Oscillatory solutions: Analysis and applications

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Introduction, Euler system

Prologue - Lax equivalence principle

Formulation for LINEAR problems

Peter D. Lax

- **Stability** - uniform bounds of approximate solutions
- **Consistency** - vanishing approximation error

$$\Longrightarrow$$

- **Convergence** - approximate solutions converge to exact solution

Euler system of gas dynamics

Leonhard Paul Euler 1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0,\ \mathbf{m} = \varrho \mathbf{u}$$

Momentum equation – Newton's second law

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0,\ p(\varrho) = a\varrho^\gamma$$

Impermeability and/or periodic boundary condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \Omega \subset R^d,\ \text{or } \Omega = \mathbb{T}^d$$

Far field conditions for unbounded domains

$$\mathbf{m} \to \mathbf{m}_\infty,\ \varrho \to \varrho_\infty \text{ as } |x| \to \infty$$

Initial state

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{m}(0,\cdot) = \mathbf{m}_0$$

Classical solutions

- **Local existence.** Classical solutions exist locally in time as long as the initial data are regular and the initial density strictly positive
- **Finite time blow–up.** Classical solutions develop singularity (become discontinuous) in a *finite* time for a fairly generic class of initial data

Mythology concerning Euler equations in several dimensions

- **Existence.** The long time existence of (possibly weak) solutions is not known
- **Uniqueness.** The is no (known) selection criterion to identify a unique solution (semiflow)
- **Computation.** Oscillatory solutions cannot be visualized by numerical simulation (weak convergence)

Weak (distributional) solutions

Jacques Hadamard 1865–1963

Laurent Schwartz 1915–2002

Mass conservation

$$\int_B \Big[\varrho(t_2,\cdot) - \varrho(t_1,\cdot) \Big] \mathrm{d}x = - \int_{t_1}^{t_2} \int_{\partial B} \varrho \mathbf{u} \cdot \mathbf{n} \ \mathrm{dS}_x \mathrm{d}t$$

$$\left[\int_\Omega \varrho\varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \Big] \ \mathrm{d}x\mathrm{d}t, \ \mathbf{m} \equiv \varrho \mathbf{u}$$

Momentum balance

$$\int_B \Big[\mathbf{m}(t_2,\cdot) - \mathbf{m}(t_1,\cdot) \Big] \ \mathrm{d}x$$

$$= - \int_{t_1}^{t_2} \int_{\partial B} \Big[\mathbf{m} \otimes \mathbf{u} \cdot \mathbf{n} + p(\varrho)\mathbf{n} \Big] \ \mathrm{dS}_x \ \mathrm{d}t$$

$$\left[\int_\Omega \mathbf{m} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

Time irreversibility – energy dissipation

Energy

$$\mathcal{E} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

$$p' \geq 0 \Rightarrow [\varrho, \mathbf{m}] \mapsto \begin{cases} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0 \\ P(\varrho) \text{ if } |\mathbf{m}| = 0,\ \varrho \geq 0 \\ \infty \text{ otherwise} \end{cases} \quad \text{is convex l.s.c}$$

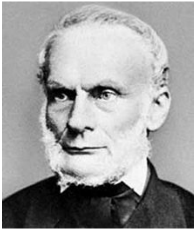
Energy balance (conservation)

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E} \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) = 0$$

Energy dissipation

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E} \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) \boxed{\leq} 0$$

$$E = \int_\Omega \mathcal{E} \ \mathrm{d}x,\ \partial_t E \leq 0,\ E(0+) = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$



Rudolf Clausius 1822–1888

Ill posedness

Theorem [A.Abbatiello, EF 2019]

Anna Abbatiello (TU Berlin)

Let $d = 2, 3$. Let ϱ_0, $\mathbf{m}_0$ be given such that

$$\varrho_0 \in \mathcal{R},\ 0 \le \underline{\varrho} \le \varrho_0 \le \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R},\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R},\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^{\infty} \subset (0, T)$ be an arbitrary (countable dense) set of times.
Then the Euler problem admits infinitely many weak solutions ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that

$$\varrho \in C_{\rm weak}([0, T]; L^{\gamma}(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

but

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)]\ \boxed{\text{is not}}\ \text{strongly continuous at any } \tau_i$$

Lecture II

Numerical analysis

FV numerical scheme

$$(\varrho_h^0, \mathbf{u}_h^0) = (\Pi_{\mathcal{T}} \varrho_0, \Pi_{\mathcal{T}} \mathbf{u}_0)$$

$$D_t \varrho_K^k + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} F_h(\varrho_h^k, \mathbf{u}_h^k) = 0$$

$$D_t(\varrho_h^k \mathbf{u}_h^k)_K + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \left(\mathbf{F}_h(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) + \overline{p(\rho_h^k)}\mathbf{n} - h^\beta [[\mathbf{u}_h^k]] \right) = 0.$$

Discrete time derivative

$$D_t r_K^k = \frac{r_K^k - r_K^{k-1}}{\Delta t}$$

Upwind, fluxes

$$\mathrm{Up}[r, \mathbf{v}] = \overline{r}\ \overline{\mathbf{v}} \cdot \mathbf{n} - \frac{1}{2} |\overline{\mathbf{v}} \cdot \mathbf{n}| [[r]]$$

$$F_h(r, \mathbf{v}) = \mathrm{Up}[r, \mathbf{v}] - h^\alpha [[r]]$$

Mária Lukáčová (Mainz)

Hana Mizerová (Bratislava)

Consistent approximation

Continuity equation

$$\int_0^T \int_\Omega [\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi] \, \mathrm{d}x\mathrm{d}t = - \int_\Omega \varrho_{0,n} \varphi(0, \cdot) \, \mathrm{d}x + e_{1,n}[\varphi]$$

for any $\varphi \in C_c^1([0, T) \times \overline{\Omega})$

Momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + 1_{\varrho_n > 0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \boldsymbol{\varphi} + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_{0,n} \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x + e_{2,n}[\boldsymbol{\varphi}]$$

for any $\boldsymbol{\varphi} \in C_c^1([0, T) \times \overline{\Omega}; R^d) \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$

Energy dissipation

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] (\tau, \cdot) \mathrm{d}x \leq \mathcal{E}_{0,n}$$

Stability and Consistency

Stability

$$\limsup_{n\to\infty} \mathcal{E}_{0,n} < \infty$$

Data compatibility

$$\int_\Omega \varrho_{0,n}\varphi \, \mathrm{d}x \to \int_\Omega \varrho_0 \varphi \, \mathrm{d}x \text{ for any } \varphi \in C_c^\infty(\Omega)$$

$$\int_\Omega \mathbf{m}_{0,n} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \to \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x \text{ for any } \boldsymbol{\varphi} \in C_c^\infty(\Omega; R^d)$$

$$\limsup_{n\to\infty} \mathcal{E}_{0,n} \leq \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \mathrm{d}x$$

Vanishing approximation error

$$e_{1,n}[\varphi] \to 0 \text{ as } n \to \infty \text{ for any } \phi \in C_c^\infty([0,T) \times \overline{\Omega})$$

$$e_{2,n}[\boldsymbol{\varphi}] \to 0 \text{ as } n \to \infty \text{ for any } \boldsymbol{\varphi} \in C_c^\infty([0,T) \times \overline{\Omega}; R^d), \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Weak vs strong convergence

Weak convergence

$$\varrho_n \to \varrho \text{ weakly-(*) } L^\infty(0,T;L^\gamma_{\rm loc}(\overline{\Omega}))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ weakly-(*) } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}_{\rm loc}(\overline{\Omega};R^d))$$

Strong convergence (Theorem EF, M.Hofmanová)

- Suppose

$$\Omega \subset R^d \setminus B, \ \ B \text{ convex}$$

$$\varrho \to \varrho_\infty, \ \ \mathbf{m} \to \mathbf{m}_\infty \text{ as } |x| \to \infty$$

- Then the following is equivalent:

$$\varrho, \mathbf{m} \text{ weak solution to the Euler system}$$

$$\Leftrightarrow$$

$$\varrho_n \to \varrho, \ \ \mathbf{m}_n \to \mathbf{m} \text{ strongly (pointwise) in } \Omega$$

Martina Hofmanová (Bielefeld)

Identifying the limit system, weak convergence

Isentropic pressure

$$p(\varrho) = a\varrho^{\gamma},\ \gamma > 1$$

Energy bounds

$$\varrho_n \text{ bounded in } L^{\infty}(0,T;L^{\gamma}(\Omega)),\ \mathbf{m}_n \text{ bounded in } L^{\infty}(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d))$$

Convergence (up to a subsequence)

$$\varrho_{n_k} \to\ \varrho \text{ weakly-(*) in } L^{\infty}(0,T;L^{\gamma}(\Omega))$$

$$\mathbf{m}_{n_k} \to\ \mathbf{m} \text{ weakly-(*) in } L^{\infty}(0,T;L^{\gamma}(\Omega;R^d))$$

$$E_{n_k} = \frac{1}{2}\frac{|\mathbf{m}_{n_k}|^2}{\varrho_{n_k}} + P(\varrho_{n_k}) \to \overline{\left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right)} \text{ weakly-(*) in } L^{\infty}(0,T;\mathcal{M}^+(\overline{\Omega}))$$

$$1_{\varrho_{n_k}>0}\frac{\mathbf{m}_{n_k}\otimes\mathbf{m}_{n_k}}{\varrho_{n_k}} \to \overline{\left(1_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right)} \text{ weakly-(*) in } L^{\infty}(0,T;\mathcal{M}^+(\overline{\Omega};R^{d\times d}_{\mathrm{sym}}))$$

$$p(\varrho_{n_k}) \to \overline{p(\varrho)} \text{ weakly-(*) in } L^{\infty}(0,T;\mathcal{M}^+(\overline{\Omega}))$$

Convergence via Young Measures

Identification

$$(\varrho_n, \mathbf{m}_n)(t,x) \approx \delta_{\varrho_{n(t,x)}, \mathbf{m}_{n(t,x)}} = \mathcal{V}_n, \ \mathcal{V}_n : (0,T) \times \Omega \mapsto \mathfrak{P}(R^{d+1})$$

$$\mathcal{V}_n \in L^\infty_{\mathrm{weak}-(*)}((0,T) \times \Omega; \mathcal{M}^+(R^{d+1}))$$

$$\mathcal{V}_{n_k} \to \mathcal{V} \text{ weakly-(*) in } L^\infty_{\mathrm{weak}-(*)}((0,T) \times \Omega; \mathcal{M}^+(R^{d+1}))$$

$$\Leftrightarrow$$

Young measure

$$b(\varrho_{n_k}, \mathbf{m}_{n_k}) \to \overline{b(\varrho, \mathbf{m})} \text{ weakly-(*) in } L^\infty((0,T) \times \Omega) \text{ for any } b \in C_c(R^{d+1})$$

$$\langle \mathcal{V}_{t,x}; b(\widetilde{\varrho}, \widetilde{\mathbf{m}}) \rangle = \overline{b(\varrho, \mathbf{m})}(t,x) \text{ for a.a. } (t,x) \in (0,T) \times \Omega$$

Basic properties:

$$\mathcal{V}_{t,x} \in \mathfrak{P}(R^{d+1}) \text{ for a.a. } (t,x) \in (0,T) \times \Omega$$

$$\mathcal{V}_{t,x} \text{ admits finite first moments and barycenter } \varrho = \langle \mathcal{V}; \widetilde{\varrho} \rangle, \ \mathbf{m} = \langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle$$

Limit problem, I

Continuity equation

$$\int_0^T \int_\Omega \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \, \mathrm{d}x = - \int_\Omega \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x$$

$$\int_0^T \int_\Omega \left[\langle \mathcal{V}_{t,x}; \widetilde{\varrho} \rangle \, \partial_t \varphi + \langle \mathcal{V}_{t,x}; \widetilde{\mathbf{m}} \rangle \cdot \nabla_x \varphi \right] \, \mathrm{d}x = - \int_\Omega \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x$$

for any $\varphi \in C^1_c([0, T) \times \overline{\Omega})$

Energy inequality

$$\int_{\overline{\Omega}} \mathrm{d} \overline{\left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right)} (\tau) \leq \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \, \mathrm{d}x \text{ for a.a. } \tau \geq 0$$

$$\int_\Omega \left\langle \mathcal{V}_{\tau,x}; \left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right) \right\rangle \, \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) \leq \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \, \mathrm{d}x$$

Energy concentration defect

$$\mathfrak{E}_{\rm conc} = \overline{\left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right)} - \left\langle \mathcal{V}_{\tau,x}; \left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right) \right\rangle \geq 0$$

Limit problem, II

Momentum equation

$$\int_0^T \int_\Omega \left[\langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}; 1_{\widetilde{\varrho}>0} \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}; p(\widetilde{\varrho}) \rangle \, \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x\mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x - \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}(t) \mathrm{d}t$$

Reynolds concentration defect

$$\mathfrak{R}_{\rm conc} = \overline{\left(1_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\widetilde{\varrho}} \right)} - \left\langle \mathcal{V}; 1_{\widetilde{\varrho}>0} \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right\rangle + \left(\overline{p(\varrho)} - \langle \mathcal{V}; p(\widetilde{\varrho}) \rangle \right) \mathbb{I}$$

$$\mathfrak{R}_{\rm conc} : (\xi \otimes \xi)$$

$$= \overline{\left(1_{\varrho>0} \frac{|\mathbf{m} \cdot \xi|^2}{\widetilde{\varrho}} \right)} - \left\langle \mathcal{V}; 1_{\widetilde{\varrho}>0} \frac{|\widetilde{\mathbf{m}} \cdot \xi|^2}{\widetilde{\varrho}} \right\rangle + \left(\overline{p(\varrho)} - \langle \mathcal{V}; p(\widetilde{\varrho}) \rangle \right) |\xi|^2 \geq 0$$

$$\Rightarrow \ \mathfrak{R}_{\rm conc} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\rm sym}))$$

Defect compatibility

$$\min \left\{ \gamma - 1; \frac{1}{2} \right\} \mathfrak{E}_{\rm conc} \leq \mathrm{trace}[\mathfrak{R}_{\rm conc}] \leq \max\{\gamma - 1; 2\} \mathfrak{E}_{\rm conc}$$

Dissipative measure–valued (DMV) solutions

Continuity equation

$$\int_0^T \int_\Omega \Big[\langle \mathcal{V}; \widetilde{\varrho} \rangle \, \partial_t \varphi + \langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle \cdot \nabla_x \varphi \Big] \, \mathrm{d}x = - \int_\Omega \mathbf{m}_0 \varphi(0, \cdot) \, \mathrm{d}x$$

Momentum equation

$$\int_0^T \int_\Omega \left[\langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}; 1_{\widetilde{\varrho} > 0} \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}; p(\widetilde{\varrho}) \rangle \, \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x - \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_{\rm conc}(t) \mathrm{d}t$$

Energy inequality

$$\int_\Omega \left\langle \mathcal{V}_{\tau,x}; \left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right) \right\rangle \, \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}_{\rm conc}(\tau) \leq \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \, \mathrm{d}x$$

Defect compatibility

$$\underline{d} \mathfrak{E}_{\rm conc} \leq \mathrm{trace}[\mathfrak{R}_{\rm conc}] \leq \overline{d} \mathfrak{E}_{\rm conc}$$

Oscillation defect

Energy oscillation defect

$$\mathfrak{E}_{\rm osc} = \left\langle \mathcal{V}; \left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right) \right\rangle - \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \geq 0 \in L^\infty(0, T; L^1(\Omega))$$

Reynolds oscillation defect

$$\mathfrak{R}_{\rm osc} = \left\langle \mathcal{V}; 1_{\widetilde{\varrho}>0} \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right\rangle - 1_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\widetilde{\varrho}} + \left(\langle \mathcal{V}; p(\widetilde{\varrho}) \rangle - p(\varrho) \right) \mathbb{I}$$

Convexity:

$$\mathfrak{R}_{\rm osc} : (\xi \otimes \xi) = \left\langle \mathcal{V}; 1_{\widetilde{\varrho}>0} \frac{|\widetilde{\mathbf{m}} \cdot \xi|^2}{\widetilde{\varrho}} \right\rangle - 1_{\varrho>0} \frac{|\mathbf{m} \cdot \xi|^2}{\widetilde{\varrho}}$$

$$+ \left(\langle \mathcal{V}; p(\widetilde{\varrho}) \rangle - p(\varrho) \right) |\xi|^2 \geq 0$$

Defect compatibility

$$\min \left\{ \gamma - 1; \frac{1}{2} \right\} \mathfrak{E}_{\rm osc} \leq {\rm trace}[\mathfrak{R}_{\rm osc}] \leq \max\{\gamma - 1; 2\} \mathfrak{E}_{\rm osc}$$

Dissipative solutions

Continuity equation

$$\int_0^T \int_\Omega \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \, \mathrm{d}x = - \int_\Omega \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x$$

Momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + 1_{\varrho > 0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x - \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}(t) \mathrm{d}t$$

Energy inequality

$$\int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \, \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) \leq \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \, \mathrm{d}x$$

Defect compatibility

$$\underline{d} \mathfrak{E} \leq \mathrm{trace}[\mathfrak{R}] \leq \overline{d} \mathfrak{E}$$

Basic properties of dissipative solutions

Well posedness, weak strong uniqueness

- **Existence.** Dissipative solutions exist globally in time for any finite energy initial data
- **Limits of consistent approximations** Limits of consistent approximations are dissipative solutions, in particular limits of consistent numerical schemes.
- **Compatibility.** Any C^1 dissipative solution $[\varrho, \mathbf{m}]$, $\varrho > 0$ is a classical solution of the Euler system
- **Weak–strong uniqueness.** If $[\widetilde{\varrho}, \widetilde{\mathbf{m}}]$ is a classical solution and $[\varrho, \mathbf{m}]$ a dissipative solution starting from the same initial data, then $\mathfrak{R}_v = \mathfrak{R}_p = 0$ and $\varrho = \widetilde{\varrho}$, $\mathbf{m} = \widetilde{\mathbf{m}}$.

Dissipative solutions – limits of numerical schemes

Dominic Breit (Edinburgh)

Martina Hofmanová (Bielefeld)

Equation of continuity

$$\partial_t \boxed{\varrho} + \mathrm{div}_x \mathbf{m} = 0$$

Momentum balance

$$\partial_t \boxed{\mathbf{m}} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \left(\mathfrak{R}_v + \mathfrak{R}_p \mathbb{I} \right)$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq 0, \ \ E(t) \leq E_0, \ \ E_0 = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$

$$\boxed{E} \equiv \left(\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d} \frac{1}{2} \mathrm{trace}[\mathfrak{R}_v] + \int_{\overline{\Omega}} \mathrm{d} \frac{1}{\gamma - 1} \mathfrak{R}_p \right)$$

Turbulent defect measures

$$\mathfrak{R}_v \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\mathrm{sym}})), \ \ \mathfrak{R}_p \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}))$$

Semiflow selection

Set of data

$$\mathcal{D} = \left\{ \varrho, \mathbf{m}, E \ \Big| \ \int_\Omega \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \ \mathrm{d}x \leq E \right\}$$

Set of trajectories

$$\mathcal{T} = \left\{ \varrho(t,\cdot), \mathbf{m}(t,\cdot), E(t-,\cdot) \Big| t \in (0,\infty) \right\}$$

Solution set

$$\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] = \left\{ [\varrho, \mathbf{m}, E] \ \Big| [\varrho, \mathbf{m}, E] \text{ dissipative solution} \right.$$

$$\left. \varrho(0,\cdot) = \varrho_0, \ \mathbf{m}(0,\cdot) = \mathbf{m}_0, \ E(0+) \leq E_0 \right\}$$

Semiflow selection – semigroup

$$U[\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0], \ [\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{D}$$

$$U(t_1 + t_2)[\varrho_0, \mathbf{m}_0, E_0] = U(t_1) \circ \Big[U(t_2)[\varrho_0, \mathbf{m}_0, E_0] \Big], \ t_1, t_2 > 0$$

Andrej Markov (1856–1933)

N. V. Krylov

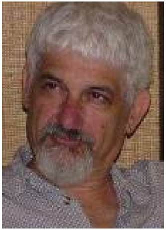
Strong instead of weak (numerics)

Komlos theorem (a variant of Strong Law of Large Numbers)

$$\{U_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q)$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} U_{n_k} \to \overline{U} \text{ a.a. in } Q \text{ as } N \to \infty$$



Janos Komlos (Ruthers Univ.)

Convergence of numerical solutions - EF, M.Lukáčová, H.Mizerová 2018

$$\frac{1}{N}\sum_{k=1}^{N} \varrho_{n_k} \to \varrho \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{m}_{n_k} \to \mathbf{m} \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \left[\frac{1}{2}\frac{|\mathbf{m}_{n,k}|^2}{\varrho_{n,k}} + P(\varrho_{n,k})\right] \to \overline{\mathcal{E}} \in L^1((0,T)\times\Omega) \text{ a.a. in } (0,T)\times\Omega$$

Computing defect – Young measure

Generating Young measure

$$\mathbf{U}_n = [\varrho_n, \mathbf{m}_n] \in R^{d+1} \text{ phase space}$$

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q; R^d) \approx \nu^n_{t,x} = \delta_{\mathbf{U}_n(t,x)}$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} \nu^{n_k}_{t,x} \to \nu_{t,x} \text{ narrowly } \boxed{a.a.} \text{ in } Q \text{ as } N \to \infty$$

Young measure

$$(t,x) \in Q \mapsto \nu_{t,x} \in \mathcal{P}[R^{d+1}] \text{ weakly-(*) measurable mapping}$$

Erich J. Balder (Utrecht)

$$\mathfrak{R}_p \approx \langle \nu; p(\varrho)\rangle - p(\varrho)$$

$$\mathfrak{R}_v \approx \left\langle \nu; \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} \right\rangle - \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}$$

Computing defect numerically -EF, M.Lukáčová, B.She

Monge–Kantorowich (Wasserstein) distance

$$\left\| \mathrm{dist} \left(\frac{1}{N} \sum_{k=1}^{N} \nu_{t,x}^{n_k}; \nu_{t,x} \right) \right\|_{L^q(Q)} \to 0$$

for some $q > 1$

Convergence in the first variation

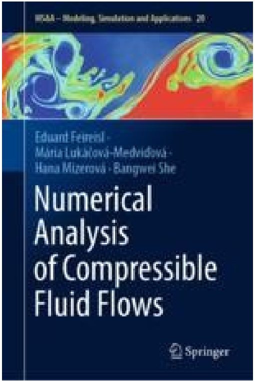
$$\frac{1}{N} \sum_{k=1}^{N} \left\langle \nu_{t,x}^{n_k}; \left| \widetilde{\mathbf{U}} - \frac{1}{N} \sum_{k=1}^{N} \mathbf{U}_n \right| \right\rangle \to \left\langle \nu_{t,x}; \left| \widetilde{\mathbf{U}} - \mathbf{U} \right| \right\rangle$$

in $L^1(Q)$

Mária Lukáčová (Mainz)

Bangwei She (CAS Praha)

A bit of publicity



Mária Lukáčová (Mainz)

Bangwei She (CAS Praha)

Hana Mizerová (Bratislava)

Lecture III

Statistical solutions

General setting

Random (stochastic) process

$$t \in I \times \omega \in \Omega \mapsto \mathbf{U}(t, \omega) \in X$$

$$I \subset R \text{ time interval}$$

$$\Omega = (\Omega, \mathcal{B}, \mathfrak{P}) \text{ probability space (basis)}$$

$$X \text{ phase space}$$

Applications to evolutionary problems

$$\mathbf{U} = \mathbf{U}(t, \mathbf{U}_0),\ \ \mathbf{U}_0 \in \mathcal{D} - \text{data space}$$

$$\omega \in \Omega \mapsto \mathbf{U}_0(\omega) \in \mathcal{D} \text{ random variable}$$

Statistical solution of an evolutionary problem $\mathcal{P}$

$\approx$

random process that solves $\mathcal{P}$ $\mathfrak{P}$-a.s.

Rayleigh–Bénard problem

Field equations - Navier–Stokes–Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x G$$

$$\partial_t(\varrho e(\varrho, \vartheta)) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u}) + \nabla_x \mathbf{q} = \mathbb{S} : \mathbb{D}_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}$$

Boundary conditions

$$\mathbf{u}|_{x_3=0} = \mathbf{u}|_{x_3=1} = 0$$

$$\vartheta|_{x_3=0} = \Theta_B, \ \vartheta|_{x_3=1} = \Theta_U$$

Newton's rheological law

$$\mathbb{S}(\vartheta, \mathbb{D}_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Fourier's law

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta$$

Why weak solutions?

far from equilibrium (not "small")
global in time solutions ≈ weak solutions

Possible formulation of the energy balance:

Internal energy balance ≈ "heat equation"

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \boxed{p \mathrm{div}_x \mathbf{u}}$$

Energy balance ≈ First law

$$\partial_t E + \boxed{\mathrm{div}_x(E\mathbf{u})} + \mathrm{div}_x(p\mathbf{u}) + \mathrm{div}_x \mathbf{q} - \boxed{\mathrm{div}_x(\mathbb{S}\cdot\mathbf{u})} = \varrho \nabla_x G \cdot \mathbf{u}$$

Entropy balance ≈ Second law

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) \boxed{=} \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

Weak solutions – basic idea

Entropy inequality $\approx$ Second law

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) \boxed{\geq} \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

Total energy balance $\approx$ First law

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E \ \mathrm{d}x \leq \int_\Omega \varrho \mathbf{g} \ \mathrm{d}x + \boxed{\text{boundary energy flux}}$$

Weak solutions - basic definition

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \nabla_x G$$

Entropy inequality

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Some form of total energy balance must be added
for the system to be (formally) well posed

Main problem with the Dirichlet b.c. for the temperature

Boundary heat flux in the energy balance

$$\int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} \, \mathrm{d}\sigma_x$$

Solution – compensation with the entropy flux

$$\int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} \, \mathrm{d}\sigma_x = \int_{\partial\Omega} \frac{\mathbf{q} \cdot \mathbf{n}}{\vartheta} \vartheta_B \mathrm{d}\sigma_x, \ \vartheta|_{\partial\Omega} = \vartheta_B$$

$$\Leftrightarrow$$

Replace energy by ballistic energy!

Energy balance – Dirichlet b.c. for temperature

Ballistic energy inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e - \vartheta_B \varrho s \right] \ \mathrm{d}x \\
&\quad + \int_\Omega \frac{\vartheta_B}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \\
&\quad \leq \int_\Omega \varrho \mathbf{u} \cdot \nabla_x G \ \mathrm{d}x \\
&\qquad - \int_\Omega \left[\varrho s \left(\partial_t \vartheta_B + \mathbf{u} \cdot \nabla_x \vartheta_B \right) + \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \vartheta_B \right] \ \mathrm{d}x.
\end{aligned}
$$

Long–time behavior, turbulence

Closed system:

$$\Theta_B = \Theta_U$$

$$\mathbf{U}(t) \equiv [\varrho, \mathbf{m} = \varrho \mathbf{u}, S = \varrho s] \to [\varrho_s, 0, S_s] \text{ as } t \to \infty \text{ static equilibrium}$$

Open system:

$$\Theta_B >> \Theta_U$$

bounded energy? $\int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right] \ \mathrm{d}x \leq \mathcal{E}_\infty$

stationary measure?

Space of global trajectories

$$\langle \varrho(\tau-,\cdot);\phi\rangle \equiv \lim_{\delta\to 0+} \frac{1}{\delta}\int_{\tau-\delta}^{\tau}\int_\Omega \varrho(t,\cdot)\phi \ \mathrm{d}x\mathrm{d}t$$

$$\langle \varrho(\tau+,\cdot);\phi\rangle \equiv \lim_{\delta\to 0+} \frac{1}{\delta}\int_{\tau}^{\tau+\delta}\int_\Omega \varrho(t,\cdot)\phi \ \mathrm{d}x\mathrm{d}t$$

$$\langle \mathbf{m}(\tau-,\cdot);\boldsymbol{\varphi}\rangle \equiv \lim_{\delta\to 0+} \frac{1}{\delta}\int_{\tau-\delta}^{\tau}\int_\Omega \mathbf{m}(t,\cdot)\cdot\boldsymbol{\varphi} \ \mathrm{d}x\mathrm{d}t$$

$$\langle \mathbf{m}(\tau+,\cdot);\boldsymbol{\varphi}\rangle \equiv \lim_{\delta\to 0+} \frac{1}{\delta}\int_{\tau}^{\tau+\delta}\int_\Omega \mathbf{m}(t,\cdot)\cdot\boldsymbol{\varphi} \ \mathrm{d}x\mathrm{d}t,$$

$$\langle S(\tau-,\cdot);\phi\rangle \equiv \lim_{\delta\to 0+} \frac{1}{\delta}\int_{\tau-\delta}^{\tau}\int_\Omega \varrho s(t,\cdot)\phi \ \mathrm{d}x\mathrm{d}t$$

$$\langle S(\tau+,\cdot);\phi\rangle \equiv \lim_{\delta\to 0+} \frac{1}{\delta}\int_{\tau}^{\tau+\delta}\int_\Omega \varrho s(t,\cdot)\phi \ \mathrm{d}x\mathrm{d}t$$

$$\varrho \in C_{\rm weak}(R; L^\gamma(\Omega)),\ \mathbf{m} = \varrho\mathbf{u} \in C_{\rm weak}(R; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^3))$$

$S \in D(R; L^\beta)$, $\beta < \gamma -$ Skorokhod space (weakly càglàd)

Trajectory space

$$\mathcal{T} = \cup_{L=1}^{\infty} \mathcal{T}_L,$$

$$\mathcal{T}_L = \Big\{ (\varrho, S, \mathbf{m}) \ \Big| \ \varrho \in L^\infty(R; W^{-k,2}(\Omega)),\ \langle \varrho; \phi_n \rangle \in C(R),\ n = 1, 2, \dots,$$
$$\sup_{t \in R} \| \varrho(t, \cdot) \|_{W^{-k,2}(\Omega)} \leq L,$$
$$\mathbf{m} \in L^\infty(R; W^{-k,2}(\Omega; R^3)),\ \langle \mathbf{m}; \boldsymbol{\varphi}_n \rangle \in C(R),\ n = 1, 2, \dots,$$
$$\sup_{t \in R} \| \mathbf{m}(t, \cdot) \|_{W^{-k,2}(\Omega; R^3)} \leq L,$$
$$S \in L^\infty(R; W^{-k,2}(\Omega)),\ \langle S; \phi_n \rangle \ \text{càglàd in } R,\ n = 1, 2, \dots,$$
$$\sup_{t \in R} \| S(t, \cdot) \|_{W^{-k,2}(\Omega)} \leq L \Big\}.$$

Krylov–Bogolyubov method

Ergodic averages

$$\frac{1}{T}\int_0^T F\Big([\varrho, \mathbf{m}, S](\cdot + t)\Big)\mathrm{d}t,\ \ F \in C_c(\mathcal{T})$$

Invariant measure

$$\frac{1}{T_n}\int_0^{T_n} F\Big([\varrho, \mathbf{m}, S](\cdot + t)\Big)\mathrm{d}t \to \langle \mathcal{V}; F\rangle$$

$\mathcal{V} \in \mathcal{P}[\mathcal{T}] -$ probability measure on the set of global trajectories

$\mathcal{V} -$ time shift invariant

Stationary statistical solution – Birkhoff-Khinchin theorem

Probability basis

$$(\mathcal{T}, \mathcal{B}, \mathcal{V})$$

Random process (stationary statistical solution)

$$\omega = (\varrho, \mathbf{m}, S) \times t \in R \mapsto (\varrho, \mathbf{m}, S)(t, \cdot)$$

Birkhoff–Khinchin theorem

$$\int_{\mathcal{T}} |F(\varrho(0,\cdot), S(0,\cdot), \mathbf{m}(0,\cdot)| \ \mathrm{d}\mathcal{V} < \infty$$

$$F - \text{Borel measurable}$$

$$\Rightarrow$$

$$\frac{1}{T} \int_0^T F(\varrho(t,\cdot), S(t,\cdot), \mathbf{m}(t,\cdot)) \mathrm{d}t \to \overline{F} \text{ as } T \to \infty$$

$$\mathcal{V} - \text{a.s. and in } L^1(\mathcal{T}, \mathcal{V})$$