Compressible fluid flows driven by temperature gradient

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Problem formulation

Navier–Stokes–Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \mathbb{D}_x \mathbf{u}) + \boxed{\varrho \nabla_x G}$$

$$\partial_t(\varrho e(\varrho, \vartheta)) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u}) + \nabla_x \mathbf{q}(\vartheta, \nabla_x \vartheta) = \mathbb{S} : \mathbb{D}_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}$$

Boundary conditions

$$\Omega \subset R^d,\ d = 2, 3 \text{ bounded, regular}$$

$$\mathbf{u}|_{\partial\Omega} = 0,\ \boxed{\vartheta|_{\partial\Omega} = \vartheta_B}$$

$$G = G(x),\ \vartheta_B = \vartheta_B(x)$$

$$\mathbb{S}(\vartheta, \mathbb{D}_x \mathbf{u}) = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}$$

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta)\nabla_x \vartheta$$

Crucial problems

- **Global existence.** Global-in-time existence of solutions "far from equilibrium"
- **Levinson dissipativity or bounded absorbing set.** Any global–in–time weak solution to the Navier–Stokes–Fourier system in a domain with impermeable boundary endowed with the Dirichlet boundary conditions for the temperature enters eventually a bounded absorbing set.
- **Asymptotic compactness.** Any bounded family of global solutions is precompact in a suitable topology of the trajectory space, whereas any of its accumulation points represents a weak solution of the same problem.

Problem with conservative boundary conditions

Conservative boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \ \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

Driving force

$$\varrho\mathbf{f}, \ \mathbf{f} = \mathbf{f}(x)$$

- $$\mathbf{f} = \nabla_x G, \ G = G(x) \ \Rightarrow \ \varrho \to \varrho_S, \ \vartheta \to \vartheta_S, \ \mathbf{m} = \varrho\mathbf{u} \to 0 \text{ as } t \to \infty$$
  $$\vartheta_S \text{ - a positive constant}, \ \nabla_x p(\varrho_S, \vartheta_S) = \varrho_S \nabla_x G$$
- $$\mathbf{f} \neq \nabla_x G \ \Rightarrow \ \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) \right] \ \mathrm{d}x \to \infty \text{ as } t \to \infty$$

Static solutions

$$\mathbf{u}_S = 0$$

$$\mathbf{u}_S = 0 \ \Rightarrow \ \mathrm{div}_x(\kappa(\vartheta_S)\nabla_x \vartheta_S) = 0, \ \vartheta_S|_{\partial\Omega} = \vartheta_B$$

$$\nabla_x p(\varrho_S, \vartheta_S) = \varrho_S \nabla_x G \ \Rightarrow \ \mathbf{curl}_x(\varrho_S \nabla_x G) = \nabla_x \varrho_S \times \nabla_x G = 0$$

$$\partial_\varrho p(\varrho_S, \vartheta_S)\nabla_x \varrho_S + \partial_\vartheta p(\varrho, \vartheta)\nabla_x \vartheta_S = \varrho_S \nabla_x G$$

$$\Rightarrow$$

$$\nabla_x \vartheta_S \times \nabla_x G = 0$$

Weak solutions, I

Equation of continuity

$$\int_T^\infty \int_\Omega \left[\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \mathrm{d}t = 0,$$

$$\int_T^\infty \int_\Omega \Big[b(\varrho) \partial_t \varphi + b(\varrho) \mathbf{u} \cdot \nabla_x \varphi + \Big(b(\varrho) - b'(\varrho) \varrho \Big) \mathrm{div}_x \mathbf{u} \varphi \Big] \ \mathrm{d}x \mathrm{d}t = 0$$

for any $\varphi \in C^1_c((T, \infty) \times \overline{\Omega})$, and any $b \in C^1(R)$, $b' \in C_c(R)$

Momentum equation

$$\int_T^\infty \int_\Omega \left[\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \mathrm{d}t$$
$$= \int_T^\infty \int_\Omega \left[\mathbb{S} : \nabla_x \boldsymbol{\varphi} - \varrho \nabla_x G \cdot \boldsymbol{\varphi} \right] \ \mathrm{d}x \mathrm{d}t,$$

for any $\boldsymbol{\varphi} \in C^1_c((T, \infty) \times \Omega; R^3)$

Weak solutions, II

Entropy inequality

$$-\int_T^\infty \int_\Omega \left[\varrho s \partial_t \varphi + \varrho s \mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \mathrm{d}t$$
$$\geq \int_T^\infty \int_\Omega \frac{\varphi}{\vartheta} \left[\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right] \ \mathrm{d}x \mathrm{d}t$$

for any $\varphi \in C^1_c((T, \infty) \times \Omega)$, $\varphi \geq 0$

Ballistic energy

$$E_{\widetilde{\vartheta}} = \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e - \widetilde{\vartheta} \varrho s \right]$$

$$\widetilde{\vartheta} > 0, \ \widetilde{\vartheta}|_{\partial\Omega} = \vartheta_B$$

Ballistic energy boundary flux

$$\mathbf{q} \cdot \mathbf{n} - \frac{\widetilde{\vartheta}}{\vartheta} \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Weak solutions, III

Ballistic energy balance

$$
\begin{aligned}
-\int_T^\infty \partial_t \psi \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e - \widetilde{\vartheta} \varrho s \right] \ \mathrm{d}x\mathrm{d}t \\
+ \int_T^\infty \psi \int_\Omega \frac{\widetilde{\vartheta}}{\vartheta} \left[\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right] \ \mathrm{d}x\mathrm{d}t \\
\leq \int_T^\infty \psi \int_\Omega \left[\varrho \mathbf{u} \cdot \nabla_x G - \varrho s \mathbf{u} \cdot \nabla_x \widetilde{\vartheta} - \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \widetilde{\vartheta} \right] \ \mathrm{d}x
\end{aligned}
$$

for any $\psi \in C^1_c(T;\infty)$, $\psi \geq 0$, and any $\widetilde{\vartheta} \in C^1([T;\infty) \times \overline{\Omega})$, $\widetilde{\vartheta}|_{\partial\Omega} = \vartheta_B$

- **Compatibility [Chaudhuri–EF 2021].** Smooth weak solutions are classical solutions
- **Weak–strong uniqueness [Chaudhuri–EF 2021].** A weak solution coincides with the strong solution as long as the latter exists

Constitutive theory, I

Gibbs' law

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

Thermodynamic stability

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \ \ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0$$

Pressure EOS

$$p(\varrho,\vartheta) = p_{\rm m}(\varrho,\vartheta) + p_{\rm rad}(\vartheta)$$

$$p_{\rm m}(\varrho,\vartheta) = \frac{2}{3}\varrho e_{\rm m}(\varrho,\vartheta), \ \ p_{\rm rad}(\vartheta) = \frac{a}{3}\vartheta^4, \ \ a > 0$$

Internal energy

$$e(\varrho,\vartheta) = e_{\rm m}(\varrho,\vartheta) + e_{\rm rad}(\varrho,\vartheta), \ \ e_{\rm rad}(\varrho,\vartheta) = \frac{a}{\varrho}\vartheta^4.$$

Constitutive theory, II

- $$\text{Gibbs' law} \ \Rightarrow \ p_m(\varrho, \vartheta) = \vartheta^{\frac{5}{2}} P \left(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \right)$$

- $$\text{Thermodynamic stability} \ \Rightarrow \ \frac{P(Z)}{Z^{\frac{5}{3}}} \searrow p_\infty > 0 \text{ as } Z \to \infty$$

- **Entropy.**
$$s(\varrho, \vartheta) = \mathcal{S} \left(\frac{\varrho}{\vartheta^{\frac{3}{2}}} \right) + \frac{4a}{3} \frac{\vartheta^3}{\varrho}, \ \mathcal{S}' < 0$$

- **Third law of thermodynamics.**
$$\mathcal{S}(Z) \searrow 0 \text{ as } Z \to \infty$$

Constitutive theory, III

Viscosity.

$$0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta),\ |\mu'(\vartheta)| \leq \overline{\mu}$$

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1+\vartheta)$$

Thermal conductivity.

$$0 < \underline{\kappa}(1+\vartheta^\beta) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^\beta),\ \beta > 6$$

Global existence [Chaudhuri, EF, 2021].
Under the above constitutive restrictions, the problem admits global–in–time weak solutions for any finite energy initial data and any sufficiently smooth boundary data

Bounded absorbing set

Bounded absorbing set [EF - A. Świerczewska-Gwiazda]
For any global–in–time weak solution $(\varrho, \vartheta, \mathbf{u})$ defined on a time interval (T, ∞), there exists a constant $\mathcal{E}_\infty$ that depends only on the boundary data and the total mass of the fluid

$$M = \int_\Omega \varrho \ \mathrm{d}x,$$

such that

$$\mathrm{ess} \limsup_{t \to \infty} \int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ \mathrm{d}x \leq \mathcal{E}_\infty, \ E(\varrho, \vartheta, \mathbf{u}) \equiv \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)$$

If, moreover,

$$\mathrm{ess} \limsup_{t \to T+} \int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ \mathrm{d}x \leq \mathcal{E}_0 < \infty,$$

then the convergence is uniform in $\mathcal{E}_0$. Specifically, for any $\varepsilon > 0$, there exists a time $\tau(\varepsilon, \mathcal{E}_0)$ such that

$$\mathrm{ess} \sup_{t > T + \tau(\varepsilon, \mathcal{E}_0)} \int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ \mathrm{d}x \leq \mathcal{E}_\infty + \varepsilon.$$

Principal difficulties

"Elastic pressure": $p(\varrho, \vartheta) \approx \varrho^{\frac{5}{3}}$ for $\vartheta \to 0$

Ballistic energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e - \widetilde{\vartheta} \varrho s - \varrho G \right) \, \mathrm{d}x$$

$$+ c_1(\vartheta_B) \left(\|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)} + \|\vartheta^{\frac{\beta}{2}}\|^2_{W^{1,2}(\Omega)} + \|\log(\vartheta)\|^2_{W^{1,2}(\Omega)} \right)$$

$$\leq c_2(\vartheta_B) \mathcal{S}(r) \int_\Omega \varrho |\mathbf{u}| \, \mathrm{d}x + \Lambda(\vartheta_B, r),$$

where $c_1 > 0$, $c_2 > 0$, $\Lambda(r) \to \infty$ as $r \to \infty$

$\boxed{\mathcal{S}(r) \to 0}$ as $r \to \infty$.

Asymptotic compactness

Asymptotic compactness [EF - A. Świerczewska-Gwiazda 2021]
Let $(\varrho_n, \vartheta_n, \mathbf{u}_n)_{n=1}^{\infty}$ be a sequence of weak solutions defined on the time intervals

$$(T_n, \infty),\ \ T_n \geq -\infty,\ \ T_n \to -\infty \text{ as } n \to \infty,$$

such that

$$\mathrm{ess}\sup_{t \to T_n} \int_\Omega E(\varrho_n, \vartheta_n, \mathbf{u}_n)(t, \cdot)\ \mathrm{d}x \leq \mathcal{E}_0.\ \int_\Omega \varrho\ \mathrm{d}x = M > 0,$$

Then there is a subsequence (not relabelled) such that

$$\varrho_n \to \varrho \text{ in } C_{\rm weak}([-M, M]; L^{\frac{5}{3}}(\Omega)) \cap C([-M, M]; L^1(\Omega)),$$
$$\vartheta_n \to \vartheta \text{ in } L^q((-M, M); L^4(\Omega)) \text{ for any } 1 \leq q < \infty,$$
$$\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^2((-M, M); W^{1,2}(\Omega; R^3))$$

for any $M > 0$, where the limit $(\varrho, \vartheta, \mathbf{u})$ is an entire weak solution defined for all $t \in R$ and satisfying

$$\int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot)\ \mathrm{d}x \leq \mathcal{E}_\infty \text{ for a.a. } t \in R.$$

Principal difficulties

Weak convergence of densities

$$\varrho_n \to \varrho \text{ in } C_{\rm weak}([-M, M]; L^{\frac{5}{3}}(\Omega))$$

$$\varrho_n(-M, \cdot) \ \boxed{\text{not compact in general}}$$

Oscillations defect

$$\mathcal{D}(t) = \int_\Omega \left[\overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \right] (t, \cdot) \ {\rm d}x$$

$$\frac{\rm d}{{\rm d}t} \mathcal{D} + \Psi(\mathcal{D}) \leq 0, \ \Psi(Z)Z > 0 \text{ for } Z \neq 0, \ t \in R$$

$$\mathcal{D} \text{ bounded } \Rightarrow \ \mathcal{D} \equiv 0$$

Strong convergence

$$\varrho_n \to \varrho \text{ in } C([-M, M]; L^1(\Omega))$$

Trajectory space, attractor

Trajectory space

$$\mathcal{T} = \cup_{L=1}^{\infty} \mathcal{T}_L,\ (\phi_n)_{n\geq 1},\ (\boldsymbol{\varphi}_n)_{n\geq 1}$$

where

$$\begin{aligned}\mathcal{T}_L = \Big\{(\varrho, S, \mathbf{m}) \ \Big| \ & \varrho \in L^\infty(R; W^{-k,2}(\Omega)),\ \langle \varrho; \phi_n \rangle \in C(R),\ n = 1, 2, \dots, \\ & \sup_{t\in R} \|\varrho(t,\cdot)\|_{W^{-k,2}(\Omega)} \leq L, \\ & \mathbf{m} \in L^\infty(R; W^{-k,2}(\Omega; R^3)),\ \langle \mathbf{m}; \boldsymbol{\varphi}_n \rangle \in C(R),\ n = 1, 2, \dots, \\ & \sup_{t\in R} \|\mathbf{m}(t,\cdot)\|_{W^{-k,2}(\Omega; R^3)} \leq L, \\ & S \in L^\infty(R; W^{-k,2}(\Omega)),\ \langle S; \phi_n \rangle \ \text{càglàd in } R,\ n = 1, 2, \dots, \\ & \sup_{t\in R} \|S(t,\cdot)\|_{W^{-k,2}(\Omega)} \leq L \Big\}.\end{aligned}$$

Attractor

$$\mathcal{A} = \Big\{(\varrho, S, \mathbf{m}) \ \Big| \ (\varrho, S, \mathbf{m}) \text{ a weak solution of the Navier–Stokes–Fourier system on the time interval } t \in R \Big\} \subset \mathcal{T}_L,\ L \text{ large enough}$$

Attractor

Trajectory attractor [EF - A. Świerczewska-Gwiazda 2021]
Let $M > 0$, $\mathcal{E}_0$ be given. Let $\mathcal{F}[M, \mathcal{E}_0]$ be a family of weak solutions to the Navier–Stokes–Fourier system on the time interval $(0, \infty)$ satisfying

$$\int_\Omega \varrho \ \mathrm{d}x = M, \ \operatorname{ess} \limsup_{\tau \to 0+} \int_\Omega E(\varrho, S, \mathbf{m})(\tau, \cdot) \ \mathrm{d}x \leq \mathcal{E}_0.$$

We identify the set $\mathcal{F}[M, \mathcal{E}_0]$ with a subset of the trajectory space $\mathcal{T}$ extending them by constant values for $\tau < 0$.

Then for any $\varepsilon > 0$, there exists a time $T(\varepsilon)$ such that

$d_{\mathcal{T}}[(\varrho, S, \mathbf{m})(\cdot + T); \mathcal{A}] < \varepsilon$ for any $(\varrho, S, \mathbf{m}) \in \mathcal{F}[M, \mathcal{E}_0]$ and any $T > T(\varepsilon)$.

Stationary statistical solutions

Stationary statistical solutions [EF - A. Świerczewska-Gwiazda 2021]
Let $\mathcal{U} \subset \mathcal{A}$ be a non-empty time–shift invariant set, meaning

$$(\varrho, S, \mathbf{m}) \in \mathcal{U} \ \Rightarrow \ (\varrho, S, \mathbf{m})(\cdot + T) \in \mathcal{U} \text{ for any } T \in R.$$

Then there exists a stationary statistical solution $\mathcal{V}$ supported by $\overline{\mathcal{U}}$:

- $\mathcal{V}$ is a Borel probability measure, $\mathcal{V} \in \mathfrak{P}(\overline{\mathcal{U}})$;
- $\mathrm{supp}\mathcal{V} \subset \overline{\mathcal{U}}$, where the closure of a $\mathcal{U}$ is a compact invariant set;
- $\mathcal{V}$ is shift invariant, i.e., $\mathcal{V}[\mathfrak{B}] = \mathcal{V}[\mathfrak{B}(\cdot + T)]$ for any Borel set $\mathfrak{B} \subset \mathcal{T}$ and any $T \in R$.

$$\Big(\mathcal{T}, \mathcal{B}(\mathcal{T}), \mathcal{V}\Big) \ - \text{probability basis}$$

$$\omega = (\varrho, S, \mathbf{m}), \ t \in R \times \omega \in \mathcal{T} \mapsto \omega(t) \text{ stationary process}$$

Ergodic means

Phase space

$$H = W^{-k,2}(\Omega) \times W^{-k,2}(\Omega) \times W^{-k,2}(\Omega; R^3).$$

Convergence of ergodic means [application of Birkhoff–Khinchin ergodic theorem]
Let $\mathcal{V}$ be a stationary statistical solution and $(\varrho, S, \mathbf{m})$ the associated stationary process. Let $F : H \to R$ be a Borel measurable function such that

$$\int_{\mathcal{T}} |F(\varrho(0,\cdot), S(0,\cdot), \mathbf{m}(0,\cdot)| \ \mathrm{d}\mathcal{V} < \infty.$$

Then there exists a measurable function $\overline{F}$,

$$\overline{F} : (\mathcal{T}, \mathcal{V}) \to R$$

such that

$$\frac{1}{T} \int_0^T F(\varrho(t,\cdot), S(t,\cdot), \mathbf{m}(t,\cdot)) \mathrm{d}t \to \overline{F} \text{ as } T \to \infty$$

$\mathcal{V}-$a.s. and in $L^1(\mathcal{T}, \mathcal{V})$.

Open questions

- Global existence of smooth solution vs. blow-up in a finite time, cf. Merle et al. 2019, Buckmaster et al. 2021
- The existence of bounded absorbing sets for more general state equations
- Extension to specific class of solutions, cf. EF, Gwiazda, Świerczewska–Gwiazda - time periodic solutions