{\varepsilon}{4}\right\}\,.$$

Set $J := \{z \in \mathcal{J}^p \,:\, K_z \ne \emptyset\}$. The set J is indeed finite and we have $K = \bigcup_{z\in J} K_z$, hence we may fix one representative $f_z \in K_z$ for each $z \in J$. For any $f \in K_z$ and $x \in A$ we find x_i such that $\|x - x_i\|_X < \delta$, and estimate

$$\begin{aligned} \|f(x) - f_z(x)\|_Y \ &\le \ \|f(x) - f(x_i)\|_Y + \|f(x_i) - y_{z_i}\|_Y + \|f_z(x_i) - y_{z_i}\|_Y + \|f_z(x) - f_z(x_i)\|_Y \\ &< \ \varepsilon\,, \end{aligned}$$

which we wanted to prove. Since every finite set of mappings in $C(A;B)$ is equicontinuous, the fact that that relatively compact sets are equicontinuous follows easily. ■

As an example, consider the spaces $C(\bar\Omega)$ of continuous real functions defined on $\bar\Omega$, endowed with the norm

$$\|f\|_{C,0} = \sup\{|f(x)|\,;\, x\in\bar\Omega\}\,,$$

and $C^1(\bar\Omega)$ of continuously differentiable real functions on $\bar\Omega$, endowed with the norm

$$\|f\|_{C,1} = \sup\left\{|f(x)| + \sum_{i=1}^N \left|\frac{\partial f}{\partial x_i}(x)\right|\,;\, x\in\bar\Omega\right\}.$$

Proposition 8.2 *If Ω is a bounded domain Lipschitzian boundary, then the space $C^1(\bar\Omega)$ is compactly embedded in $C(\bar\Omega)$.*

Proof. Condition (8.1) is automatically satisfied. Furthermore, let $K\subset C^1(\bar\Omega)$ be bounded. Hence, there exists $M>0$ such that

$$\forall f\in K\ \ \forall x\in\bar\Omega : |f(x)| + \sum_{i=1}^N \left|\frac{\partial f}{\partial x_i}(x)\right| \le M\,.$$

We are thus in the situation of Theorem 8.1 with $X=\mathbb{R}^N$, $Y=\mathbb{R}$, $A=\bar\Omega$, $B=[-M,M]$, provided we check that K is equicontinuous. Let $x,y\in\bar\Omega$ be arbitrarily chosen. We find a Lipschitz continuous function $\xi:[0,1]\to\bar\Omega$ and a constant $C>0$ such that $\xi(0)=x$, $\xi(1)=y$, $|\xi'(\sigma)|\le C|x-y|$ a. e. (this is possible by the hypotheses on Ω), and use the chain rule to estimate

$$\begin{aligned}|f(x)-f(y)| &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\sigma} f(\xi(\sigma))\,\mathrm{d}\sigma\right| \\ &= \left|\int_0^1 \langle\nabla f(\xi(\sigma)),\xi'(\sigma)\rangle\,\mathrm{d}\sigma\right| \\ &\le MC|x-y|\,.\end{aligned}$$

The relative compactness now follows from Theorem 8.1. ■

For Sobolev spaces $W^{1,p}(\Omega)$, we have the following classical embedding formula.

Theorem 8.3 *Let $p,q\in[1,\infty)$ be such that*

$$\frac1p \ge \frac1q > \frac1p - \frac1N\,,$$

and set

$$\kappa \;:=\; 1 - N\left(\frac1p - \frac1q\right) \;\in\; (0,1]\,.$$

Then there exists $C_{pq}>0$ such that for every $u\in W^{1,p}(\mathbb{R}^N)$ and every $\sigma\in(0,1]$ we have

$$|u^\sigma - u|_q \;\le\; C_{pq}\,\sigma^\kappa\,|\nabla u|_p\,, \tag{8.3}$$

where u^σ is as in (4.3).

Proof. Notice first that for every $x \in \mathbb{R}^N$ and $\sigma \in (0,1)$ we obtain, integrating by parts, that

$$\begin{aligned} \frac{\partial}{\partial \sigma} u^\sigma(x) &= \sigma^{-N} \int_{\mathbb{R}^N} \sum_{i=1}^N \frac{\partial}{\partial y_i} \left(\frac{x_i - y_i}{\sigma} \varphi\left(\frac{x-y}{\sigma}\right)\right) u(y)\,\mathrm{d}y \\ &= -\sigma^{-N} \int_{\mathbb{R}^N} \left\langle \Phi\left(\frac{x-y}{\sigma}\right), \nabla u(y) \right\rangle \mathrm{d}y\,, \end{aligned} \tag{8.4}$$

where we set $\Phi(\xi) = \xi \varphi(\xi)$. This yields in particular,

$$|u^\beta(x) - u^\alpha(x)| \le \int_\alpha^\beta \sigma^{-N} \left| \int_{\mathbb{R}^N} \left\langle \Phi\left(\frac{x-y}{\sigma}\right), \nabla u(y)\right\rangle \mathrm{d}y \right| \mathrm{d}\sigma \tag{8.5}$$

for every $0 < \alpha < \beta \le 1$. To estimate the difference $u^\beta - u^\alpha$ in (8.5) in the space $L^q(\mathbb{R}^N)$, we make use of the Minkowski and Young II inequalities with r as in (2.12), using the notation $\int_X$, $\int_Y$ for $\int_{\mathbb{R}^N}$ as in Proposition 2.3. More specifically, we have

$$\begin{aligned} |u^\beta - u^\alpha|_q &\le \left(\int_X \left(\int_\alpha^\beta \sigma^{-N} \int_Y \left| \Phi\left(\frac{x-y}{\sigma}\right)\right| |\nabla u(y)|\,\mathrm{d}y\,\mathrm{d}\sigma \right)^q \mathrm{d}x \right)^{1/q} \\ &\overset{\text{Minkowski}}{\le} \int_\alpha^\beta \sigma^{-N} \left(\int_X \left(\int_Y \left| \Phi\left(\frac{x-y}{\sigma}\right)\right| |\nabla u(y)|\,\mathrm{d}y \right)^q \mathrm{d}x \right)^{1/q} \mathrm{d}\sigma \\ &\overset{\text{Young II}}{\le} \int_\alpha^\beta \sigma^{-N} \left(\int_{\mathbb{R}^N} \left| \Phi\left(\frac{y}{\sigma}\right)\right|^r \mathrm{d}y \right)^{1/r} \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p \,\mathrm{d}x \right)^{1/p} \mathrm{d}\sigma \\ &\le |\nabla u|_p \left(\int_{B_1(0)} |\Phi(x)|^r \,\mathrm{d}x \right)^{1/r} \int_\alpha^\beta \sigma^{N(1/r-1)}\,\mathrm{d}\sigma\,. \end{aligned} \tag{8.6}$$

We have $N(1/r - 1) = \kappa - 1$, hence

$$|u^\beta - u^\alpha|_q \le C_{pq} (\beta^\kappa - \alpha^\kappa) |\nabla u|_p \tag{8.7}$$

with $C_{pq} = |\Phi|_r/\kappa$. Hence, for every sequence $\sigma_i \to 0+$, u^{σ_i} is a Cauchy sequence in $L^q(\mathbb{R}^N)$. By Corollary 4.4, u^{σ_i} converge to u in $L^p(\mathbb{R}^N)$, hence $u \in L^q(\mathbb{R}^N)$ and u^{σ_i} converge to u (strongly) in $L^q(\mathbb{R}^N)$. Letting α tend to 0 and replacing β by σ, we thus obtain (8.3). ∎

Corollary 8.4 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, and let $1 \le p < \infty$, $1 \le q < \infty$ satisfy the inequality*

$$\frac{1}{q} > \frac{1}{p} - \frac{1}{N}\,.$$

Then the space $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$.

Proof. Assume first $q \ge p$, and set $u_* = E_p u$, where $E_p : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ is the prolongation operator from Theorem 5.2. By Theorems 5.2 and 8.3, there exist constants C_1 and C_2 such that for every $\sigma \in (0,1]$ we have

$$|u_*^\sigma - u_*|_q \le C_1 \sigma^\kappa |\nabla u_*|_p \le C_2 \sigma^\kappa \|u\|_{1;p,\Omega}\,. \tag{8.8}$$

By (4.4) and Theorem 5.2 we have

$$|u_*^\sigma|_q \;\le\; C_3\,\sigma^{\kappa-1}\,|u_*|_p \;\le\; C_3\,c_p\,\sigma^{\kappa-1}\,|u|_{p,\Omega}\,, \tag{8.9}$$

with $C_3 = |\varphi|_r$. Consequently, there exists a constant $C_4 > 0$ such that

$$|u|_{q,\Omega} \;\le\; |u_*|_q \;\le\; C_4\left(\sigma^{\kappa-1}\,|u|_{p,\Omega} + \sigma^\kappa\,\|u\|_{1;p,\Omega}\right) \tag{8.10}$$

for all $\sigma \in (0,1]$. According to (8.1), $W^{1,p}(\Omega)$ is thus embedded in $L^q(\Omega)$. To see that the embedding is compact, consider a bounded set $M \subset W^{1,p}(\Omega)$ and an arbitrary $\varepsilon > 0$. We fix $\sigma > 0$ such that, with the notation of Theorem 8.3, we have

$$C_{pq}\,\sigma^\kappa\,|\nabla u_*|_p \;<\; \frac{\varepsilon}{4} \quad \forall u \in M\,. \tag{8.11}$$

With this fixed σ, every element u_*^σ of the set $M_\sigma = \{u_*^\sigma\,;\, u \in M\}$ vanishes outside of the set $(1+\sigma)B_1(0) =: B_{1+\sigma}(0)$. Moreover, M_σ is bounded in $C^1(B_{1+\sigma}(0)$, hence, by Proposition 8.2, there exist $u_1,\dots,u_n \in M$ such that

$$\forall u \in M \;\; \exists k \in \{1,\dots,n\} \;\; \forall x \in B_{1+\sigma}(0): \;\; |u_*^\sigma(x) - u_k^\sigma(x)| < \frac{\varepsilon}{4\mathrm{meas}\,(B_{1+\sigma}(0))}\,. \tag{8.12}$$

We then have, by (8.11), (8.12), and Theorem 8.3, that

$$|u_* - u_k^\sigma|_q \le |u_*^\sigma - u_k^\sigma|_q + \frac{\varepsilon}{4} < \frac{\varepsilon}{2}\,. \tag{8.13}$$

For $k = 1,\dots,n$ set $M_k = \{u \in M\,;\, |u_* - u_k^\sigma|_q < \varepsilon/2\}$, and $J = \{k \in \{1,\dots,n\}\,;\, M_k \ne \emptyset\}$. For every $k \in J$ we fix one representative $\hat u_k \in M_k$, so that for every $u \in M_k$ we have $|u - \hat u_k|_{q,\Omega} < \varepsilon$ and $M = \bigcup_{k\in J} M_k$. The proof is thus complete for $q \ge p$. Let now $q < p$. Hölder's inequality yields

$$|u_*^\sigma - u_*|_q \;\le\; (\mathrm{meas}\,(B_{1+\sigma}(0)))^{1/q-1/p}|u_*^\sigma - u_*|_p\,,$$

hence the above argument remains valid. ∎

Corollary 8.5 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, and let $p > N$. Then the space $W^{1,p}(\Omega)$ is compactly embedded in $C(\bar\Omega)$.*

Proof. We repeat the argument of the proof of Theorem 8.3 and Corollary 8.4, putting

$$\kappa \;:=\; 1 - \frac{N}{p} \;\in\; (0,1]\,.$$

A computation analogous to (8.6) yields for every $x \in \mathbb{R}^N$ that

$$\begin{aligned} |u_*^\beta(x) - u_*^\alpha(x)| \;&\le\; \int_\alpha^\beta \sigma^{-N}\int_Y \left|\Phi\left(\frac{x-y}{\sigma}\right)\right| |\nabla u_*(y)|\,\mathrm{d}y\,\mathrm{d}\sigma \\ &\overset{\text{Hölder}}{\le} \int_\alpha^\beta \sigma^{-N}\left(\int_{\mathbb{R}^N}\left|\Phi\left(\frac{y}{\sigma}\right)\right|^{p'}\mathrm{d}y\right)^{1/p'}\left(\int_{\mathbb{R}^N}|\nabla u_*(x)|^p\,\mathrm{d}x\right)^{1/p}\mathrm{d}\sigma \\ &\le\; |\nabla u_*|_p\left(\int_{B_1(0)}|\Phi\,(x)|^{p'}\,\mathrm{d}x\right)^{1/p'}\int_\alpha^\beta \sigma^{-N/p}\,\mathrm{d}\sigma\,, \end{aligned} \tag{8.14}$$

and we proceed as above. ∎

9 Limit cases and counterexamples

Proposition 9.1 *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open bounded connected set with Lipschitzian boundary, and let*

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{N}\,. \tag{9.1}$$

Then the space $W^{1,p}(\Omega)$ is embedded in $L^q(\Omega)$.

Proof. We proceed in principle as in the proof of Theorem 8.3. The main difference is that the number κ is zero here and we have to proceed more carefully. We represent $x \in \mathbb{R}^N$ as $x = (x', x_N)$, $x' \in \mathbb{R}^{N-1}$, and rewrite inequality (8.5) as

$$|u^\beta(x', x_N) - u^\alpha(x', x_N)| \leq \int_\alpha^\beta \sigma^{-N} \int_{\mathbb{R}^N} \left|\Phi\left(\frac{x'-y'}{\sigma}, \frac{x_N - y_N}{\sigma}\right)\right| |\nabla u(y', y_N)| \,\mathrm{d}y' \,\mathrm{d}y_N \,\mathrm{d}\sigma\,. \tag{9.2}$$

With $r = N/(N-1)$, we now repeat the computation from (8.6), restricted to the component x', to obtain

$$\begin{aligned} &|u^\beta(\cdot, x_N) - u^\alpha(\cdot, x_N)|_q \\ \leq\ & \left(\int_{\mathbb{R}^{N-1}} \left(\int_\alpha^\beta \sigma^{-N} \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} \left|\Phi\left(\frac{x'-y'}{\sigma}, \frac{x_N-y_N}{\sigma}\right)\right| |\nabla u(y', y_N)| \,\mathrm{d}y' \,\mathrm{d}y_N \,\mathrm{d}\sigma\right)^q \mathrm{d}x\right)^{1/q} \\ \overset{\text{Minkowski}}{\leq}\ & \int_{\mathbb{R}} \int_\alpha^\beta \sigma^{-N} \left(\int_{\mathbb{R}^{N-1}} \left(\int_{\mathbb{R}^{N-1}} \left|\Phi\left(\frac{x'-y'}{\sigma}, \frac{x_N-y_N}{\sigma}\right)\right| |\nabla u(y', y_N)| \,\mathrm{d}y'\right)^q \mathrm{d}x'\right)^{1/q} \mathrm{d}\sigma \,\mathrm{d}y_N \\ \overset{\text{Young II}}{\leq}\ & \int_{\mathbb{R}} \int_\alpha^\beta \sigma^{-N} \left(\int_{\mathbb{R}^{N-1}} \left|\Phi\left(\frac{y'}{\sigma}, \frac{x_N-y_N}{\sigma}\right)\right|^r \mathrm{d}y'\right)^{1/r} \left(\int_{\mathbb{R}^{N-1}} |\nabla u(x', y_N)|^p \,\mathrm{d}x'\right)^{1/p} \mathrm{d}\sigma \,\mathrm{d}y_N \\ \leq\ & \int_{\mathbb{R}} \int_\alpha^\beta \sigma^{-N+N-1/r} |\nabla u(\cdot, y_N)|_p \left|\Phi\left(\cdot, \frac{x_N - y_N}{\sigma}\right)\right|_r \mathrm{d}\sigma \,\mathrm{d}y_N\,. \end{aligned} \tag{9.3}$$

The function $\left|\Phi\left(\cdot, \frac{x_N-y_N}{\sigma}\right)\right|_r$ vanishes if $\sigma < |x_N - y_N|$. Moreover, Φ is bounded by a constant $\Phi_0 > 0$. Hence, using the fact that $-N + N - 1/r = -2 + 1/N$, we have

$$\int_\alpha^\beta \sigma^{-N+N-1/r} \left|\Phi\left(\cdot, \frac{x_N - y_N}{\sigma}\right)\right|_r \mathrm{d}\sigma \leq \Phi_0 \int_{|x_N-y_N|}^\infty \sigma^{-2+1/N} \,\mathrm{d}\sigma = \Phi_0\, r\, |x_N - y_N|^{-1/r}\,.$$

We thus have

$$|u^\beta(\cdot, x_N) - u^\alpha(\cdot, x_N)|_q \leq \Phi_0\, r \int_{\mathbb{R}} |\nabla u(\cdot, y_N)|_p \,|x_N - y_N|^{-1/r} \,\mathrm{d}y_N\,.$$

At this point, we use the Hardy-Littlewood inequality (3.1), with q replaced by q'. Indeed, $1/q' + 1/p + 1/r = 2$. Hence, for every function $g \in L^{q'}(\mathbb{R})$ we have by Proposition 3.1 that

$$\int_{\mathbb{R}} |u^\beta(\cdot, x_N) - u^\alpha(\cdot, x_N)|_q \, g(x_N) \,\mathrm{d}x_N \leq C|g|_{q'} |\nabla u|_p$$

with some constant $C > 0$, hence, by the reverse Hölder inequality (2.11), we have

$$|u^\beta - u^\alpha|_q \le C|\nabla u|_p . \tag{9.4}$$

Since u^σ converge strongly to u in $L^p(\mathbb{R}^N)$ and their L^q norms are bounded, we conclude that they converge strongly in $L^q(\mathbb{R}^N)$ as well and the embedding formula follows. ■

Corollary 9.2 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, and let*

$$q \ge q^* := \frac{Np - p}{N - p}.$$

Then the trace operator F_p defined in Theorem 6.1 is continuous from $W^{1,p}(\Omega)$ to $L^q(\partial\Omega)$. If moreover $q > q^*$, then the trace operator is compact.

Proof. With the notation from the proof of Theorem 6.1 we have

$$\begin{aligned}
&\int_{\Delta_k} |v_k(y', a_k(y'))|^q \left(1 + \sum_{i=1}^N (\partial_i a_k(y'))^2\right)^{1/2} \mathrm{d}y' \\
&\quad \le C \int_{\Delta_k} \int_{a_k(y')-\delta}^{a_k(y')} |v_k(y', y_N)|^{q-1}\, |\partial_N v_k(y', y_N)| \,\mathrm{d}y_N \,\mathrm{d}y' \\
&\quad \le C \left(\int_{Q_k^-} |v_k(y)|^{p'(q-1)} \,\mathrm{d}y\right)^{1/p'} \left(\int_{Q_k^-} |\partial_N v_k(y)|^p \,\mathrm{d}y\right)^{1/p},
\end{aligned}$$

where we have used Hölder's inequality with p' as in (2.3). We have

$$p'(q-1) \ge p(q^* - 1) = \frac{Np}{N-p}.$$

From Proposition 9.1 it follows that the norm $|v_k|_{q,G_k^-}$ is bounded above by the norm $\|v_k\|_{1;p,G_k^-}$, and the assertion follows from the partition of unity. The compactness of the embedding for $q > q^*$ is a consequence of Corollary 8.4. ■

We now show a few examples to illustrate that the embedding inequalities are (at least qualitatively) optimal.

(i) To see that the embedding in Proposition 9.1 is not compact, and that $W^{1,p}(\Omega)$ is not embedded in $L^q(\Omega)$ if

$$\frac{1}{q} < \frac{1}{p} - \frac{1}{N}, \tag{9.5}$$

it suffices to fix any open set Ω, some $x_0 \in \Omega$, find $s_0 > 0$ such that $x_0 + s_0 B_1(0) \subset \Omega$, and consider the family of functions

$$u_s(x) = s^{1-N/p} \varphi\left(\frac{x - x_0}{s}\right), \quad s \in (0, s_0), \tag{9.6}$$

with φ as in (1.1). We have

$$|u_s|_{p,\Omega} = s|\varphi|_p\,, \quad \left|\frac{\partial u_s}{\partial x_i}\right|_{p,\Omega} = \left|\frac{\partial \varphi}{\partial x_i}\right|_p\,, \quad |u_s|_{q,\Omega} = s^\alpha|\varphi|_q \quad \forall s \in (0, s_0)\,,$$

where $\alpha = 1 - N(1/p - 1/q)$. In the case (9.1), we have $\alpha = 0$. Using the fact that u_s converge to 0 in $L^p(\Omega)$ as $s \to 0+$, we conclude that the family $\{u_s\}$, having constant nonzero norm in $L^q(\Omega)$, does not contain any convergent subsequence in $L^q(\Omega)$, hence the embedding is not compact. In the case (9.5), we have $\alpha < 0$, hence the family $\{u_s\}$ is unbounded in $L^q(\Omega)$ and no embedding takes place.

(ii) In another limit case

$$p = N\,, \tag{9.7}$$

the space $W^{1,p}(\Omega)$ is embedded in $L^\infty(\Omega)$ if and only if $p = N = 1$, and the embedding is not compact. For $N \geq 2$, it suffices to consider $\Omega = B_1(0)$, and

$$u(x) = \left(-\log\left(\frac{|x|}{2}\right)\right)^\alpha\,,$$

for any $0 < \alpha < 1 - 1/N$. Then u is unbounded, but belongs to $W^{1,N}(B_1(0))$. For $N = 1$, the embedding of $W^{1,1}(\Omega)$ into $C(\bar{\Omega})$ (hence $L^\infty(\Omega)$) for every bounded interval Ω is obvious. To see that it is not compact, we may consider for $n \in \mathbb{N}$ the sequence

$$u_n(x) = \begin{cases} \sin\frac{1}{x} & \text{for } x \in \left[\frac{1}{(n+1)\pi}, \frac{1}{n\pi}\right] \\ 0 & \text{otherwise.} \end{cases}$$

It is bounded in $W^{1,1}(0, 1/\pi)$, but $\sup|u_n(x) - u_m(x)| = 1$ for all $m \neq n$, hence it is not precompact in $L^\infty(0, 1/\pi)$.

(iii) The assumption on the Lipschitzian boundary is substantial. We show that there exists an open simply connected set $\Omega \subset \mathbb{R}^2$ such that $W^{1,p}(\Omega)$ is not embedded in $L^q(\Omega)$ for any $q > p \geq 1$. This set can be defined as (see Fig. 5)

$$\Omega \;=\; \left\{x = (x_1, x_2) \in \mathbb{R}^2\,;\, 0 < x_1 < 1,\, 0 < x_2 < \mathrm{e}^{\,-1/x_1}\right\}\,.$$

For any $q > p$ we set

$$u_{pq}(x) = \mathrm{e}^{\,2/(p+q)x_1}\,.$$

Then $u_{pq} \in W^{1,p}(\Omega)$, but $u_{pq} \notin L^q(\Omega)$.

10 Anisotropic embeddings

In evolution problems, one deals with functions which depend on a space variable $x \in \Omega$ and time $t \in \omega$, where $\omega \subset \mathbb{R}$ is an open interval corresponding to the time of the process. For $1 \leq p, q < \infty$, we introduce the spaces

$$L^p(\omega\,;\, L^q(\Omega)) \;=\; \left\{u \in L^1(\Omega \times \omega)\,;\, |u|_{p,q,\Omega,\omega} := \left(\int_\omega |u(\cdot,t)|^p_{q,\Omega}\,\mathrm{d}t\right)^{1/p} < \infty\right\}\,, \tag{10.1}$$

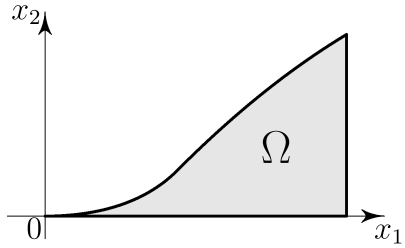

Figure 5: Non-Lipschitzian boundary.

with obvious modifications for $p = \infty$ or $q = \infty$.

We state explicitly one possible embedding result for such spaces, without going into much detail in the proof, which is fully analogous to the above ones.

Theorem 10.1 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, let ω be a bounded open interval, and let $W^{(1);p_0,q_0;p_1,q_1}(\omega;\Omega)$ be the space*

$$W^{1,(p_0,q_0);(p_1,q_1)}(\omega;\Omega) \;=\; \Big\{u \in L^1(\Omega\times\omega)\,;\ \frac{\partial u}{\partial t} \in L^{p_0}(\omega\,;\,L^{q_0}(\Omega)), \quad \frac{\partial u}{\partial x_i} \in L^{p_1}(\omega\,;\,L^{q_1}(\Omega)) \text{ for } i = 1,\dots,N\Big\}.$$

If $q_2 \ge \max\{q_0,q_1\}$, $p_2 \ge \max\{p_0,p_1\}$, and

$$\left(1-\frac{1}{p_0}+\frac{1}{p_2}\right)\left(\frac{1}{N}-\frac{1}{q_1}+\frac{1}{q_2}\right) > \left(\frac{1}{p_1}-\frac{1}{p_2}\right)\left(\frac{1}{q_0}-\frac{1}{q_2}\right), \tag{10.2}$$

then the space $W^{(1);p_0,q_0;p_1,q_1}(\omega;\Omega)$ is compactly embedded in $L^{p_2}(\omega\,;\,L^{q_2}(\Omega))$. If moreover (10.2) holds for $q_2 = \infty$ with the convention $1/\infty = 0$, that is,

$$\left(1-\frac{1}{p_0}+\frac{1}{p_2}\right)\left(\frac{1}{N}-\frac{1}{q_1}\right) > \frac{1}{q_0}\left(\frac{1}{p_1}-\frac{1}{p_2}\right), \tag{10.3}$$

then $W^{(1);p_0,q_0;p_1,q_1}(\omega;\Omega)$ is compactly embedded in $L^{p_2}(\omega\,;\,C(\bar\Omega))$. If (10.2) holds for $p_2 = q_2 = \infty$, that is,

$$\frac{p_0'}{p_1 q_0} + \frac{1}{q_1} < \frac{1}{N}, \tag{10.4}$$

then the space $W^{1,(p_0,q_0);(p_1,q_1)}(\omega;\Omega)$ is compactly embedded in $C(\bar\Omega\times\bar\omega)$.

Hint for the proof. Consider as before the extensions to the space $W^{1,(p_0,q_0);(p_1,q_1)}(\mathbb{R};\mathbb{R}^N)$, where the norms $|\cdot|_{p_i,q_i,\Omega,\omega}$ are denoted again for simplicity as $|\cdot|_{p_i,q_i}$, $i = 0,1$. For $\sigma \in (0,1]$ and $u \in W^{(1);p_0,q_0;p_1,q_1}(\mathbb{R};\mathbb{R}^N)$, we define regularizations analogous to (4.3) in the form

$$u^\sigma(x,t) \;=\; \sigma^{-N-\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\varphi\left(\frac{x-y}{\sigma},\frac{t-s}{\sigma^\lambda}\right) u(y,s)\,\mathrm{d}y\,\mathrm{d}s\,, \tag{10.5}$$

where φ is a smooth nonnegative function on $\mathbb{R}^{N+1}$, which vanishes outside $B_1(0)\times(-1,1)$, and

$$\int_{-1}^1\int_{B_1(0)}\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t \;=\; 1\,.$$

The number λ is to be chosen as

$$\lambda \;=\; \frac{1+N\left(\frac{1}{q_0}-\frac{1}{q_1}\right)}{\frac{1}{p_0'}+\frac{1}{p_1}}\,. \tag{10.6}$$

Note that $\lambda>0$ by (10.2). A computation similar to (8.4)–(8.6) yields

$$\begin{aligned}\frac{\partial}{\partial\sigma}u^\sigma(x,t) &= -\lambda\sigma^{-N-1}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\Phi_0\left(\frac{x-y}{\sigma},\frac{t-s}{\sigma^\lambda}\right)\frac{\partial u}{\partial s}(y,s)\,\mathrm{d}y\,\mathrm{d}s\\ &\quad -\sigma^{-N-\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\left\langle\Phi_1\left(\frac{x-y}{\sigma},\frac{t-s}{\sigma^\lambda}\right),\nabla_y u(y,s)\right\rangle\mathrm{d}y\,\mathrm{d}s\,,\end{aligned} \tag{10.7}$$

where $\Phi_0(\xi,\tau)=\tau\varphi(\xi,\tau)$, $\Phi_1(\xi,\tau)=\xi\varphi(\xi,\tau)$, hence

$$|u^\beta(x,t)-u^\alpha(x,t)| \;\le\; \lambda\mathcal{I}_0(x,t)+\mathcal{I}_1(x,t) \tag{10.8}$$

for $0<\alpha<\beta\le 1$, where

$$\left.\begin{aligned}\mathcal{I}_0(x,t) &= \int_\alpha^\beta\sigma^{-N-1}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\left|\Phi_0\left(\tfrac{x-y}{\sigma},\tfrac{t-s}{\sigma^\lambda}\right)\right|\left|\tfrac{\partial u}{\partial s}(y,s)\right|\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\sigma\,,\\ \mathcal{I}_1(x,t) &= \int_\alpha^\beta\sigma^{-N-\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\left|\Phi_1\left(\tfrac{x-y}{\sigma},\tfrac{t-s}{\sigma^\lambda}\right)\right|\left|\nabla_y u(y,s)\right|\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\sigma\,.\end{aligned}\right\} \tag{10.9}$$

Let (10.2) hold. With the intention to use Young's inequality for convolutions again, we introduce the numbers r_0,s_0,r_1,s_1 by the identities

$$\frac{1}{r_0}=1-\frac{1}{q_0}+\frac{1}{q_2}\,,\quad \frac{1}{s_0}=1-\frac{1}{p_0}+\frac{1}{p_2}\,,\quad \frac{1}{r_1}=1-\frac{1}{q_1}+\frac{1}{q_2}\,,\quad \frac{1}{s_1}=1-\frac{1}{p_1}+\frac{1}{p_2}\,. \tag{10.10}$$

We use again the notation $\int_X \mathrm{d}x$, $\int_Y \mathrm{d}y$ for $\int_{\mathbb{R}^N}\mathrm{d}x$, $\int_{\mathbb{R}^N}\mathrm{d}y$, and $\int_T \mathrm{d}t$, $\int_S \mathrm{d}s$ for $\int_{\mathbb{R}}\mathrm{d}t$, $\int_{\mathbb{R}}\mathrm{d}s$. For $t\in\mathbb{R}$, we have

$$\begin{aligned}|\mathcal{I}_0(\cdot,t)|_{q_2} &\overset{\text{Minkowski}}{\le} \int_\alpha^\beta\sigma^{-N-1}\int_S\left(\int_X\left(\int_Y\left|\Phi_0\left(\frac{x-y}{\sigma},\frac{t-s}{\sigma^\lambda}\right)\right|\left|\frac{\partial u}{\partial s}(y,s)\right|\mathrm{d}y\right)^{q_2}\mathrm{d}x\right)^{1/q_2}\mathrm{d}s\,\mathrm{d}\sigma\\ &\overset{\text{Young II}}{\le} \int_\alpha^\beta\sigma^{-N-1}\int_S\left|\Phi_0\left(\frac{\cdot}{\sigma},\frac{t-s}{\sigma^\lambda}\right)\right|_{r_0}\left|\frac{\partial u}{\partial s}(\cdot,s)\right|_{q_0}\mathrm{d}s\,\mathrm{d}\sigma\\ &= \int_\alpha^\beta\sigma^{-N-1+N/r_0}\int_S\left|\Phi_0\left(\cdot,\frac{t-s}{\sigma^\lambda}\right)\right|_{r_0}\left|\frac{\partial u}{\partial s}(\cdot,s)\right|_{q_0}\mathrm{d}s\,\mathrm{d}\sigma\,,\end{aligned} \tag{10.11}$$

hence

$$\begin{aligned}|\mathcal{I}_0|_{p_2,q_2} &\overset{\text{Minkowski}}{\le} \int_\alpha^\beta\sigma^{-N-1+N/r_0}\left(\int_T\left(\int_S\left|\Phi_0\left(\cdot,\frac{t-s}{\sigma^\lambda}\right)\right|_{r_0}\left|\frac{\partial u}{\partial s}(\cdot,s)\right|_{q_0}\mathrm{d}s\right)^{p_2}\mathrm{d}t\right)^{1/p_2}\mathrm{d}\sigma\\ &\overset{\text{Young II}}{\le} \int_\alpha^\beta\sigma^{-N-1+N/r_0}\left|\Phi_0\left(\cdot,\frac{\cdot}{\sigma^\lambda}\right)\right|_{s_0,r_0}\left|\frac{\partial u}{\partial s}\right|_{p_0,q_0}\mathrm{d}\sigma\\ &= |\Phi_0|_{s_0,r_0}\left|\frac{\partial u}{\partial s}\right|_{p_0,q_0}\int_\alpha^\beta\sigma^{-N-1+N/r_0+\lambda/s_0}\,\mathrm{d}\sigma\,.\end{aligned} \tag{10.12}$$

Similarly,

$$
\begin{aligned}
|\mathcal{I}_1(\cdot,t)|_{q_2} &\overset{\text{Minkowski}}{\leq} \int_\alpha^\beta \sigma^{-N-\lambda} \int_S \left(\int_X \left(\int_Y \left| \Phi_1 \left(\frac{x-y}{\sigma}, \frac{t-s}{\sigma^\lambda} \right) \right| |\nabla_y u(y,s)| \, \mathrm{d}y \right)^{q_2} \mathrm{d}x \right)^{1/q_2} \mathrm{d}s \, \mathrm{d}\sigma \\
&\overset{\text{Young II}}{\leq} \int_\alpha^\beta \sigma^{-N-\lambda} \int_S \left| \Phi_1 \left(\frac{\cdot}{\sigma}, \frac{t-s}{\sigma^\lambda} \right) \right|_{r_1} |\nabla_y u(\cdot,s)|_{q_1} \, \mathrm{d}s \, \mathrm{d}\sigma \\
&= \int_\alpha^\beta \sigma^{-N-\lambda+N/r_1} \int_S \left| \Phi_1 \left(\cdot, \frac{t-s}{\sigma^\lambda} \right) \right|_{r_1} |\nabla_y u(\cdot,s)|_{q_1} \, \mathrm{d}s \, \mathrm{d}\sigma \,, \qquad (10.13)
\end{aligned}
$$

hence

$$
\begin{aligned}
|\mathcal{I}_1|_{p_2,q_2} &\overset{\text{Minkowski}}{\leq} \int_\alpha^\beta \sigma^{-N-\lambda+N/r_1} \left(\int_T \left(\int_S \left| \Phi_1 \left(\cdot, \frac{t-s}{\sigma^\lambda} \right) \right|_{r_1} |\nabla_y u(\cdot,s)|_{q_1} \, \mathrm{d}s \right)^{p_2} \mathrm{d}t \right)^{1/p_2} \mathrm{d}\sigma \\
&\overset{\text{Young II}}{\leq} \int_\alpha^\beta \sigma^{-N-\lambda+N/r_1} \left| \Phi_1 \left(\cdot, \frac{\cdot}{\sigma^\lambda} \right) \right|_{s_1,r_1} |\nabla_y u|_{p_1,q_1} \, \mathrm{d}\sigma \\
&= |\Phi_1|_{s_1,r_1} |\nabla_y u|_{p_1,q_1} \int_\alpha^\beta \sigma^{-N-\lambda+N/r_1+\lambda/s_1} \, \mathrm{d}\sigma \,. \qquad (10.14)
\end{aligned}
$$

Set

$$
\kappa = N \left(\frac{1}{p_0'} + \frac{1}{p_1} \right)^{-1} \left(\left(1 - \frac{1}{p_0} + \frac{1}{p_2} \right) \left(\frac{1}{N} - \frac{1}{q_1} + \frac{1}{q_2} \right) - \left(\frac{1}{q_0} - \frac{1}{q_2} \right) \left(\frac{1}{p_1} - \frac{1}{p_2} \right) \right) .
$$

Then $\kappa > 0$ by (10.2), and we have

$$
-N - 1 + \frac{N}{r_0} + \frac{\lambda}{s_0} = -N - \lambda + \frac{N}{r_1} + \frac{\lambda}{s_1} = \kappa - 1 \,.
$$

Combining (10.8) with (10.9), (10.12), and (10.14) yields

$$
|u^\beta - u^\alpha|_{p_2,q_2} \leq C_{p_0,p_1,p_2,q_0,q_1,q_2} (\beta^\kappa - \alpha^\kappa) \left(\left| \frac{\partial u}{\partial t} \right|_{p_0,q_0} + |\nabla_x u|_{p_1,q_1} \right) , \qquad (10.15)
$$

and we obtain the result similarly as in Corollaries 8.4 or 8.5. ■

We can consider a higher degree of anisotropy, where also the degree of differentiability is different in different directions. General formulas can again be found in [2, 3]. Here we show one example which is typical for parabolic PDEs.

Theorem 10.2 *Let $\Omega \subset \mathbb{R}^N$ be an open bounded connected set with Lipschitzian boundary, let ω be a bounded open interval, and let $W^{1,(p_0,q_0);2,(p_1,q_1)}(\omega;\Omega)$ be the space*

$$
\begin{aligned}
W^{1,(p_0,q_0);2,(p_1,q_1)}(\omega;\Omega) = \Big\{ & u \in L^1(\Omega \times \omega) \,;\ \frac{\partial u}{\partial t} \in L^{p_0}(\omega\,;\, L^{q_0}(\Omega)), \\
& \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^{p_1}(\omega\,;\, L^{q_1}(\Omega)) \ \text{for} \ i,j = 1,\dots,N \Big\} .
\end{aligned}
$$

If $q_2 \geq \max\{q_0, q_1\}$, $p_2 \geq \max\{p_0, p_1\}$, *and*

$$\left(1 - \frac{1}{p_0} + \frac{1}{p_2}\right)\left(\frac{1}{N} - \frac{1}{q_1} + \frac{1}{q_2}\right) > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)\left(\frac{1}{N} + \frac{1}{q_0} - \frac{1}{q_2}\right), \tag{10.16}$$

then for every bounded set $B \subset W^{1,(p_0,q_0);2,(p_1,q_1)}(\omega;\Omega)$ *the set* $\{\nabla_x u; u \in B\}$ *is precompact in* $L^{p_2}(\omega\,;\,L^{q_2}(\Omega))$.

Sketch the proof. We choose

$$\lambda \;=\; \frac{2 + N\left(\frac{1}{q_0} - \frac{1}{q_1}\right)}{\frac{1}{p_0'} + \frac{1}{p_1}}. \tag{10.17}$$

and repeat the computations from the proof of Theorem 10.1 with u^σ as in (10.5) for $\sigma \in (0,1]$, and $u \in B$. Note that for $j \in \{1,\dots,N\}$ we have

$$\frac{\partial u^\sigma}{\partial x_j}(x,t) \;=\; \sigma^{-N-\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\varphi\left(\frac{x-y}{\sigma}, \frac{t-s}{\sigma^\lambda}\right)\frac{\partial u}{\partial y_j}(y,s)\,\mathrm{d}y\,\mathrm{d}s\,, \tag{10.18}$$

and

$$\begin{aligned}
\frac{\partial}{\partial\sigma}\left(\frac{\partial u^\sigma}{\partial x_j}\right)(x,t) \;&=\; -\lambda\sigma^{-N-1}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\Phi_0\left(\frac{x-y}{\sigma}, \frac{t-s}{\sigma^\lambda}\right)\frac{\partial^2 u}{\partial s\partial y_j}(y,s)\,\mathrm{d}y\,\mathrm{d}s\\
&\quad -\sigma^{-N-\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\left\langle\Phi_1\left(\frac{x-y}{\sigma}, \frac{t-s}{\sigma^\lambda}\right), \frac{\partial\nabla_y u}{\partial y_j}(y,s)\right\rangle\mathrm{d}y\,\mathrm{d}s\\
&=\; \lambda\sigma^{-N-2}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\Phi_j\left(\frac{x-y}{\sigma}, \frac{t-s}{\sigma^\lambda}\right)\frac{\partial u}{\partial s}(y,s)\,\mathrm{d}y\,\mathrm{d}s\\
&\quad -\sigma^{-N-\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\left\langle\Phi_1\left(\frac{x-y}{\sigma}, \frac{t-s}{\sigma^\lambda}\right), \frac{\partial\nabla_y u}{\partial y_j}(y,s)\right\rangle\mathrm{d}y\,\mathrm{d}s\,,
\end{aligned} \tag{10.19}$$

where $\Phi_j(\xi,\tau) = \frac{\partial\Phi_0}{\partial\xi_j}(\xi\tau)$, hence

$$\left|\frac{\partial u^\beta}{\partial x_j}(x,t) - \frac{\partial u^\alpha}{\partial x_j}(x,t)\right| \;\leq\; \lambda\mathcal{I}_j(x,t) + \mathcal{I}_1(x,t) \tag{10.20}$$

for $0 < \alpha < \beta \leq 1$, where

$$\left.\begin{aligned}
\mathcal{I}_j(x,t) \;&=\; \int_\alpha^\beta \sigma^{-N-2}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\left|\Phi_j\left(\tfrac{x-y}{\sigma}, \tfrac{t-s}{\sigma^\lambda}\right)\right|\left|\tfrac{\partial u}{\partial s}(y,s)\right|\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\sigma\,,\\
\mathcal{I}_1(x,t) \;&=\; \int_\alpha^\beta \sigma^{-N-\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}^N}\left|\Phi_1\left(\tfrac{x-y}{\sigma}, \tfrac{t-s}{\sigma^\lambda}\right)\right|\left|\tfrac{\partial\nabla_y u}{\partial y_j}(y,s)\right|\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}\sigma\,,
\end{aligned}\right\} \tag{10.21}$$

and we proceed as in the proof of Theorem 10.1. ■

Note that the order of integration in (10.1) cannot be reversed. For $p \geq q$ we have by Remark 2.5 that $L^q(\Omega\,;\,L^p(\omega))$ is embedded into $L^p(\omega\,;\,L^q(\Omega))$, but the opposite inclusion does not hold, see Example 2.4. On the other hand, denoting

$$\begin{aligned}
W^{1,(q_0,p_0);(q_1,p_1)}(\Omega;\omega) \;&=\; \Big\{u \in L^1(\Omega\times\omega)\,;\, \frac{\partial u}{\partial t} \in L^{q_0}(\Omega\,;\,L^{p_0}(\omega)),\\
&\qquad \frac{\partial u}{\partial x_i} \in L^{q_1}(\Omega\,;\,L^{p_1}(\omega)) \text{ for } i = 1,\dots,N\Big\},
\end{aligned}$$

we may repeat the computations in (10.11)–(10.14) with reversed order of integration, to check that conditions (10.2), (10.4) remain valid for the compact embedding of $W^{1,(q_0,p_0);(q_1,p_1)}(\Omega;\omega)$ into $L^{q_2}(\Omega; L^{p_2}(\omega))$ and $C(\bar\Omega \times \bar\omega)$, respectively. Let us mention one important particular case which frequently occurs in applications. We omit the proof which is the same as for the other cases.

Corollary 10.3 *If* $q_2 \geq \max\{q_0, q_1\}$, *and*

$$\frac{1}{p_0'}\left(\frac{1}{N} - \frac{1}{q_1} + \frac{1}{q_2}\right) > \frac{1}{p_1}\left(\frac{1}{q_0} - \frac{1}{q_2}\right), \tag{10.22}$$

then the space $W^{1,(q_0,p_0);(q_1,p_1)}(\Omega;\omega)$ *is compactly embedded in* $L^{q_2}(\Omega\,;\,C(\bar\omega))$.

Embeddings of function spaces that are fully anisotropic with respect to all variables can be treated in the same way. For a vector $\mathbf{p} = (p_1, \dots, p_N)$, $1 \leq p_i \leq \infty$, we define the space $L^{\mathbf{p}}_R(\mathbb{R}^N)$ as the subspace of $L^1_R(\mathbb{R}^N)$ of functions u of variable $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ with support in the ball $B_R(0) \subset \mathbb{R}^N$ and such that the norm

$$\|u\|_{\mathbf{p}} = \left(\int_{\mathbb{R}}\left(\dots\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|u(x)|^{p_1}\,\mathrm{d}x_1\right)^{p_2/p_1}\mathrm{d}x_2\,\dots\right)^{p_N/p_{N-1}}\mathrm{d}x_N\right)^{1/p_N} \tag{10.23}$$

is finite, where the expression

$$\left(\int_{\mathbb{R}} U^{p_i}(x_i)\,\mathrm{d}x_i\right)^{1/p_i} \qquad \text{for } p_i = \infty$$

has to be interpreted as

$$\sup_{x_i \in \mathbb{R}} \operatorname{ess} U(x_i)\,.$$

For a matrix $\mathbf{P} = (P_{ij})_{i,j=1}^N$, $P_{ij} = 1/p_{ij}$, $1 \leq p_{ij} \leq \infty$, we define the anisotropic Sobolev space

$$W_R^{1,\mathbf{P}}(\mathbb{R}^N) = \left\{u \in L^1_R(\mathbb{R}^N) : \frac{\partial u}{\partial x_i} \in L_R^{\mathbf{p}_i}(\mathbb{R}^N), i = 1, \dots, N\right\}, \tag{10.24}$$

where $\mathbf{p}_i = (p_{i1}, \dots, p_{iN})$. We denote by $\mathbf{I}$ the identity $N \times N$ matrix, and by $\mathbf{1}$ the vector $\mathbf{1} = (1, 1, \dots, 1)$. The spectral radius $\varrho(\mathbf{P})$ of $\mathbf{P}$ is defined as

$$\varrho(\mathbf{P}) = \max\{|\lambda| : \lambda \in \mathbb{C},\ \det(\mathbf{P} - \lambda\mathbf{I}) = 0\} = \limsup_{n\to\infty} |\mathbf{P}^n|^{1/n}. \tag{10.25}$$

Theorem 10.4 *Let* $\mathbf{P}$ *be a matrix as above, and let* $\mathbf{q} = (q_1, \dots, q_N)$, $1 \leq q_j \leq \infty$ *be a given vector such that* $q_j \geq p_{ij}$ *for all* $i, j = 1, \dots, N$. *We define the matrix* $\mathbf{P^q} = (P_{ij}^q)_{i,j=1}^N$ *with entries*

$$P_{ij}^q = \frac{1}{p_{ij}} - \frac{1}{q_j}$$

Let $\varrho(\mathbf{P^q}) < 1$. *Then* $W_R^{1,\mathbf{P}}(\mathbb{R}^N)$ *is compactly embedded in* $L_R^{\mathbf{q}}(\mathbb{R}^N)$.

The proof of Theorem 10.4 can be found in [9], see also [8]. The idea is to put

$$\mathbf{b} = (\mathbf{I} - \mathbf{P}^{\mathbf{q}})^{-1}\mathbf{1} = (\mathbf{I} + \mathbf{P}^{\mathbf{q}} + (\mathbf{P}^{\mathbf{q}})^2 + \dots)\,\mathbf{1}, \quad \mathbf{b} = (b_1, \dots, b_N),$$

choose the regularizations in the form

$$u^\sigma(x) = \sigma^{-|\mathbf{b}|} \int_{\mathbb{R}^N} \varphi\Big(\frac{x-y}{\sigma^{\mathbf{b}}}\Big) u(y)\,\mathrm{d}y \tag{10.26}$$

with φ as in (1.1), where we denote $|\mathbf{b}| = \sum_{i=1}^N b_i$ and

$$\frac{x-y}{\sigma^{\mathbf{b}}} = \left(\frac{x_1 - y_1}{\sigma^{b_1}}, \dots, \frac{x_N - y_N}{\sigma^{b_N}}\right),$$

and use the Minkowski and the Young inequality for convolutions as above to obtain the estimate

$$|u - u^\sigma|_{\mathbf{q}} \le C\sigma \sum_{i=1}^N |\partial_i u|_{\mathbf{p}_i} \tag{10.27}$$

with a constant $C > 0$ independent of σ.

11 Interpolations

We first recall the following classical interpolation result in L^p spaces.

Proposition 11.1 *Let $\Omega \subset \mathbb{R}^N$ be an open set (bounded or unbounded), and let $1 \le p_0 < p_1 \le \infty$ be given. If $u \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$, then $u \in L^p(\Omega)$ for all $p \in [p_0, p_1]$, and we have*

$$|u|_{p,\Omega} \ \le \ |u|_{p_0,\Omega}^{1-\alpha} |u|_{p_1,\Omega}^{\alpha}$$

for all $u \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$, where

$$\alpha = \frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \frac{1}{p_1}}\,.$$

Proof. Set $q = p_1/\alpha p$. Then $q' = p_0/(1-\alpha)p$, and we may use Hölder's inequality to obtain

$$\begin{aligned} |u|_{p,\Omega} &= \left(\int_\Omega |u(x)|^{(1-\alpha)p} |u(x)|^{\alpha p}\,\mathrm{d}x\right)^{1/p} \\ &\le \left(\int_\Omega |u(x)|^{(1-\alpha)pq'}\,\mathrm{d}x\right)^{1/pq'} \left(\int_\Omega |u(x)|^{\alpha pq}\,\mathrm{d}x\right)^{1/pq} \\ &= |u|_{p_0,\Omega}^{1-\alpha} |u|_{p_1,\Omega}^{\alpha}\,. \end{aligned}$$

■

We now establish an interpolation formula between L^p spaces and Sobolev spaces.

Theorem 11.2 *Let* $p, q, s \in (1, \infty)$ *be such that*

$$\frac{1}{s} > \frac{1}{q} > \frac{1}{p} - \frac{1}{N},$$

and set

$$\kappa := 1 - N\left(\frac{1}{p} - \frac{1}{q}\right), \quad \gamma = N\left(\frac{1}{s} - \frac{1}{q}\right).$$

Then there exists $C_{pqs} > 0$ *such that for every* $u \in W^{1,p}(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$ *and every* $\sigma \in (0, 1]$ *we have*

$$|u|_q \leq C_{pqs}\left(\sigma^{-\gamma}|u|_s + \sigma^{\kappa}\,|\nabla u|_p\right). \tag{11.1}$$

Proof. The assertion follows from (4.4) and (8.3) provided $q \geq p$. In particular, for $q = p$ we have $\kappa = 1$, $\gamma = \gamma_0 := N(1/s - 1/p)$, and

$$|u|_p \leq C_{pps}\left(\sigma^{-\gamma_0}|u|_s + \sigma\,|\nabla u|_p\right). \tag{11.2}$$

Let now $q < p$. By Proposition 11.1 we have

$$|u|_q \leq |u|_s^{1-\alpha}|u|_p^{\alpha}, \quad \text{with } \alpha = \frac{\frac{1}{s} - \frac{1}{q}}{\frac{1}{s} - \frac{1}{p}}.$$

This yields

$$|u|_p \leq C_{pps}^{\alpha}\left(\sigma^{-\alpha\gamma_0}|u|_s + \sigma^{\alpha}\,|u|_s^{1-\alpha}\,|\nabla u|_p^{\alpha}\right).$$

We now use inequality (2.5) with p replaced by $1/\alpha$, and with $x = \mu\sigma^{\alpha}|\nabla u|_p^{\alpha}$, $y = |u|_s^{1-\alpha}/\mu$, where we set $\mu = \sigma^{(1-\alpha)\alpha\gamma_0}$, and obtain

$$\sigma^{\alpha}\,|u|_s^{1-\alpha}\,|\nabla u|_p^{\alpha} \leq \alpha\sigma^{1+(1-\alpha)\gamma_0}\,|\nabla u|_p + (1-\alpha)\sigma^{-\alpha\gamma_0}\,|u|_s.$$

Hence,

$$|u|_p \leq 2C_{pps}^{\alpha}\left(\sigma^{-\alpha\gamma_0}|u|_s + \sigma^{1+(1-\alpha)\gamma_0}\,|\nabla u|_p\right),$$

which is precisely (11.1). ■

We conclude this text with the famous Gagliardo-Nirenberg inequality.

Corollary 11.3 ***(Gagliardo-Nirenberg inequality)*** *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded connected set with Lipschitzian boundary, and let*

$$\frac{1}{s} > \frac{1}{q} > \frac{1}{p} - \frac{1}{N}.$$

Set

$$\varrho = \frac{\frac{1}{s} - \frac{1}{q}}{\frac{1}{N} + \frac{1}{s} - \frac{1}{p}}.$$

Then there exists a constant $K_{pqs} > 0$ *such that for every* $u \in W^{1,p}(\Omega)$ *we have*

$$|u|_{q,\Omega} \leq K_{pqs}\left(|u|_{s,\Omega} + |u|_{s,\Omega}^{1-\varrho}\|u\|_{1;p,\Omega}^{\varrho}\right). \tag{11.3}$$

Proof. As in the proof of Corollary 8.4, we set $u_* = E_p u$. By Theorem 11.2, we have

$$|u_*|_q \;\leq\; C_{pqs}\left(\sigma^{-\gamma}|u_*|_s + \sigma^{\kappa}\,|\nabla u_*|_p\right)\,. \tag{11.4}$$

If $|\nabla u_*|_p > |u_*|_s$, then we set

$$\sigma = \left(\frac{|u_*|_s}{|\nabla u_*|_p}\right)^{1/(\gamma+\kappa)}\,,$$

otherwise we choose $\sigma = 1$. In both cases we obtain

$$|u_*|_q \;\leq\; 2C_{pqs}\left(|u_*|_s + |u_*|_s^{\kappa/(\gamma+\kappa)}\,|\nabla u_*|_p^{\gamma/(\gamma+\kappa)}\right)\,. \tag{11.5}$$

We have $\kappa/(\gamma+\kappa) = 1-\varrho$, $\gamma/(\gamma+\kappa) = \varrho$, and the desired result follows from Theorem 5.2. ∎

It is in principle possible to derive from (10.27) the corresponding interpolation inequalities also for anisotropic spaces. We leave the particular cases to the reader.

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