MATHEMATICAL THEORY OF COMPRESSIBLE VISCOUS FLUIDS
Lecture I: Fluid equations in continuum mechanics

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Literature

- Feireisl, Eduard; Karper, Trygve G.; Pokorný, Milan: *Mathematical theory of compressible viscous fluids. Analysis and numerics.* Advances in Mathematical Fluid Mechanics. Lecture Notes in Mathematical Fluid Mechanics. Birkhäuser/Springer, Cham, 2016
- Lecture notes: www.math.cas.cz http://www.math.cas.cz/homepage/main_ page.php?id_membre=37&id_menu=5& id_submenu=15
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LECTURE I: FLUID EQUATIONS IN CONTINUUM MECHANICS

Basic principles

Physical domain: $\Omega \subset R^d$, d=1,2,3
Eulerian variables: time $t \in I = (0, T)$, $x \in \Omega$
Fields: $\mathbf{U} = \mathbf{U}(t, x) : I \times \Omega \to R^m$

- **volume densities of observables:** d = d(t,x)
- **fluxes:** $\mathbf{F} = \mathbf{F}$ (t,x)
- **sources:** s = s(t,x)

Field equations - balance laws:

$$\left[\int_B d(t,x)\mathrm{d}x\right]_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2}\int_{\partial B} \mathbf{F}(t,x)\cdot\mathbf{n}\ \mathrm{d}S_x\mathrm{d}t + \int_{t_1}^{t_2}\int_B s(t,x)\ \mathrm{d}x\mathrm{d}t$$

for any

$B \subset \Omega$, $t_1 < t_2$

Balance laws as PDEs

Balance law (strong form):

- **scalar:**

$$\partial_t d(t,x) + \mathrm{div}_x \mathbf{F}(t,x) = s(t,x) \text{ for any } t, x$$

- **vectorial:**

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F} = \mathbf{s}(t,x)$$

Weak (distributional) form:

$$\int_I \int_\Omega \Big[d(t,x)\partial_t \varphi(t,x) + \mathbf{F}(t,x)\cdot\nabla_x\varphi(t,x) \Big] \,\mathrm{d}x\mathrm{d}t = -\int_I \int_\Omega s(t,x)\varphi(t,x)\,\mathrm{d}x\mathrm{d}t$$

$$\text{for any } \varphi \in C^1_c(I\times\Omega)$$

Initial and/or boundary conditions:

$$\int_{t_1}^{t_2} \int_\Omega \Big[d\partial_t\varphi + \mathbf{F}\cdot\nabla_x\varphi \Big] \,\mathrm{d}x\mathrm{d}t$$

$$= \left[\int_\Omega d\varphi \,\mathrm{d}x \right]_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \int_{\partial\Omega} \mathbf{F}_b \cdot \mathbf{n}\varphi \,\mathrm{d}S_x \mathrm{d}t - \int_{t_1}^{t_2} \int_\Omega s\varphi \,\mathrm{d}x\mathrm{d}t$$

Physical principles

Mass conservation – equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

ϱ .... mass density
$\mathbf{u}$ .... fluid (bulk) velocity

Momentum conservation – Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x \mathbb{T} + \varrho \mathbf{g}$$

$\mathbb{T}$ .... Cauchy stress
$\mathbf{g}$ .... external volume force

Energy balance – First law of thermodynamics

$$\partial_t E + \mathrm{div}_x(E\mathbf{u}) = \mathrm{div}_x\Big(\mathbb{T} \cdot \mathbf{u} - \mathbf{q}\Big) + \varrho \mathbf{g} \cdot \mathbf{u}$$

E .... total energy
$\mathbf{q}$ .... (internal) energy flux

Energy balance

Total energy

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e$$

Kinetic energy balance

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 \right) + \mathrm{div}_x \left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u} \right) = \mathrm{div}_x(\mathbb{T}\cdot\mathbf{u}) - \mathbb{T} : \nabla_x\mathbf{u} + \varrho\mathbf{g}\cdot\mathbf{u}$$

Internal energy balance

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e\mathbf{u}) + \mathrm{div}_x\mathbf{q} = \mathbb{T} : \nabla_x\mathbf{u}$$

e .... internal energy

Constitutive relations

Stokes' law (mathematical definition of fluid)

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}$$

$\mathbb{S}$ .... viscous stress
p .... pressure

Gibbs' law – entropy

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

s .... entropy
ϑ .... (absolute) temperature

Second law of thermodynamics – entropy equation

Internal energy balance

$$\varrho\partial_t e + \varrho\mathbf{u}\cdot\nabla_x e + \mathrm{div}_x\mathbf{q} = \mathbb{S}:\nabla_x\mathbf{u} - p\,\mathrm{div}_x\mathbf{u} \qquad \Big|\frac{1}{\vartheta}$$

$$\text{Gibbs' law} \;\Rightarrow\; \frac{1}{\vartheta}\varrho\partial_t e = \varrho\partial_t s + \frac{1}{\vartheta\varrho}p\partial_t\varrho = \varrho\partial_t s - \frac{1}{\vartheta\varrho}p\,\mathrm{div}_x(\varrho\mathbf{u})$$

$$= \varrho\partial_t s - \frac{1}{\vartheta\varrho}p\nabla_x\varrho\cdot\mathbf{u} - \frac{1}{\vartheta}p\,\mathrm{div}_x\mathbf{u}$$

$$\frac{1}{\vartheta}\varrho\mathbf{u}\cdot\nabla_x e = \varrho\mathbf{u}\cdot\nabla_x s + \frac{1}{\vartheta\varrho}p\mathbf{u}\cdot\nabla_x\varrho$$

Entropy balance

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \frac{1}{\vartheta}\Big(\mathbb{S}:\nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\Big)$$

Entropy production rate

$$\frac{1}{\vartheta}\left(\mathbb{S}:\nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right) \geq 0$$

Newtonian (linearly viscous) fluids

Newton's rheological law

$$\mathbb{S}(\mathbb{D}_x\mathbf{u}) = \mu\left(\nabla_x\mathbf{u} + \nabla_x\mathbf{u}^t - \frac{2}{d}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta\mathrm{div}_x\mathbf{u}\mathbb{I},\ \mathbb{D}_x \equiv \frac{1}{2}\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u}\right)$$

μ .... shear viscosity coefficient
η .... bulk viscosity coefficient

Fourier's law

$$\mathbf{q} = -\kappa\nabla_x\vartheta$$

κ .... heat conductivity coefficient

Entropy production rate

$$\frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right) \geq 0$$

$$\Rightarrow$$

$$\mu \geq 0,\ \eta \geq 0$$

Perfect fluids

$$\mathbb{S} \equiv 0, \ \mathbf{q} \equiv 0$$

$$\mathbf{m} \equiv \varrho \mathbf{u}, \ E = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e$$

Leonhard Paul Euler 1707–1783

Euler system of gas dynamics

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = 0$$

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0$$

(Incomplete) equation of state (gases)

$$p = (\gamma - 1) \varrho e$$

γ .... adiabatic coefficient

Perfect fluids–entropy

Entropy transport

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) = 0 \ \Rightarrow \ \partial_t s + \mathbf{u} \cdot \nabla_x s = 0$$

$$s = \overline{s} - \text{constant} \ \Rightarrow \ \text{isentropic EOS } p = p(\varrho)$$

Barotropic (isentropic) Euler system

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \ \partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + p(\varrho) = 0$$

"Nobody is perfect"

Entropy inequality

$$\partial_t(\varrho s) + \mathrm{div}_x(s\mathbf{m}) \geq 0$$

Renormalized entropy inequality

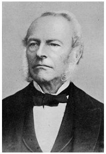
$$\partial_t(\varrho F(s)) + \mathrm{div}_x(F(s)\mathbf{m}) \geq 0$$

$$F' \geq 0$$

Navier–Stokes–Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u})$$



$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) - \mathrm{div}_x(\kappa \nabla_x \vartheta) = \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u}$$

Entropy balance equation

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) - \mathrm{div}_x \left(\frac{\kappa \nabla_x \vartheta}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S}(\mathbb{D}_x \mathbf{u}) : \nabla_x \mathbf{u} + \frac{\kappa |\nabla_x \vartheta|^2}{\vartheta} \right)$$

Entropies for viscous and perfect fluids

Renormalized entropy equation for the Euler system

$$\partial_t(\varrho F(s)) + \mathrm{div}_x(F(s)\mathbf{m}) = (\geq)0$$

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Entropy minimum principle

$$s(t,x) \geq \inf_{y \in \Omega} s(0,y) \ \ t \geq 0$$

Entropy minimum principle for weak solutions

$$\int_\Omega \varrho F_n(s)(t,\cdot)\ \mathrm{d}x \geq \int_\Omega \varrho F_n(s)(0,\cdot)\ \mathrm{d}x$$

$$F_n(s) = n \min\left\{s - \underline{s}, 0\right\},\ \ \underline{s} = \min_{y \in \Omega} s(0,y)$$

Absence of a large class of entropies for the Navier–Stokes–Fourier system
The composition $F(s)$ is in general not and entropy for the Navier–Stokes–Fourier system. The minimum principle may not hold either.

Boundary conditions for closed systems

Perfect fluid – impermeability of the boundary

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Real fluid

- **Impermeability of the boundary, no heat flux**

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

- **no slip or complete slip**

$$\mathbf{u}|_{\partial\Omega} = 0 \text{ or } (\mathbb{S} \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0$$

Periodic boundary conditions

$$\Omega = \mathbb{T}^d, \ \mathbb{T}^d = \left([-1, 1]|_{\{-1;1\}}\right)^d - \text{flat torus}$$

Boundary conditions for open systems – real fluids

Boundary velocity

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B$$

Boundary decomposition

$$\partial\Omega = \Gamma_{\rm in} \cup \Gamma_{\rm out} \cup \Gamma_{\rm wall}$$

$$\mathbf{u}_B \cdot \mathbf{n} < 0 \text{ on } \Gamma_{\rm in},\ \mathbf{u}_B \cdot \mathbf{n} > 0 \text{ on } \Gamma_{\rm out},\ \mathbf{u}_B \cdot \mathbf{n} = 0 \text{ on } \Gamma_{\rm wall}$$

Flux conditions

- **Mass flux**

$$\varrho|_{\Gamma_{\rm in}} = \varrho_B$$

- **Heat flux**

$$\left[\varrho_B e \mathbf{u}_B + \mathbf{q}\right] \cdot \mathbf{n}|_{\Gamma_{\rm in}} = F_B \quad \text{(Robin boundary conditions)}$$

$$\mathbf{q} \cdot \mathbf{n}|_{\Gamma_{\rm out} \cup \Gamma_{\rm wall}} = 0$$

Local well posedness for smooth data

Initial data

- **Smoothness in Sobolev class**

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \vartheta(0,\cdot) = \vartheta_0,\ \varrho_0, \mathbf{u}_0, \vartheta_0 \in W^{k,2},\ k > d$$

- **Smoothness of the boundary data**

$$\partial\Omega \text{ of class } C^{2+\nu}$$

$$\mathbf{u}_B,\ \varrho_B, F_B \in C^k,\ k \geq 2$$

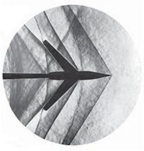
- **Absence of vacuum**

$$\varrho_0,\ \varrho_B > 0,\ \vartheta_0 > 0$$

- **Compatibility conditions**
  Equations and their derivatives satisfied up to $t = 0$

Local existence
Both Euler and Navier–Stokes system admit smooth solutions existing on a maximal time interval $[0, T_{\max}),\ T_{\max} > 0$

Blow up for 1-d Euler system



Isentropic Euler system (constant entropy)

$$\partial_t \varrho + \partial_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \partial_x(\varrho \mathbf{u}^2) + a\partial_x \varrho^\gamma = 0$$

Step 1: Lagrange mass coordinates

$$t = t,\ y(t,x) = \int_{-\infty}^{x} \varrho(t,z)\ \mathrm{d}z$$

$$\partial_t V - \partial_y w = 0,\ \partial_t w + a\partial_y V^{-\gamma} = 0$$

$$V = \varrho^{-1} \text{ (specific volume)},\ w\left(t, \int_{-\infty}^{x} \varrho(t,z)\ \mathrm{d}z\right) = u(t,x)$$

Blow up for 1-d Euler system, continuation

Step 2: P–system

$$\partial_t V - \partial_y w = 0,\ \partial_t w - \partial_y P(V) = 0,\ P'(V) > 0$$

$$\sqrt{P'(V)}\partial_t V - \sqrt{P'(V)}\partial_y w = 0,\ \partial_t w - \sqrt{P'(V)}\sqrt{P'(V)}\partial_y V = 0$$

Step 3: Riemann invariants

$$\partial_t Z - A(Z)\partial_y w = 0,\ \partial_t w - A(Z)\partial_y Z = 0$$

$$Z(V) = \int_0^V \sqrt{P'(z)}\,\mathrm{d}z,\ A(Z) = \sqrt{P'(V(Z))}$$

Solution:

$$Z = \pm w$$

Blow up for 1-d Euler system, continuation

Step 3:

$$Z = -w,\ \partial_t Z + A(Z)\partial_x Z = 0$$

Burger's equation

$$\partial_t U + U\partial_y U = 0,\ A(Z) = U$$

Solutions of Burger's equation

$$U(t, y + tU_0(y)) = U_0(y),\ U(0, y) = U_0(y)$$

$$U_0(y_1) > U_0(y_2),\ y_2 > y_1 \ \Rightarrow \ \text{singularity at } \tau = \frac{y_2 - y_1}{U_0(y_1) - U_0(y_2)} > 0$$

Weak solutions to Euler system

Periodic boundary conditions

$$\Omega = \mathbb{T}^d$$

$$\int_0^T \int_{\mathbb{T}^d} \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \, \mathrm{d}x = - \int_{\mathbb{T}^d} \varrho_0 \varphi \, \mathrm{d}x$$

$$\varphi \in C^1_c([0, T) \times \mathbb{T}^d)$$

$$\int_0^T \int_{\mathbb{T}^d} \left[\mathbf{m} \cdot \partial \boldsymbol{\varphi} + 1_{\varrho > 0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x\mathrm{d}t = - \int_{\mathbb{T}^d} \mathbf{m}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x$$

$$\boldsymbol{\varphi} \in C^1_c([0, T) \times \mathbb{T}^d; R^d)$$

$$\int_0^T \int_{\mathbb{T}^d} \left[E \partial_t \varphi + (E + p) \frac{\mathbf{m}}{\varrho} \cdot \nabla_x \boldsymbol{\varphi} \right] \, \mathrm{d}x\mathrm{d}t = - \int_{\mathbb{T}^d} E_0 \varphi \, \mathrm{d}x$$

$$\varphi \in C^1_c([0, T) \times \mathbb{T}^d)$$

Entropy admissibility condition

$$\int_0^T \int_{\mathbb{T}^d} \left[\varrho s \partial_t \varphi + s \mathbf{m} \cdot \nabla_x \varphi \right] \, \mathrm{d}x\mathrm{d}t \le - \int_{\mathbb{T}^d} \varrho_0 s_0 \varphi \, \mathrm{d}x, \ \varphi \ge 0$$

Ill–posedness of Euler system

Theorem:
Let $d = 2, 3$. Let $\varrho_0 > 0$, $\vartheta_0 > 0$ be piecewise constant, arbitrary functions. Then there exists $\mathbf{u}_0 \in L^\infty$ such that the Euler system admits infinitely many *admissible* weak solutions in $(0, T) \times \mathbb{T}^d$ with the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$.

Remarks:

- There are examples of Lipschitz initial data for which the Euler system admits infinitely many admissible weak solutions
- The result can be extended to the Euler system driven by stochastic forcing

Weak solutions to barotropic Euler system

Periodic boundary conditions

$$\Omega = \mathbb{T}^d$$

$$\int_0^T \int_{\mathbb{T}^d} \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \, \mathrm{d}x = - \int_{\mathbb{T}^d} \varrho_0 \varphi \, \mathrm{d}x$$

$$\varphi \in C^1_c([0,T) \times \mathbb{T}^d)$$

$$\int_0^T \int_{\mathbb{T}^d} \left[\mathbf{m} \cdot \partial \boldsymbol{\varphi} + 1_{\varrho > 0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \mathrm{d}t = - \int_{\mathbb{T}^d} \mathbf{m}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x$$

$$\boldsymbol{\varphi} \in C^1_c([0,T) \times \mathbb{T}^d; R^d)$$

Energy inequality

$$\int_0^T \partial_t \psi \int_{\mathbb{T}^d} E(t) \, \mathrm{d}x \mathrm{d}t \geq - \int_{\mathbb{T}^d} E_0 \varphi \, \mathrm{d}x, \ \psi \in C^1_c[0,T), \ \psi \geq 0$$

$$E \equiv \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho), \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

Ill posedness for barotropic Euler system

Riemann integrable functions
$\mathcal{R}(Q)$ – the class of functions on Q that are Riemann integrable $\Leftrightarrow$ the functions are continuous at a.a. Lebesgue point

Theorem: Let $d = 2, 3$. Suppose that the initial data belong to the class

- $$\varrho_0 \in \mathcal{R}(\mathbb{T}^d),\ 0 < \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho}$$

- $$\mathbf{m}_0 \in \mathcal{R}(\mathbb{T}^d; R^d),\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R}(\mathbb{T}^d)$$

Let $\mathcal{E} \in \mathcal{R}(0, T)$ be given, $0 \leq \mathcal{E} \leq \overline{\mathcal{E}}$.
Then there exists a constant $\mathcal{E}_\infty \geq 0$ such that the barotropic Euler system admits infinitely many weak solutions with the initial data $[\varrho_0, \mathbf{m}_0]$ and the energy profile

$$\int_{\mathbb{T}^d} \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (t, \cdot)\ \mathrm{d}x = \mathcal{E} + \mathcal{E}_\infty$$

for a.a. $t \in (0, T)$.

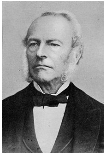
Barotropic Navier–Stokes system

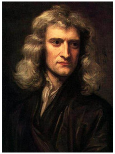
Mass conservation – equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u})$$





Newton's rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Energy balance equation

$$\partial_t E + \mathrm{div}_x(E\mathbf{u} + p\mathbf{u}) - \mathrm{div}_x\left(\mathbb{S}\mathbf{u}\right) = -\mathbb{S}(\mathbb{D}_x \mathbf{u}) : \nabla_x \mathbf{u}$$

$$E = \frac{1}{2}\varrho |\mathbf{u}|^2 + P(\varrho),\ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

Energy balance

1 Write

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) = \mathbf{u}(\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u})) + \varrho\left(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla_x\mathbf{u}\right)$$

$$= \varrho\left(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla_x\mathbf{u}\right)$$

2 Take the scalar product of the resulting expression with $\mathbf{u}$

$$\varrho\left(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla_x\mathbf{u}\right)\cdot\mathbf{u} = \frac{1}{2}\varrho\left(\partial_t|\mathbf{u}|^2 + \mathbf{u}\cdot\nabla_x|\mathbf{u}|^2\right)$$

$$= \partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right) + \mathrm{div}_x\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right) - \frac{1}{2}|\mathbf{u}|^2(\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}))$$

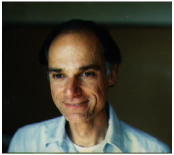
$$= \partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right) + \mathrm{div}_x\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right)$$

3 Similarly,

$$\mathrm{div}_x\mathbb{S}\cdot\mathbf{u} = \mathrm{div}_x(\mathbb{S}\cdot\mathbf{u}) - \mathbb{S} : \nabla_x\mathbf{u}$$

$$p\mathbf{u} = \mathrm{div}_x(p\mathbf{u}) - p\,\mathrm{div}_x\mathbf{u}$$

Internal energy balance



[R. DiPerna]

1 Write the equation of continuity as

$$\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho + \varrho \mathrm{div}_x \mathbf{u}$$

2 Multiply by $b'(\varrho)$

$$\partial_t b(\varrho) + \mathbf{u} \cdot \nabla_x b(\varrho) + b'(\varrho) \varrho \mathrm{div}_x \mathbf{u}$$

Renormalized equation of continuity

$$= \partial_t b(\varrho) + \mathrm{div}_x (b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

3

$$-p(\varrho)\mathrm{div}_x \mathbf{u} = \partial_t P(\varrho) + \mathrm{div}_x (P(\varrho)\mathbf{u})$$

[P.L. Lions]

LECTURE II: STABILITY AND APPROXIMATIONS

Prologue - Lax equivalence principle

Peter D. Lax

Formulation for LINEAR problems

- **Stability** - uniform bounds of approximate solutions
- **Consistency** - vanishing approximation error

$$\Longleftrightarrow$$

- **Convergence** - approximate solutions converge to exact solution

Euler system of gas dynamics

Leonhard Paul Euler 1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \ \mathbf{m} = \varrho \mathbf{u}$$

Momentum equation – Newton's second law

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0, \ p(\varrho) = a \varrho^{\gamma}$$

Impermeability and/or periodic boundary condition

$$\mathbf{m} \cdot \mathbf{n}|_{\partial \Omega} = 0, \ \Omega \subset R^d, \ \text{or } \Omega = \mathbb{T}^d$$

Initial state

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0$$

Weak (distributional) solutions

Jacques Hadamard 1865–1963

Laurent Schwartz 1915–2002

Mass conservation

$$\int_B \Big[\varrho(t_2,\cdot) - \varrho(t_1,\cdot)\Big] \mathrm{d}x = - \int_{t_1}^{t_2} \int_{\partial B} \varrho \mathbf{u} \cdot \mathbf{n} \ \mathrm{dS}_x \mathrm{d}t$$

$$\left[\int_\Omega \varrho \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \Big] \ \mathrm{d}x\mathrm{d}t, \ \ \mathbf{m} \equiv \varrho \mathbf{u}$$

Momentum balance

$$\int_B \Big[\mathbf{m}(t_2,\cdot) - \mathbf{m}(t_1,\cdot) \Big] \ \mathrm{d}x$$

$$= - \int_{t_1}^{t_2} \int_{\partial B} \Big[\mathbf{m} \otimes \mathbf{u} \cdot \mathbf{n} + p(\varrho)\mathbf{n} \Big] \ \mathrm{dS}_x \ \mathrm{d}t$$

$$\left[\int_\Omega \mathbf{m} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

Time irreversibility – energy dissipation

Energy

$$E = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

$$p' \geq 0 \Rightarrow [\varrho, \mathbf{m}] \mapsto \left\{ \begin{array}{l} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0 \\ P(\varrho) \text{ if } |\mathbf{m}| = 0,\ \ \varrho \geq 0 \\ \infty \text{ otherwise} \end{array} \right. \quad \text{is convex l.s.c}$$

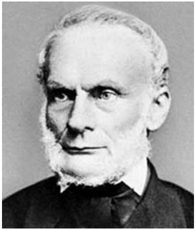
Energy balance (conservation)

$$\partial_t E + \mathrm{div}_x \left(E \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) = 0$$

Energy dissipation

$$\partial_t E + \mathrm{div}_x \left(E \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) \boxed{\leq} 0$$

$$\mathcal{E} = \int_\Omega E \ \mathrm{d}x,\ \ \partial_t \mathcal{E} \leq 0,\ \ \mathcal{E}(0+) = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$



Rudolf Clausius 1822–1888

Weak and strong continuity

$$U : [0, T] \to X$$

Strong continuity

$$\|U(t_1, \cdot) - U(t_2, \cdot)\|_X \to 0 \text{ if } |t_1 - t_2| \to 0$$

Weak continuity

$$t \mapsto \langle F; U(t, \cdot) \rangle \text{ continuous for any } F \in X^*$$

$$U \in C_{\rm weak}([0, T]; L^p(\Omega)), \ t \mapsto \int_\Omega U(t, x)\phi(x) \, {\rm d}x \text{ continuous for any (smooth) } \phi$$

Ill posedness for barotropic Euler system

Riemann integrable functions

$\mathcal{R}(Q)$ – the class of functions on Q that are Riemann integrable $\Leftrightarrow$ the functions are continuous at a.a. Lebesgue point

Weak continuity vs. strong continuity

$$\varrho \in C_{\rm weak}([0,T]; L^\gamma(\mathbb{T}^d)),\ \mathbf{m} \in C_{\rm weak}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d; R^d))$$

Theorem: Let $d = 2, 3$. Let $\{\tau_j\}_{j=1}^\infty \subset (0, T)$ be a countable (possible dense) set of times. Suppose that the initial data belong to the class

- $$\varrho_0 \in \mathcal{R}(\mathbb{T}^d),\ 0 < \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho}$$
- $$\mathbf{m}_0 \in \mathcal{R}(\mathbb{T}^d; R^d),\ {\rm div}_x \mathbf{m}_0 \in \mathcal{R}(\mathbb{T}^d)$$

Then the barotropic Euler system admits infinitely many weak solutions $[\varrho, \mathbf{m}]$ with strictly decreasing total energy and such that

$$t \mapsto [\varrho(t,\cdot), \mathbf{m}(t,\cdot)] \text{ is NOT strongly continuous at any } \tau_j,\ j = 1, 2, \dots$$

Consistent approximation

Continuity equation

$$\int_0^T \int_\Omega [\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi] \,\mathrm{d}x\mathrm{d}t = - \int_\Omega \varrho_{0,n} \varphi(0,\cdot) \ \mathrm{d}x + e_{1,n}[\varphi]$$

for any $\varphi \in C^1_c([0,T) \times \overline{\Omega})$

Momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + 1_{\varrho_n > 0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \boldsymbol{\varphi} + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_{0,n} \cdot \boldsymbol{\varphi}(0,\cdot) \ \mathrm{d}x + e_{2,n}[\boldsymbol{\varphi}]$$

for any $\boldsymbol{\varphi} \in C^1_c([0,T) \times \overline{\Omega}; R^d)$ $\boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$

Energy dissipation

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] (\tau, \cdot) \mathrm{d}x \leq \mathcal{E}_{0,n}$$

Stability and Consistency

Stability

$$\limsup_{n\to\infty} \mathcal{E}_{0,n} < \infty$$

Data compatibility

$$\int_\Omega \varrho_{0,n}\varphi \ \mathrm{d}x \to \int_\Omega \varrho_0\varphi \ \mathrm{d}x \text{ for any } \varphi \in C_c^\infty(\Omega)$$

$$\int_\Omega \mathbf{m}_{0,n} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \to \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi} \ \mathrm{d}x \text{ for any } \boldsymbol{\varphi} \in C_c^\infty(\Omega; R^d)$$

$$\limsup_{n\to\infty} \mathcal{E}_{0,n} \leq \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \mathrm{d}x$$

Vanishing approximation error

$$e_{1,n}[\varphi] \to 0 \text{ as } n \to \infty \text{ for any } \phi \in C_c^\infty([0,T) \times \overline{\Omega})$$

$$e_{2,n}[\boldsymbol{\varphi}] \to 0 \text{ as } n \to \infty \text{ for any } \boldsymbol{\varphi} \in C_c^\infty([0,T) \times \overline{\Omega}; R^d), \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Problems with convergence: Oscillations

Oscillatory sequence

$$g(x+a) = g(x) \text{ for all } x \in R, \quad \int_0^a g(x)\mathrm{d}x = 0,$$

$$g_n(x) = g(nx), \quad n = 1, 2, \ldots$$

Weak convergence (convergence in integral averages)

$$\int_R g_n(x)\varphi(x)\ \mathrm{d}x, \quad \text{where } \varphi \in C_c^\infty(R).$$

$$G(x) = \int_0^x g(z)\ \mathrm{d}z$$

$$\int_R g_n(x)\varphi(x)\ \mathrm{d}x = \int_R g(nx)\varphi(x)\ \mathrm{d}x = -\frac{1}{n}\int_R G(nx)\partial_x\varphi(x)\ \mathrm{d}x \to 0$$

Beware

$$g_n \rightharpoonup g \ \boxed{\textit{does not}} \text{ imply } H(g_n) \rightharpoonup H(g) \text{ if } H \text{ is not linear.}$$

Problem with convergence: Concentrations

Concentrating sequence

$$g_n(x) = ng(nx)$$

$$g \in C_c^\infty(-1,1),\ g(-x) = g(x),\ g \geq 0,\ \int_R g(x)\ \mathrm{d}x = 1.$$

- $$g_n(x) \to 0 \text{ as } n \to \infty \text{ for any } x \neq 0,\ \text{in particular } g_n \to 0 \text{ a.a. in } R;$$

- $$\|g_n\|_{L^1(R)} = \int_R g_n(x)\ \mathrm{d}x = \int_R g(x)\ \mathrm{d}x = 1 \text{ for any } n = 1,2,\ldots.$$

Convergence in the space of measures

$$\int_R g_n(x)\varphi(x)\ \mathrm{d}x = \int_{-1/n}^{1/n} g_n(x)\varphi(x)\ \mathrm{d}x$$

$$\in \left[\min_{x\in[-1/n,1/n]} \varphi(x),\ \max_{x\in[-1/n,1/n]} \varphi(x)\right] \to \varphi(0)\ \Rightarrow\ g_n \to \delta_0$$

Identifying the limit system, weak convergence

Isentropic pressure

$$p(\varrho) = a\varrho^{\gamma},\ \gamma > 1$$

Energy bounds

$$\varrho_n \text{ bounded in } L^{\infty}(0,T;L^{\gamma}(\Omega)),\ \mathbf{m}_n \text{ bounded in } L^{\infty}(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d))$$

Convergence (up to a subsequence)

$$\varrho_{n_k} \to \ \varrho \text{ weakly-(*) in } L^{\infty}(0,T;L^{\gamma}(\Omega))$$

$$\mathbf{m}_{n_k} \to \ \mathbf{m} \text{ weakly-(*) in } L^{\infty}(0,T;L^{\gamma}(\Omega;R^d))$$

$$E_{n_k} = \frac{1}{2}\frac{|\mathbf{m}_{n_k}|^2}{\varrho_{n_k}} + P(\varrho_{n_k}) \to \overline{\left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right)} \text{ weakly-(*) in } L^{\infty}(0,T;\mathcal{M}^+(\overline{\Omega}))$$

$$1_{\varrho_{n_k}>0}\frac{\mathbf{m}_{n_k}\otimes\mathbf{m}_{n_k}}{\varrho_{n_k}} \to \overline{\left(1_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right)} \text{ weakly-(*) in } L^{\infty}(0,T;\mathcal{M}^+(\overline{\Omega};R^{d\times d}_{\rm sym}))$$

$$p(\varrho_{n_k}) \to \overline{p(\varrho)} \text{ weakly-(*) in } L^{\infty}(0,T;\mathcal{M}^+(\overline{\Omega}))$$

Convergence via Young Measures

Identification

$$(\varrho_n, \mathbf{m}_n)(t,x) \approx \delta_{\varrho_n(t,x), \mathbf{m}_{n(t,x)}} = \mathcal{V}_n,\ \mathcal{V}_n : (0,T) \times \Omega \mapsto \mathfrak{P}(R^{d+1})$$

$$\mathcal{V}_n \in L^\infty_{\rm weak-(*)}((0,T) \times \Omega; \mathcal{M}^+(R^{d+1}))$$

$$\mathcal{V}_{n_k} \to \mathcal{V} \text{ weakly-(*) in } L^\infty_{\rm weak-(*)}((0,T) \times \Omega; \mathcal{M}^+(R^{d+1}))$$

$$\Leftrightarrow$$

Young measure

$$b(\varrho_{n_k}, \mathbf{m}_{n_k}) \to \overline{b(\varrho, \mathbf{m})} \text{ weakly-(*) in } L^\infty((0,T) \times \Omega) \text{ for any } b \in C_c(R^{d+1})$$

$$\langle \mathcal{V}_{t,x}; b(\widetilde{\varrho}, \widetilde{\mathbf{m}}) \rangle = \overline{b(\varrho, \mathbf{m})}(t,x) \text{ for a.a. } (t,x) \in (0,T) \times \Omega$$

Basic properties:

$$\mathcal{V}_{t,x} \in \mathfrak{P}(R^{d+1}) \text{ for a.a. } (t,x) \in (0,T) \times \Omega$$

$\mathcal{V}_{t,x}$ admits finite first moments and barycenter $\varrho = \langle \mathcal{V}; \widetilde{\varrho} \rangle$, $\mathbf{m} = \langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle$

Limit problem, I

Continuity equation

$$\int_0^T \int_\Omega \Big[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \Big] \, \mathrm{d}x = - \int_\Omega \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x$$

$$\int_0^T \int_\Omega \Big[\langle \mathcal{V}_{t,x}; \widetilde{\varrho} \rangle \, \partial_t \varphi + \langle \mathcal{V}_{t,x}; \widetilde{\mathbf{m}} \rangle \cdot \nabla_x \varphi \Big] \, \mathrm{d}x = - \int_\Omega \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x$$

for any $\varphi \in C^1_c([0,T) \times \overline{\Omega})$

Energy inequality

$$\int_{\overline{\Omega}} \mathrm{d} \overline{\left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right)}(\tau) \leq \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \, \mathrm{d}x \text{ for a.a. } \tau \geq 0$$

$$\int_\Omega \left\langle \mathcal{V}_{\tau,x}; \left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right) \right\rangle \, \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) \leq \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \, \mathrm{d}x$$

Energy concentration defect

$$\mathfrak{E}_{\mathrm{conc}} = \overline{\left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right)} - \left\langle \mathcal{V}_{\tau,x}; \left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right) \right\rangle \geq 0$$

Limit problem, II

Momentum equation

$$\int_0^T \int_\Omega \left[\langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}; 1_{\widetilde{\varrho}>0} \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}; p(\widetilde{\varrho}) \rangle \, \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi} \ \mathrm{d}x - \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}(t)\mathrm{d}t$$

Reynolds concentration defect

$$\mathfrak{R}_{\rm conc} = \overline{\left(1_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\widetilde{\varrho}} \right)} - \left\langle \mathcal{V}; 1_{\widetilde{\varrho}>0} \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right\rangle + \left(\overline{p(\varrho)} - \langle \mathcal{V}; p(\widetilde{\varrho}) \rangle \right) \mathbb{I}$$

$$\mathfrak{R}_{\rm conc} : (\xi \otimes \xi)$$

$$= \overline{\left(1_{\varrho>0} \frac{|\mathbf{m} \cdot \xi|^2}{\widetilde{\varrho}} \right)} - \left\langle \mathcal{V}; 1_{\widetilde{\varrho}>0} \frac{|\widetilde{\mathbf{m}} \cdot \xi|^2}{\widetilde{\varrho}} \right\rangle + \left(\overline{p(\varrho)} - \langle \mathcal{V}; p(\widetilde{\varrho}) \rangle \right) |\xi|^2 \geq 0$$

$$\Rightarrow \ \mathfrak{R}_{\rm conc} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\rm sym}))$$

Defect compatibility

$$\min \left\{ \gamma - 1; \frac{1}{2} \right\} \mathfrak{E}_{\rm conc} \leq {\rm trace}[\mathfrak{R}_{\rm conc}] \leq \max\{\gamma - 1; 2\} \mathfrak{E}_{\rm conc}$$

Dissipative measure–valued (DMV) solutions

Continuity equation

$$\int_0^T \int_\Omega \Big[\langle \mathcal{V}; \widetilde{\varrho} \rangle \, \partial_t \varphi + \langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle \cdot \nabla_x \varphi \Big] \ \mathrm{d}x = - \int_\Omega \mathbf{m}_0 \varphi(0, \cdot) \ \mathrm{d}x$$

Momentum equation

$$\int_0^T \int_\Omega \left[\langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}; 1_{\widetilde{\varrho} > 0} \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}; p(\widetilde{\varrho}) \rangle \, \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi} \ \mathrm{d}x - \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_{\rm conc}(t) \mathrm{d}t$$

Energy inequality

$$\int_\Omega \left\langle \mathcal{V}_{\tau,x}; \left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right) \right\rangle \ \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}_{\rm conc}(\tau) \leq \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \ \mathrm{d}x$$

Defect compatibility

$$\underline{d} \mathfrak{E}_{\rm conc} \leq \mathrm{trace}[\mathfrak{R}_{\rm conc}] \leq \overline{d} \mathfrak{E}_{\rm conc}$$

Convexity and weak convergence

$\mathbf{U}_n \to \mathbf{U}$ weakly in $L^\beta(Q)$, $\beta > 1$, $F : R^m \to [0, \infty]$ convex l.s.c

- **Weak lower semicontinuity.**

$$F(\mathbf{U}) \leq \liminf_{n\to\infty} F(\mathbf{U}_n) \text{ a.a. in } Q$$

- **Jensen's inequality.**

$$\left\langle \mathcal{V}; F(\widetilde{\mathbf{U}}) \right\rangle \equiv \int_{R^d} F(\widetilde{\mathbf{U}}) \mathrm{d}\mathcal{V} \geq F\left(\int_{R^d} \widetilde{\mathbf{U}} \mathrm{d}\mathcal{V} \right) \equiv F\left(\left\langle \mathcal{V}; \widetilde{\mathbf{U}} \right\rangle \right)$$

$$F \text{ strictly convex} \Rightarrow \Big(\text{equality holds } \Leftrightarrow \ \mathcal{V} = \delta_{\langle \mathcal{V}; \widetilde{\mathbf{U}} \rangle}\Big)$$

Oscillation defect

Energy oscillation defect

$$\mathfrak{E}_{\rm osc} = \left\langle \mathcal{V}; \left(\frac{1}{2} \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right) \right\rangle - \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \geq 0 \in L^\infty(0,T;L^1(\Omega))$$

Reynolds oscillation defect

$$\mathfrak{R}_{\rm osc} = \left\langle \mathcal{V}; 1_{\widetilde{\varrho}>0} \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right\rangle - 1_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\widetilde{\varrho}} + \left(\langle \mathcal{V}; p(\widetilde{\varrho}) \rangle - p(\varrho) \right) \mathbb{I}$$

Convexity:

$$\mathfrak{R}_{\rm osc} : (\xi \otimes \xi) = \left\langle \mathcal{V}; 1_{\widetilde{\varrho}>0} \frac{|\widetilde{\mathbf{m}} \cdot \xi|^2}{\widetilde{\varrho}} \right\rangle - 1_{\varrho>0} \frac{|\mathbf{m} \cdot \xi|^2}{\widetilde{\varrho}}$$

$$+ \left(\langle \mathcal{V}; p(\widetilde{\varrho}) \rangle - p(\varrho) \right) |\xi|^2 \geq 0$$

Defect compatibility

$$\min \left\{ \gamma - 1; \frac{1}{2} \right\} \mathfrak{E}_{\rm osc} \leq {\rm trace}[\mathfrak{R}_{\rm osc}] \leq \max\{\gamma - 1; 2\} \mathfrak{E}_{\rm osc}$$

Dissipative solutions

Continuity equation

$$\int_0^T \int_\Omega \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \, \mathrm{d}x = - \int_\Omega \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x$$

Momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + 1_{\varrho > 0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x - \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}(t) \mathrm{d}t$$

Energy inequality

$$\int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \, \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) \leq \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \, \mathrm{d}x$$

Defect compatibility

$$\underline{d} \mathfrak{E} \leq \mathrm{trace}[\mathfrak{R}] \leq \overline{d} \mathfrak{E}$$

(DMV)/dissipative solutions – summary

- Any stable consistent approximation generates (up to a subsequence) a DMV solution
- Any convex combination of DMV solutions (with the same initial data) is a DMV solutions
- Barycenter of any DMV solution is a dissipative solution

Defects

$$\mathfrak{R} = \mathfrak{R}_{\rm conc} + \mathfrak{R}_{\rm osc} \ \ \mathfrak{E} = \mathfrak{E}_{\rm conc} + \mathfrak{E}_{\rm osc}$$

$$\mathfrak{E}_{\rm conc} = 0 \Rightarrow \mathfrak{R}_{\rm conc} = 0$$

$\Rightarrow$ total energy of the generating sequence is equi–integrable

$$\mathfrak{E}_{\rm osc} = 0 \Rightarrow \mathfrak{R}_{\rm osc} = 0$$

$\Rightarrow$ the generating sequence converges strongly in $L^1((0,T) \times \Omega)$

Bregman distance – relative energy

Bregman distance

$$E \text{ convex: } d_B(\mathbf{U};\mathbf{V}) = E(\mathbf{U}) - \partial E(\mathbf{V})(\mathbf{U}-\mathbf{V}) - E(\mathbf{V}) \geq 0$$

Relative energy

$$E\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{M}\right) = 1_{\varrho>0}\frac{1}{2}\varrho\left|\frac{\mathbf{m}}{\varrho} - \frac{\mathbf{M}}{r}\right|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r)$$

$$E\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right) = 1_{\varrho>0}\frac{1}{2}\varrho\left|\frac{\mathbf{m}}{\varrho} - \mathbf{U}\right|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r),\ r\mathbf{U} = \mathbf{M}$$

$$E\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right) = \underbrace{\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)}_{\text{energy}} + \underbrace{\left(\frac{1}{2}|\mathbf{U}|^2 - P'(r)\right)\varrho}_{\text{computable from continuity equation}}$$

$$- \underbrace{\mathbf{m}\cdot\mathbf{U}}_{\text{computable from momentum equation}} + \underbrace{p(r)}_{\text{reference pressure}}$$

Gronwall's lemma

Thomas Hakon Gronwall (1877–1932)

Gronwall Lemma

$$0 \leq U(\tau) \leq a + \int_0^\tau \chi(t)U(t)\ \mathrm{d}t$$
$$\Leftarrow$$
$$U(\tau) \leq a \exp\left(\int_0^\tau \chi(t)\mathrm{d}t\right)$$

Relative energy inequality

$$\int_\Omega E\left(\varrho, \mathbf{m} \,\Big|\, r, \mathbf{U}\right)(\tau, \cdot)\ \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) \le \int_\Omega E\left(\varrho_0, \mathbf{m}_0 \,\Big|\, r(0,\cdot), \mathbf{U}(0,\cdot)\right)\ \mathrm{d}x$$

$$- \int_0^\tau \int_\Omega \varrho \left(\mathbf{U} - \frac{\mathbf{m}}{\varrho}\right) \cdot \mathbb{D}_x \mathbf{U} \cdot \left(\mathbf{U} - \frac{\mathbf{m}}{\varrho}\right)\ \mathrm{d}x\mathrm{d}t$$

$$- \int_0^\tau \int_\Omega \Big(p(\varrho) - p'(r)(\varrho - r) - p(r)\Big) \mathrm{div}_x \mathbf{U}\ \mathrm{d}x\mathrm{d}t$$

$$+ \int_s^\tau \int_\Omega \Big[\partial_t (r\mathbf{U}) + \mathrm{div}_x\,(r\mathbf{U} \otimes \mathbf{U}) + \nabla_x p(r)\Big] \cdot \frac{1}{r}\Big(\varrho \mathbf{U} - \mathbf{m}\Big)\ \mathrm{d}x\mathrm{d}t$$

$$+ \int_s^\tau \int_\Omega \Big[\partial_t r + \mathrm{div}_x (r\mathbf{U})\Big] \left[\left(1 - \frac{\varrho}{r}\right) p'(r) + \frac{1}{r}\mathbf{U} \cdot (\mathbf{m} - \varrho \mathbf{U})\right]\ \mathrm{d}x\mathrm{d}t$$

$$- \int_0^\tau \int_{\overline{\Omega}} \mathbb{D}_x \mathbf{U} : \mathrm{d}\mathfrak{R}(t)\mathrm{d}t$$

Weak–strong uniqueness and other applications

- **Weak–strong uniqueness.** Suppose that the Euler system admits a C^1(Lipschitz) solution $[\varrho, \mathbf{m}]$ in $[0, T) \times \overline{\Omega}$. Then $\mathfrak{E} = 0$, $\mathfrak{R} = 0$, and

$$\mathcal{V} = \delta_{[\varrho,\mathbf{m}]}$$

  for any DMV solution starting from the same initial data.

- **Compatibility.** Let $\mathcal{V}$ be a DMV solution. Suppose its barycenter $[\varrho, \mathbf{m}] \in C^1[0, T) \times \overline{\Omega}$. Then $[\varrho, \mathbf{m}]$ is a classical solution of the Euler system and $\mathfrak{E} = 0$, $\mathfrak{R} = 0$.

- **Lax equivalence principle for the Euler system**
  Suppose that the Euler system admits a C^1-solution. Then any stable consistent approximation converges strongly, specifically

$$\varrho_n \to \varrho \text{ in } L^q(0, T; L^\gamma(\Omega))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ in } L^q(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)) \text{ for any } 1 \leq q < \infty.$$

  There is no need to extract a subsequence.

- The above results can be extended to weak solutions satisfying a "one sided Lipschitz condition" $\mathbb{D}_x \mathbf{U} > -C$

Convergence to weak solution, I

Spatial domain, far field conditions

$$\Omega = R^d,\ \varrho \to \varrho_\infty,\ \mathbf{u} = \mathbf{u}_\infty$$

$$E\left(\varrho, \mathbf{u} \Big| \varrho_\infty, \mathbf{u}_\infty\right) = \varrho |\mathbf{u} - \mathbf{u}_\infty|^2 + P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty)$$

Consistency

$$\int_0^T \int_{R^d} \left[\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi\right] \mathrm{d}x\mathrm{d}t = e_{1,n}[\varphi]$$

$e_{1,n}[\varphi] \to 0$ for any $\varphi \in C^1_c((0,T) \times R^d)$

$$\int_0^T \int_{R^d} \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + 1_{\varrho_n > 0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \boldsymbol{\varphi} + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi}\right] \mathrm{d}x\mathrm{d}t = e_{2,n}[\boldsymbol{\varphi}]$$

$e_{2,n}[\boldsymbol{\varphi}] \to 0$ for any $\boldsymbol{\varphi} \in C^1_c((0,T) \times R^d; R^d)$

Stability

$$\sup_{n>0} \int_{R^d} E\left(\varrho_n, \mathbf{m}_n \ \Big| \ \varrho_\infty, \mathbf{m}_\infty\right) \mathrm{d}x < \infty$$

$\varrho_\infty,\ \mathbf{m}_\infty$ far field conditions (not necessarily constant)

Convergence to weak solution, II

Weak convergence

$$\varrho_n \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma_{\rm loc}(R^d))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}_{\rm loc}(R^d;R^d))$$

Weak solution

$$0 = \int_0^T \int_{R^d} \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + 1_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}\right] \mathrm{d}x\mathrm{d}t$$

$$= -\int_0^T \int_{R^d} \nabla_x\boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}(t)\mathrm{d}t$$

$$\mathrm{trace}[\mathfrak{R}] \approx \overline{E(\varrho,\mathbf{m})} - E(\varrho,\mathbf{m})$$

$$= \overline{E(\varrho,\mathbf{m})} - \lim_{n\to\infty} \partial E[\varrho_\infty,\mathbf{m}_\infty]\cdot(\varrho_n-\varrho,\mathbf{m}_n-\mathbf{m}) - E(\varrho,\mathbf{m})$$

$$= \overline{E(\varrho,\mathbf{m})} - \lim_{n\to\infty} \partial E[\varrho_\infty,\mathbf{m}_\infty]\cdot(\varrho_n-\varrho_\infty,\mathbf{m}_n-\mathbf{m}_\infty) - E(\varrho_\infty,\mathbf{m}_\infty)$$

$$-\Big[E(\varrho,\mathbf{m}) - \lim_{n\to\infty} \partial E[\varrho_\infty,\mathbf{m}_\infty]\cdot(\varrho-\varrho_\infty,\mathbf{m}-\mathbf{m}_\infty) - E(\varrho_\infty,\mathbf{m}_\infty)\Big]$$

$$= \overline{E\left(\varrho,\mathbf{m}\Big|\varrho_\infty,\mathbf{m}_\infty\right)} - E\left(\varrho,\mathbf{m}\Big|\varrho_\infty,\mathbf{m}_\infty\right) \geq 0$$

Convergence to weak solution, II

Limit problem

$$\mathrm{div}_x \mathfrak{R} = 0 \text{ in } \mathcal{D}'(R^d),\ \|\mathrm{trace}[\mathfrak{R}]\|_{\mathcal{M}(R^d)} < \infty,\ \mathfrak{R} \in \mathcal{M}^+(R^d; R^{d\times d}_{\mathrm{sym}})$$

$$\Rightarrow$$

$$\mathfrak{R} \equiv 0$$

Weak formulation

$$\int_{R^d} \nabla_x \varphi : \mathrm{d}\mathfrak{R} = 0 \text{ for any } \varphi \in C_c^\infty(R^d)$$

Larger class of test functions

Weak formulation

$$\int_{R^d} \nabla_x \varphi : \mathrm{d}\mathfrak{R} = 0 \text{ for any } \varphi \in C_c^\infty(R^d)$$

for any $\varphi \in C_c^\infty(R^d)$

Cut-off

$$0 \le \psi_R \le 1,\ \psi_R \in C_c^\infty(R^d)$$

$$\psi_R(Y) = 1 \text{ if } |Y| < r,\ \psi_R(Y) = 0 \text{ if } |Y| > 2r,\ |\nabla_x \psi_R| \le \frac{2}{R}$$

Globally Lipschitz test functions

$$0 = \int_{R^d} \nabla_x(\psi_R \varphi) : \mathrm{d}\mathfrak{R} = \int_{R^d} \psi_R \nabla_x \varphi : \mathrm{d}\mathfrak{R} + \int_{R^d} (\nabla_x \psi_R \otimes \varphi) : \mathrm{d}\mathfrak{R}$$

$$= \int_{|x|<R} \nabla_x \varphi : \mathrm{d}\mathfrak{R} + \int_{|x| \ge R} [\psi_R \nabla_x \varphi + (\nabla_x \psi_R \otimes \varphi)] : \mathrm{d}\mathfrak{R}$$

Conclusion

Extending the class of text functions

$$\int_{R^d} \nabla_x \varphi : \mathrm{d}\mathfrak{R} = 0$$

for any $\varphi \in C^\infty(R^d)$, $|\nabla_x \varphi| \leq c$

Special test function

$$\varphi, \ \varphi_i = \sum_{j=1}^{N} \xi_i \xi_j x_j$$

Conclusion

$$\int_{R^d} (\xi \otimes \xi) : \mathrm{d}\mathfrak{R} = 0 \Rightarrow (\xi \otimes \xi) : \mathfrak{R} = 0 \Rightarrow \mathfrak{R} = 0$$

Conclusion weak vs. strong

- If the limit is a weak solution of the Euler system, then the convergence is strong,

$$\varrho_n \to \varrho \text{ in } L^q(0,T; L^{\gamma}_{\mathrm{loc}}(R^d))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ in } L^q(0,T; L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(R^d; R^d))$$

  for any $1 \leq q < \infty$

- If the convergence is weak, then the limit IS NOT a weak solution of the Euler system

FV numerical scheme

$$(\varrho_h^0, \mathbf{u}_h^0) = (\Pi_{\mathcal{T}} \varrho_0, \Pi_{\mathcal{T}} \mathbf{u}_0)$$

$$D_t \varrho_K^k + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} F_h(\varrho_h^k, \mathbf{u}_h^k) = 0$$

$$D_t (\varrho_h^k \mathbf{u}_h^k)_K + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \left(\mathbf{F}_h(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) + \overline{p(\rho_h^k)} \mathbf{n} - h^\beta \left[\left[\mathbf{u}_h^k \right]\right] \right) = 0.$$

Mária Lukáčová (Mainz)

Hana Mizerová (Bratislava)

Discrete time derivative

$$D_t r_K^k = \frac{r_K^k - r_K^{k-1}}{\Delta t}$$

Upwind, fluxes

$$\mathrm{Up}[r, \mathbf{v}] = \overline{r}\ \overline{\mathbf{v}} \cdot \mathbf{n} - \frac{1}{2} |\overline{\mathbf{v}} \cdot \mathbf{n}| \, [[r]]$$

$$F_h(r, \mathbf{v}) = \mathrm{Up}[r, \mathbf{v}] - h^\alpha \, [[r]]$$

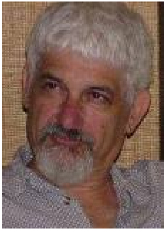
Strong instead of weak (numerics)

Komlos theorem (a variant of Strong Law of Large Numbers)

$$\{U_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q)$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} U_{n_k} \to \overline{U} \text{ a.a. in } Q \text{ as } N \to \infty$$



Janos Komlos (Rutgers Univ.)

Convergence of numerical solutions - EF, M.Lukáčová, H.Mizerová 2018

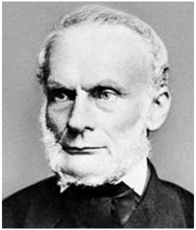
$$\frac{1}{N}\sum_{k=1}^{N} \varrho_{n_k} \to \varrho \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{m}_{n_k} \to \mathbf{m} \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \left[\frac{1}{2}\frac{|\mathbf{m}_{n,k}|^2}{\varrho_{n,k}} + P(\varrho_{n,k})\right] \to \overline{E} \in L^1((0,T)\times\Omega) \text{ a.a. in } (0,T)\times\Omega$$

LECTURE III: LARGE TIME BEHAVIOR AND TURBULENCE

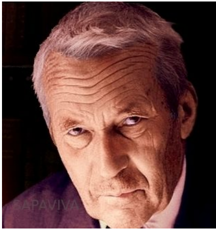
Motivation



Rudolf Clasius
1822–1888

Basic principles of thermodynamics of closed systems
The energy of the world is constant; its entropy tends to a maximum

Turbulence - ergodic hypothesis
Time averages along trajectories of the flow converge, for large enough times, to an ensemble average given by a certain probability measure



Andrey Nikolaevich Kolmogorov
1903–1987

Dynamical systems

Dynamical system

$$\mathbf{U}(t,\cdot) : [0,\infty) \times X \to X$$

- **Closed system:** $\mathbf{U}(t,\mathbf{U}_0) \to \mathbf{U}_\infty$ equilibrium solution as $t \to \infty$

- **Open system:** $\frac{1}{T}\int_0^T F(\mathbf{U}(t,\mathbf{U}_0))\mathrm{d}t \to \int_X F(X)\,\mathrm{d}\mu,\ T \to \infty,\ F \in BC(X)$

μ a.s. in $\mathbf{U}_0$

$\mu \in \mathfrak{P}[X]$ –probability measure on the phase space X

Principal mathematical problems:

- **Low regularity of global in time solutions**
  Global in time solutions necessary. For many problems in fluid dynamics – Navier–Stokes or Euler system – only weak solutions available
- **Lack of uniqueness**
  Solutions do not, or at least are not known to, depend uniquely on the initial data. Spaces of trajectories: Sell, Nečas, Temam and others
- **Propagation of oscillations**
  Realistic systems are partly hyperbolic: propagation of oscillations "from the past", singularities

Abstract setting

George Roger Sell
1937–2015

Space of entire trajectories

$$\mathcal{T} = C_{\rm loc}(R; X), \ \ t \in (-\infty, \infty)$$

ω**–limit set**

$$\mathbf{U} = \mathbf{U}(t, \mathbf{U}_0) \text{ entire trajectory } \mathbf{U}(0, \cdot) = \mathbf{U}_0$$

$$\omega[\mathbf{U}(\cdot, \mathbf{U}_0)] \subset \mathcal{T}$$

$$\omega[\mathbf{U}(\cdot, \mathbf{U}_0)] = \left\{ \mathbf{V} \in \mathcal{T} \ \middle| \ \mathbf{U}(\cdot + t_n, X_0) \to \mathbf{V} \text{ in } \mathcal{T} \text{ as } t_n \to \infty \right\}$$

Necessary ingredients

- **Dissipativity** – ultimate boundedness of trajectories
- **Compactness** – in appropriate spaces

Probability measures

X – Polish space, metrizable, complete, separable

$\mu_n \in \mathfrak{P}[X], \; n = 1, 2, \dots$ a family of probability measures

Tightness property

$$\forall \, \varepsilon > 0 \; \exists \text{ compact } K_\varepsilon \subset X, \; \mu_n(X \setminus K_\varepsilon) \leq \varepsilon \; \forall n \geq n(\varepsilon)$$

Yuri Prokhorov
1929–2013

Narrow convergence

$$\int F(\mathbf{U}) \, \mathrm{d}\mu_n(X) \to \int F(\mathbf{U}) \, \mathrm{d}\mu(X) \text{ for any } F \in BC(X)$$

Prokhorov Theorem

$$\{\mu_n\}_{n=1}^{\infty} \text{ tight } \; \Leftrightarrow \; \mu_{n_k} \to \mu \text{ weakly (narrowly) in } \mathfrak{P}[X]$$

Statistical solution

$\mathcal{T} = C(R; X)$ space of entire trajectories, $\omega[\mathbf{U}(0, \mathbf{U}_0)]$ $\omega-$ limit set $\subset \mathcal{T}$

(Global) statistical solution

- Statistical solution is

$$\boxed{\text{probability measure } \mu \in \mathfrak{P}[\mathcal{T}]}$$

  on the space of entire trajectories

- $$\mathbf{V} \in \omega[\mathbf{U}(0, \mathbf{U}_0)] \ \mu - \text{a.s. in } \mathcal{T} \ \Leftrightarrow \ \mathrm{supp}[\mu] \subset \omega[\mathbf{U}(0, \mathbf{U}_0)]$$

- **Stationarity.** μ is stationary if it is shift invariant

$$\mu[\mathcal{B}(\cdot + T)] = \mu[\mathcal{B}] \text{ for any } T \in R \text{ and any Borel set } \mathcal{B} \subset \mathcal{T}$$

- **Ergodicity.**

$$\mathcal{B}(\cdot + T) = \mathcal{B} \text{ for any } T \in R \ \Rightarrow \ \mu[\mathcal{B}] = 1 \text{ or } \mu[\mathcal{B}] = 0$$

Strong and weak ergodic hypothesis

Krylov – Bogolyubov construction

$$T \mapsto \frac{1}{T}\int_0^T \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t \text{ – a family of probability measures on } \mathcal{T}$$

$$\text{tightness in } \mathcal{T} \ \Rightarrow \ T_n \mapsto \frac{1}{T_n}\int_0^{T_n} \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

$$\textbf{Ergodic hypothesis} \ \Leftrightarrow \ \mu \text{ is unique } \Rightarrow \ T \mapsto \frac{1}{T}\int_0^T \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t \to \mu$$

$$\text{unique} \ \approx \ \text{unique on } \omega[\mathbf{U}(\cdot, X_0)]$$

Weak ergodic hypothesis

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t = \mu \text{ exists in the narrow sense in } \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

Barotropic Navier–Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \mathbf{g}$$

Constitutive equations

- barotropic (isentropic) pressure–density EOS $p = p(\varrho)$ ($p = a\varrho^\gamma$)
- Newton's rheological law

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I},\ \mu > 0,\ \eta \geq 0$$

- Gravitational external force

$$\mathbf{g} = \nabla_x F,\ F = F(x)$$

Energy

$$E(\varrho, \mathbf{m}) \equiv \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - \varrho F,\ P'(\varrho)\varrho - P(\varrho) = p(\varrho),\ \mathbf{m} = \varrho \mathbf{u}$$

Energy balance

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) = \varrho\partial_t\mathbf{u} + \varrho\mathbf{u}\cdot\nabla_x\mathbf{u}$$

$$\varrho\partial_t\mathbf{u}\cdot\mathbf{u} + \varrho\mathbf{u}\cdot\nabla_x\mathbf{u}\cdot\mathbf{u} + \mathrm{div}_x(p\mathbf{u}) - p\,\mathrm{div}_x\mathbf{u} = \mathrm{div}_x(\mathbb{S}\cdot\mathbf{u}) - \mathbb{S} : \nabla_x\mathbf{u} + \varrho\nabla_x F\cdot\mathbf{u}$$

$$\partial_t P(\varrho) + \mathrm{div}_x(P(\varrho)\mathbf{u}) = -p(\varrho)\mathrm{div}_x\mathbf{u},\ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

$$\varrho\nabla_x F\cdot\mathbf{u} = \mathrm{div}_x(\varrho F\mathbf{u}) - F\mathrm{div}_x(\varrho\mathbf{u}) = \mathrm{div}_x(\varrho F\mathbf{u}) + \partial_t(\varrho F)$$

Energy balance equation

$$\partial_t \underbrace{\left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) - \varrho F\right]}_{\textbf{total energy}} + \mathrm{div}_x\left(\left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) - \varrho F\right]\mathbf{u}\right)$$

$$+\mathrm{div}_x\left(p\mathbf{u} - \mathbb{S}\cdot\mathbf{u}\right) = -\underbrace{\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}}_{\textbf{dissipation}}$$

Energetically insulated system

Conservative boundary conditions

$\Omega \subset R^d$ bounded (sufficiently regular) domain

• **impermeability** $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$

• **no–slip** $[\mathbf{u}]_{\rm tan}|_{\partial\Omega} = 0$

Long–time behavior – Clausius scenario

- Total mass conserved

$$\int_\Omega \varrho(t, \cdot) \ {\rm d}x = M_0$$

- Total energy – Lyapunov function

$$\frac{\rm d}{{\rm d}t} \int_\Omega E(\varrho, \mathbf{m}) \ {\rm d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ {\rm d}x = (\leq) 0, \ \int_\Omega E(\varrho, \mathbf{m}) \ {\rm d}x \searrow \mathcal{E}_\infty$$

- Stationary solution

$$\mathbf{m}_\infty = 0, \ \nabla_x p(\varrho_\infty) = \varrho_\infty \nabla_x F, \ \int_\Omega \varrho_\infty \ {\rm d}x = M_0, \int_\Omega E(\varrho_\infty, 0) \ {\rm d}x = \mathcal{E}_\infty$$

Energetically open system

In/out flow boundary conditions

$$\mathbf{u} = \mathbf{u}_b \text{ on } \partial\Omega$$

$$\Gamma_{\rm in} = \left\{ x \in \partial\Omega \ \middle| \ \mathbf{u}_b(x) \cdot \mathbf{n}(x) < 0 \right\}, \ \Gamma_{\rm out} = \left\{ x \in \partial\Omega \ \middle| \ \mathbf{u}_b(x) \cdot \mathbf{n}(x) \geq 0 \right\}$$

Density (pressure) on the inflow boundary

$$\varrho = \varrho_b \text{ on } \Gamma_{\rm in}$$

Energy balance

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_b|^2 + P(\varrho) \ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x\mathrm{d}t \\ &\quad + \int_{\Gamma_{\rm in}} P(\varrho_b) \mathbf{u}_b \cdot \mathbf{n} \ \mathrm{d}S_x + \int_{\Gamma_{\rm out}} P(\varrho) \mathbf{u}_b \cdot \mathbf{n} \ \mathrm{d}S_x \\ &\quad = (\leq) - \int_\Omega [\varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho)\mathbb{I}] : \nabla_x \mathbf{u}_b \ \mathrm{d}x + \frac{1}{2} \int_\Omega \varrho \mathbf{u} \cdot \nabla_x |\mathbf{u}_b|^2 \ \mathrm{d}x\mathrm{d}t \\ &\quad + \int_\Omega \mathbb{S} : \nabla_x \mathbf{u}_b \ \mathrm{d}x\mathrm{d}t + \int_\Omega \varrho \nabla_x F \cdot (\mathbf{u} - \mathbf{u}_b) \ \mathrm{d}x \end{aligned}$$

Convergence to equilibria, I

Lyapunov function

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E(\varrho,\mathbf{m})\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\ \mathrm{d}x = (\leq)0,\ \int_\Omega E(\varrho,\mathbf{m})\ \mathrm{d}x \searrow \mathcal{E}_\infty$$

$$\int_0^\infty \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\ \mathrm{d}x\mathrm{d}t < \infty\ \Rightarrow\ \int_{T_n}^{T_n+1}\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\ \mathrm{d}x\mathrm{d}t \to 0$$

Convergence, step 1

$$\varrho_n(t,x) = \varrho(t+T_n,x),\ \mathbf{u}_n(t,x) = \mathbf{u}(t+T_n,x)$$

$$\text{Korn – Poincaré inequality}\ \Rightarrow\ \mathbf{u}_n \to 0 \text{ in } L^2((0,1)\times\Omega;R^d)$$

Compactness argument

$$\varrho_n \to \varrho_\infty \text{ in } L^1((0,T)\times\Omega)$$

$$\nabla_x p(\varrho_\infty) = \varrho_\infty \nabla_x F$$

Convergence to equilibria, II

Equilibrium solutions

$$\nabla_x p(\varrho_\infty) = \varrho_\infty \nabla_x F \ \Rightarrow \ \frac{1}{\varrho_\infty} p'(\varrho_\infty) \nabla_x \varrho_\infty = \nabla_x F$$

$$\nabla_x \mathcal{P}(\varrho_\infty) = \nabla_x F \ \ \Rightarrow \ \mathcal{P}(\varrho_\infty) = F + C, \ \varrho_\infty = \mathcal{P}^{-1}(F + C)$$

$$\mathcal{P}'(r) = \frac{p'(r)}{r} \approx \varrho^{\gamma - 2}$$

Uniqueness of equilibrium

$$\varrho_\infty > 0, \ \int_\Omega \varrho_\infty \ \mathrm{d}x = M_0 \ \Rightarrow \ \exists ! C = C(M_0)$$

Equilibria with vacuum

If merely $\varrho_\infty \geq 0$, certain geometric conditions needed

$$\left\{ x \in \Omega \ \middle| \ F(x) \geq D \right\} \text{ connected for any } D \geq 0$$

Conservative boundary conditions, summary

Hypotheses

$$\mathbf{u}|_{\partial\Omega=0} = 0,\ \mathbf{g} = \nabla_x F,\ F = F(x)$$

$$\left\{ x \in \Omega \ \middle|\ F(x) \geq D \right\} \text{ connected for any } D \geq 0$$

Conclusion

- **Convergence to equilibrium**

$$\varrho(t,\cdot) \to \varrho_\infty,\ \mathbf{m}(t,\cdot) \to 0 \text{ as } t \to \infty$$

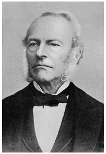
$$\nabla_x p(\varrho_\infty) = \varrho_\infty \nabla_x F,\ \int_\Omega \varrho_\infty \, \mathrm{d}x = M_0$$

- **Ergodic measure**

$$\Omega[(\varrho, \mathbf{m})] = (\varrho_\infty, 0) \ \Rightarrow\ \mu = \delta_{(\varrho_\infty, 0)} \ \Rightarrow\ \mu \text{ ergodic}$$

Complete Navier–Stokes–Fourier system

Claude Louis Marie Henri Navier [1785-1836]



George Gabriel Stokes [1819-1903]

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f}$$

Entropy production

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

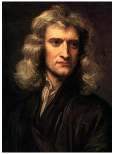
$$\sigma = (\geq) \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Constitutive relations

Joseph Fourier [1768-1830]

Fourier's law

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x\vartheta$$



Isaac Newton [1643-1727]

Newton's rheological law

$$\mathbb{S} = \mu(\vartheta)\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\right) + \eta(\vartheta)\mathrm{div}_x\mathbf{u}\mathbb{I}$$

Boundary conditions

Impermeability

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

No-slip

$$\mathbf{u}_{\tan}|_{\partial\Omega} = 0$$

No-stick

$$[\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Thermal insulation

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Long–time behavior

Dichotomy for the closed/open system

$$\mathbf{f} = \mathbf{f}(x)$$

Either

$$\mathbf{f} = \nabla_x F \;\Rightarrow\; \text{all solutions tend to a single equilibrium}$$

or

$$\mathbf{f} \neq \nabla_x F \;\Rightarrow\; \int_\Omega E(t,\cdot)\,\mathrm{d}x \to \infty \text{ as } t \to \infty$$

Nonconservative system, possible scenarios

Equilibrium solutions

$$\mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho F$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_b, \ \varrho|_{\Gamma_{\rm in}} = \varrho_b$$

Time–periodic solutions

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho F$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_b, \ \varrho|_{\Gamma_{\rm in}} = \varrho_b$$

$$(\varrho, \mathbf{u})(t + T_p, \cdot) = (\varrho, \mathbf{u})(t, \cdot), \ T_p > 0$$

High Reynolds number regime – turbulence

$$\mathbb{S}_\varepsilon(\nabla_x \mathbf{u}) \approx \varepsilon \mathbb{S}(\nabla_x \mathbf{u}), \ \varepsilon \to 0$$

Basic problems

$$T_n \to \infty$$

$$\varrho_n(t,x) = \varrho(t+T_n,x),\ \mathbf{u}_n(t,x) = \mathbf{u}(t+T_n,x)$$

Global boundedness

$$\int_\Omega \frac{1}{2}\varrho_n|\mathbf{u}_n - \mathbf{u}_b|^2 + P(\varrho_n)\ \mathrm{d}x \le E_\infty \text{ as } n \to \infty$$

Asymptotic compactness

$$\varrho_n \to \widetilde{\varrho},\ \mathbf{u}_n \to \widetilde{\mathbf{u}}$$

$\widetilde{\varrho}, \widetilde{\mathbf{u}}$ solution of the same problem?

Rigid motion of boundary

Energy balance

$$\mathbb{D}_x \mathbf{u} = \frac{1}{2}\left(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t\right)$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho|\mathbf{u}-\mathbf{u}_b|^2 + P(\varrho)\ \mathrm{d}x + \int_\Omega \mathbb{S}(\mathbb{D}_x\mathbf{u}) : \mathbb{D}_x\mathbf{u}\ \mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Gamma_{\rm in}} P(\varrho_b)\mathbf{u}_b\cdot\mathbf{n}\ \mathrm{d}S_x + \int_{\Gamma_{\rm out}} P(\varrho)\mathbf{u}_b\cdot\mathbf{n}\ \mathrm{d}S_x \\
&\quad = (\leq) - \int_\Omega [\varrho\mathbf{u}\otimes\mathbf{u} + p(\varrho)\mathbb{I}] : \boxed{\mathbb{D}_x\mathbf{u}_b}\ \mathrm{d}x + \frac{1}{2}\int_\Omega \varrho\mathbf{u}\cdot\nabla_x|\mathbf{u}_b|^2\ \mathrm{d}x\mathrm{d}t \\
&\quad + \int_\Omega \mathbb{S} : \boxed{\mathbb{D}_x\mathbf{u}_b}\ \mathrm{d}x\mathrm{d}t + \int_\Omega \varrho\nabla_x F\cdot(\mathbf{u}-\mathbf{u}_b)\ \mathrm{d}x
\end{aligned}$$

Boundary driving by rigid motion

$$\boxed{\mathbb{D}_x\mathbf{u}_b = 0,\ \nabla_x F\cdot\mathbf{u}_b = 0}$$

Convergence to equilibria

Stationary solutions

$$\mathbb{D}_x \mathbf{u}_\infty = 0$$

$$\mathrm{div}_x(\varrho_\infty \mathbf{u}_\infty) = 0$$

$$\mathrm{div}_x(\varrho_\infty \mathbf{u}_\infty \otimes \mathbf{u}_\infty) + \nabla_x p(\varrho_\infty) = \varrho_\infty \nabla_x F$$

$$\left\{ x \in \Omega \ \middle| \ F(x) \geq D \right\} \text{ connected for any } D \geq 0$$

Convergence to equilibria

Open system with the boundary conditions

$$\mathbf{u}_b = \mathbf{u}_\infty \text{ on } \partial\Omega, \ \varrho_b = \varrho_\infty \text{ on } \Gamma_{\mathrm{in}}, \ \int_\Omega \varrho_0 \ \mathrm{d}x = \int_\Omega \varrho_\infty \ \mathrm{d}x$$

$$\Rightarrow$$

$$\varrho \to \varrho_\infty, \ \mathbf{u} \ \to \mathbf{u}_\infty \text{ as } t \to \infty$$

Global bounded trajectories

Global in time weak solutions
$\mathbf{U} = [\varrho, \mathbf{m} = \varrho \mathbf{u}]$ – weak solution of the Navier–Stokes system satisfying energy inequality and defined for $t > T_0$

Bounded energy

$$\limsup_{t \to \infty} \int_\Omega E(\varrho, \mathbf{m}) \, \mathrm{d}x \leq \mathcal{E}_\infty$$

Available results

- **Existence:** T. Chang, B. J. Jin, and A. Novotný, *SIAM J. Math. Anal.*, **51**(2):1238–1278, 2019
  H. J. Choe, A. Novotný, and M. Yang *J. Differential Equations*, **266**(6):3066–3099, 2019
- **Globally bounded solutions:** F. Fanelli, E. F., and M. Hofmanová **arxiv preprint No. 2006.02278**, 2020
  J. Březina, E. F., and A. Novotný, Communications in PDE's 2020

Globally bounded energy

Energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho|\mathbf{u}-\mathbf{u}_b|^2 + P(\varrho)\ \mathrm{d}x + \boxed{\int_\Omega \mathbb{S}(\mathbb{D}_x\mathbf{u}) : \mathbb{D}_x\mathbf{u}\ \mathrm{d}x\mathrm{d}t}$$

$$+\int_{\Gamma_{\rm in}} P(\varrho_b)\mathbf{u}_b\cdot\mathbf{n}\ \mathrm{d}S_x + \int_{\Gamma_{\rm out}} P(\varrho)\mathbf{u}_b\cdot\mathbf{n}\ \mathrm{d}S_x$$

$$= (\leq) - \int_\Omega [\varrho\mathbf{u}\otimes\mathbf{u} + p(\varrho)\mathbb{I}] : \boxed{\mathbb{D}_x\mathbf{u}_b}\ \mathrm{d}x + \frac{1}{2}\int_\Omega \varrho\mathbf{u}\cdot\nabla_x|\mathbf{u}_b|^2\ \mathrm{d}x\mathrm{d}t$$

$$+\int_\Omega \mathbb{S} : \mathbb{D}_x\mathbf{u}_b\ \mathrm{d}x\mathrm{d}t + \int_\Omega \varrho\nabla_x F\cdot(\mathbf{u}-\mathbf{u}_b)\ \mathrm{d}x$$

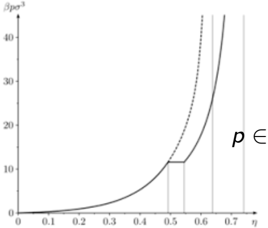
Gronwall argument

$$\frac{\mathrm{d}}{\mathrm{d}t}E + DE \stackrel{<}{\sim} C + \lambda E$$

What is needed:

- "Small" extension $\mathbf{u}_b$ inside Ω to make λ small
- $$E \stackrel{<}{\sim} \int_\Omega \mathbb{S}(\mathbb{D}_x\mathbf{u}) : \mathbb{D}_x\mathbf{u}\ \mathrm{d}x$$

Hard sphere pressure EOS



$$p \in C[0, \overline{\varrho}) \cap C^1(0, \overline{\varrho}),\ p'(\varrho) > 0 \text{ for } \varrho > 0,\ \lim_{\varrho \to \overline{\varrho}-} p(\varrho) = \infty$$

Ultimate boundedness of trajectories – bounded absorbing set

$$\limsup_{t \to \infty} \int_{\Omega} E(\varrho, \mathbf{m}) \, \mathrm{d}x \leq \mathcal{E}_\infty$$

$\mathcal{E}_\infty$ – universal constant

ω – limit sets

$$p \approx a\varrho^\gamma, \ \gamma > \frac{d}{2} \text{ or hard sphere EOS}$$

Trajectory space

$$X = \left\{ \varrho, \mathbf{m} \ \middle| \ \varrho(t,\cdot) \in L^\gamma(\Omega), \ \mathbf{m}(t,\cdot) \in L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d) \hookrightarrow W^{-k,2} \right\}$$

$$\mathcal{T} = C_{\rm loc}(R; L^1 \times W^{-k,2})$$

Fundamental result on compactness [Fanelli, EF, Hofmanová, 2020]

The ω–limit set $\omega[\varrho, \mathbf{m}]$ of each global in time trajectory with globally bounded energy is:

- $\boxed{non - empty}$
- $\boxed{compact}$ in $\mathcal{T}$
- time shift invariant
- consists of entire (defined for all $t \in R$) weak solutions of the Navier–Stokes system

Compactness of time shifts, I

Family of time shifts

$$\varrho_n(t,x) = \varrho(t+T_n,x),\ \mathbf{u}_n(t,x) = \mathbf{u}(t+T_n,x),\ \ T_n \to \infty$$

Weak convergence (up to a subsequence)

$$\varrho_n \to \varrho \text{ in } C_{\rm weak}([0,1];L^\gamma(\Omega))$$

$$\mathbf{u}_n \to \mathbf{u} \text{ weakly in } L^2([0,1];W^{1,2}(\Omega;R^d))$$

$$\mathbf{m}_n = \varrho_n \mathbf{u}_n \to \varrho\mathbf{u} \text{ in } C_{\rm weak}([0,1];L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d))$$

Lions identity

$$\overline{\varrho {\rm div}_x \mathbf{u}} - \varrho {\rm div}_x \mathbf{u} = \overline{p(\varrho)\varrho} - \overline{p(\varrho)}\ \varrho \geq 0$$

Compactness of time shifts, II

Renormalized equation of continuity

$$\partial_t b(\varrho_n) + \mathrm{div}_x (b(\varrho_n)\mathbf{u}_n) + \Big(b'(\varrho_n)\varrho_n - b(\varrho_n) \Big) \mathrm{div}_x \mathbf{u}_n = 0$$

$$\Rightarrow$$

$$\partial_t \overline{b(\varrho)} + \mathrm{div}_x (\overline{b(\varrho)}\mathbf{u}) + \overline{\Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u}} = 0$$

Renormalized equation of continuity for the limit

$$\partial_t b(\varrho) + \mathrm{div}_x (b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

Compactness of time shifts, III

Compactness of densities:

$$\varrho_n \equiv \varrho(\cdot + T_n) \to \varrho \text{ in } C_{\rm weak,loc}(R; L^\gamma(\Omega))$$

$$\varrho_n \log(\varrho_n) \to \overline{\varrho \log(\varrho)} \geq \varrho \log(\varrho)$$

oscillation defect: $D(t) \equiv \int_\Omega \overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \ {\rm d}x \geq 0$

Renormalized equation:

$$\frac{\rm d}{{\rm d}t} D + \int_\Omega \left[\overline{\varrho {\rm div}_x \mathbf{u}} - \varrho {\rm div}_x \mathbf{u} \right] \ {\rm d}x = 0, \ 0 \leq D \leq \overline{D}, \ t \in R$$

Lions' identity $\Rightarrow$

$$\frac{\rm d}{{\rm d}t} D + \int_\Omega \left[\overline{p(\varrho)\varrho} - \overline{p(\varrho)}\varrho \right] \ {\rm d}x = 0, \ 0 \leq D \leq \overline{D}, \ t \in R$$

Compactness of time shifts, IV

Strong convergence

$$D \equiv 0 \ \Rightarrow \ \text{strong converegence of } \{\varrho_n\}_{n=1}^{\infty}$$

Intuitive argument

1. F convex $\Rightarrow F(\varrho_n) - F(\varrho) \approx F'(\varrho)(\varrho_n - \varrho) + F''(\xi_n)(\varrho_n - \varrho)^2 \Rightarrow$
$\int_\Omega \overline{F(\varrho)} - F(\varrho)\ \mathrm{d}x \approx \lim_{n\to\infty} \int_\Omega |\varrho_n - \varrho|^2\ \mathrm{d}x$
2. p monotone $\Rightarrow |\varrho_n - \varrho|^2 \stackrel{<}{\sim} (p(\varrho_n) - p(\varrho))(\varrho_n - \varrho) \Rightarrow$
$\lim_{n\to\infty} \int_\Omega |\varrho_n - \varrho|^2\ \mathrm{d}x \stackrel{<}{\sim} \int_\Omega \overline{p(\varrho)\varrho} - \overline{p(\varrho)}\varrho\ \mathrm{d}x$

Crucial differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} D + \Psi(D) \leq 0,\ 0 \leq D \leq \overline{D},\ t \in R$$

$$\Psi \in C(R),\ \Psi(0) = 0,\ \Psi(Z)Z > 0 \text{ for } Z \neq 0$$

$$\Rightarrow D \equiv 0$$

Statistical stationary solutions

Application of Krylov – Bogolyubov method

$$\frac{1}{T_n}\int_0^{T_n} \delta_{\varrho(\cdot+t,\cdot),\mathbf{m}(\cdot+t,\cdot)}\ \mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}] \text{ narrowly}$$

$[\mathcal{T},\mu]$ (canonical representation) – statististical stationary solution

$\mu(t)|_X$ (marginal) independent of $t \in R$

Stationary process

$$\mathbf{U}(\omega) \in C(R;X),\ \omega \in \{\Omega,\mu\}$$

$$\mu\left\{[\mathbf{U}(t_1),\dots,\mathbf{U}(t_n)] \in \mathcal{B}\right\} = \mu\left\{[\mathbf{U}(t_1+T),\dots,\mathbf{U}(t_n+T)] \in \mathcal{B}\right\}$$

$$t_1 < t_2 \dots t_n,\ \ T > 0,\ \mathcal{B} \text{ Borel in } X^n$$

Stationary statistical solutions = stationary process

$$(\varrho,\mathbf{m}) \in \mathcal{T},\ \ t \mapsto (\varrho(t),\mathbf{m}(t)) - \text{stationary process in } [X,\mu]$$

Birkhoff–Khinchin ergodic theorem

Application of Birkhoff – Khinchin ergodic theorem

$$\frac{1}{T}\int_0^T F(\varrho(t,\cdot),\mathbf{m}(t,\cdot))\mathrm{d}t \to \overline{F} \text{ as } T \to \infty$$

F bounded Borel measurable on X for $\mu -$ a.a. $(\varrho, \mathbf{m}) \in \omega$

Related results for incompressible Navier–Stokes system with conservative boundary conditions

F.Flandoli and D. Gatarek, F.Flandoli and M.Romito (stochastic forcing),
P. Constantin and I. Procaccia, C. Foiaș, O. Manley, R. Rosa, and R. Temam,
M. Vishik and A. Fursikov etc (deterministic forcing)

Back to ergodic hypothesis – conclusion

Ergodicity

$$\mu \text{ ergodic } \Leftrightarrow \mathcal{B} \subset \omega[\varrho, \mathbf{m}] \text{ shift invariant } \Rightarrow \mu[\mathcal{B}] = 1 \text{ or } \mu[\mathcal{B}] = 0$$

$$\mu \in \text{conv}\Big\{ \text{ergodic measures on } \omega[\varrho, \mathbf{m}] \Big\}$$

State of the art for compressible Navier–Stokes system

- Each bounded energy global trajectory generates a stationary statistical solution – a shift invariant measure μ – sitting on its $\omega-$limit set $\omega[\varrho, \mathbf{m}]$
- The weak ergodic hypothesis (the existence of limits of ergodic averages for any Borel measurable F) holds on $\omega[\varrho, \mathbf{m}]$ $\mu-$a.s.
- The (strong) ergodic hypothesis definitely holds for enegetically isolated systems and a class of potential forces F, where all solutions tend to equilibrium

LECTURE IV: STATISTICAL SOLUTIONS AND TURBULENCE

Obstacle problem

Fluid domain and obstacle

$$\Omega = R^d \setminus B,\ d = 2, 3$$

$$B \text{ compact, convex}$$

Navier–Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$p(\varrho) \approx a\varrho^\gamma \text{ (convex)},\ \gamma > 1,\ \mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}$$

$$\mu > 0, \lambda \geq 0$$

Boundary and far field conditions

$$\mathbf{u}|_{\partial\Omega} = 0,\ \varrho \to \varrho_\infty,\ \mathbf{u} \to \mathbf{u}_\infty \text{ as } |x| \to \infty$$

$$\varrho_\infty \geq 0,\ \mathbf{u}_\infty \in R^d \text{ constant}$$

High Reynolds number (vanishing viscosity) limit

Vanishing viscosity

$$\varepsilon_n \searrow 0,\ \ \mu_n = \varepsilon_n \mu, \mu > 0,\ \ \lambda_n = \varepsilon_n \lambda, \lambda \geq 0$$

Questions

- Identify the limit of the corresponding solutions $(\varrho_n, \mathbf{u}_n)$ as $n \to \infty$ in the fluid domain Q
- **Yakhot and Orszak [1986]:** *"The effect of the boundary in the turbulence regime can be modeled in a statistically equivalent way by fluid equations driven by stochastic forcing"*

  Clarify the meaning of "statistically equivalent way"

  Is the (compressible) Euler system driven by a general "stochastic" force adequate to describe the limit of $(\varrho_n, \mathbf{u}_n)$?

Bounded energy solutions

Energy and relative energy

$$E(\varrho, \mathbf{u}) = \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho),\ E(\varrho, \mathbf{m}) = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ \mathbf{m} = \varrho\mathbf{u} - \text{convex}$$

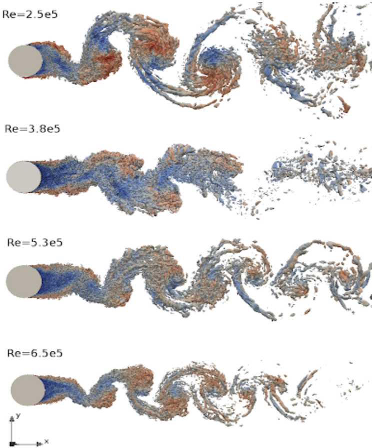
$$E\left(\varrho, \mathbf{u} \,\middle|\, \varrho_\infty, \mathbf{u}_\infty\right) = \frac{1}{2}\varrho|\mathbf{u} - \mathbf{u}_\infty|^2 + P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty)$$

$$P(\varrho) = \frac{a}{\gamma - 1}\varrho^\gamma,\ \mathbf{u}_\infty = 0 \text{ for } |x| < R_1,\ \mathbf{u}_\infty = \mathbf{u}_\infty \text{ for } |x| > R_2$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E\left(\varrho, \mathbf{u}\middle|\varrho_\infty, \mathbf{u}_\infty\right)\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\ \mathrm{d}x$$
$$\leq -\int_\Omega \Big(\varrho\mathbf{u}\otimes\mathbf{u} + p(\varrho)\mathbb{I}\Big) : \nabla_x\mathbf{u}_\infty\ \mathrm{d}x + \frac{1}{2}\int_\Omega \varrho\mathbf{u}\cdot\nabla_x|\mathbf{u}_\infty|^2\ \mathrm{d}x$$
$$+ \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}_\infty\ \mathrm{d}x.$$

Limit problem?



- **Oscillatory limit.** The sequence $(\varrho_n, \mathbf{m}_n)$ generates a Young measure. This scenario is compatible with the hypothesis that the limit is independent of the choice of $\varepsilon_n \searrow 0 \Rightarrow$ computable numerically.
- **Statistical limit.** The limit is a statistical solution of the Euler system in agreement with Kolmogorov hypothesis. This scenario is not compatible with the hypothesis that the limit is independent of $\varepsilon_n \searrow 0$ ($\Rightarrow$ numerically problematic) unless the limit is a monoatomic.

Trajectory space

(Weak) continuity

$$t \mapsto \int_\Omega \varrho \phi \ \mathrm{d}x \in C[0,T], \ t \mapsto \int_\Omega \varrho \mathbf{u} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \in C[0,T]$$

Boundedness of energy and dissipation

$$\sup_{0 \leq t \leq T} \int_\Omega E \left(\varrho, \mathbf{u} \ \middle| \varrho_\infty, \mathbf{u}_\infty \right) \ \mathrm{d}x < \infty$$

$$\boxed{\varepsilon} \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \mathrm{d}t < \infty$$

Trajectory space

$$\varrho \in C_{\rm weak, loc}([0,T]; L^\gamma(\Omega)), \ \varrho \mathbf{u} \equiv \mathbf{m} \in C_{\rm weak, loc}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

$$\mathcal{T} = \left\{ [\varrho, \mathbf{m}] \ \middle| \ [\varrho, \mathbf{m}] \in C_{\rm weak, loc}([0,T]; L^\gamma(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)) \right\}$$

Statistical solution to the Navier–Stokes system

Statistical solution

$\mathcal{V}$ Borel probability measure on the trajectory space $\mathcal{T}$

Properties

- $$\mathcal{V}\,[\text{finite energy weak solution of the NS system}] = 1$$
  $$\Leftrightarrow$$
  $$[\varrho, \mathbf{m}] \in \mathcal{T} \text{ solves NS system } a.s.$$
- **Examples.**

$$(\varrho, \mathbf{m}) - \text{a single solution } \approx \delta_{(\varrho, \mathbf{m})}$$

$$(\varrho_n, \mathbf{m}_n)_{n=1}^N \approx \sum_{n=1}^N \frac{1}{N} \delta_{(\varrho_n, \mathbf{m}_n)} - \text{convex combination of probability measures}$$

Statistical solution - random (stochastic) process

Probability space

$$\left\{ \underbrace{\mathcal{T}}_{\text{probability space}}, \underbrace{\mathfrak{B}(\mathcal{T})}_{\text{Borel sets}}, \underbrace{\mathcal{V}}_{\text{probability measure}} \right\}$$

Canonical representation

$$\omega(= [\varrho, \mathbf{m}]) \in \mathcal{T}, t \in [0, T] \mapsto (\varrho, \mathbf{m}) \in C\left([0, T] \left[L^{\gamma}(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d) \right]_{\text{weak,loc}} \right)$$

$$(\varrho, \mathbf{m}) - \text{random (stochastic proces)} : \mathcal{T} \to C(X) \text{ or } \mathcal{T} \times [0, T] \to X$$

General probability space

$$\{Q, \mathfrak{B}(Q), \mathcal{V}\}, \ (\varrho, \mathbf{m}) : Q \times [0, T] \to X$$

$$\text{alternatively } (\varrho, \mathbf{m}) : Q \to C(X)$$

High Reynolds number limit

Cesàro averages

$$(\varrho_n, \mathbf{m}_n) \in \mathcal{T} - \text{solutions of the NS system with viscosity } \varepsilon_n$$

Statistical solution

$$\mathcal{V}_N = \frac{1}{N} \sum_{n=1}^{N} \delta_{\varrho_n, \mathbf{m}_n} \in \mathfrak{P}(\mathcal{T})$$

With probability $\frac{1}{N}$ the statistical solution is $(\varrho_n, \mathbf{m}_n)$

General sequence of random processes

$$(\varrho, \mathbf{m})_N : \{Q; \nu\} \to \mathcal{T},\ \varepsilon_N : \{Q; \nu\} \to (0, \infty)$$

$$(\varrho, \mathbf{m})_N(\omega) \text{ solve NS system with vicosity } \varepsilon_N(\omega)$$

for $\nu-$a.a. ω.

Zero viscosity limit:

$$\nu \{\varepsilon_N > \delta\} \to 0 \text{ as } N \to \infty \text{ for any } \delta > 0$$

Observable quantities

$G : \mathcal{T} \to R$ Borel function, $(\varrho, \mathbf{m}) : \{Q; \nu\} \to \mathcal{T}$ random process

Expected (mean) value

$$\mathbb{E}[G(\varrho, \mathbf{m})] \equiv \int_Q G((\varrho, \mathbf{m}))\mathrm{d}\nu$$

Equivalence in law of two processes

$$(\varrho, \mathbf{m}) \approx (\widetilde{\varrho}, \widetilde{\mathbf{m}}) \;\Leftrightarrow\; \mathbb{E}[G(\varrho, \mathbf{m})] = \mathbb{E}[G(\widetilde{\varrho}, \widetilde{\mathbf{m}})]$$

$$\text{for any } G \in BC(\mathcal{T})$$

Equivalence on two different probability spaces

The concept of *equivalence* may be used for processes defined on different probability spaces but with the same range

High Reynolds number limit

Hypotheses

- **Approximations by random processes.**

$$(\varrho_N, \mathbf{m}_N) : \{Q; \nu\} \to \mathcal{T},\ \varepsilon_N : \{Q; \nu\} \to (0, \infty)$$

$(\varrho_n, \mathbf{m}_N = \varrho_N \mathbf{u}_N)$ solves NS system with viscosity ε_N $\nu-$a.s.

- **Vanishing viscosity.**

$$\nu \{\varepsilon_N > \delta\} \to 0 \text{ as } N \to \infty \text{ for any } \delta > 0$$

- **Boundedness of energy.**

$$\mathbb{E}\left[\sup_{0 \le t \le T} \int_\Omega E(\varrho_N, \mathbf{m}_N | \varrho_\infty, \mathbf{u}_\infty) \ \mathrm{d}x\right] < \overline{E}$$

- **Boundedness of energy dissipation**

$$\varepsilon_N \mathbb{E}\left[\int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_N) : \nabla_x \mathbf{u}_N \ \mathrm{d}x\right] < \overline{D}$$

Identifying the limit

The law (distribution) of the process

$$\mathcal{V}_N \in \mathfrak{P}(\mathcal{T} \times R)$$

$$\mathcal{V}_N(B) = \nu\left\{(\varrho_N, \mathbf{m}_n, \varepsilon_N)^{-1}(B)\right\}, \ B \text{ Borel in } \mathcal{T} \times R$$

$\mathcal{V}_N$ – family of probability measures

Yuri Prokhorov 1929–2013

Prokhorov Theorem

$$\{\mathcal{V}_N\}_{N=1}^{\infty} \text{ tight } \Leftrightarrow \ \mathcal{V}_{N_k} \to \mathcal{V} \text{ weakly (narrowly) in } \mathfrak{P}[\mathcal{T} \times R]$$

Tightness, I

Boundedness of energy and energy dissipation

$$\varepsilon_N \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x\mathrm{d}t < \overline{D} \ \Rightarrow \ \sqrt{\varepsilon_N} \|\nabla_x \mathbf{u}_N\|_{L^2} \stackrel{<}{\sim} \overline{D}$$

$$\left[\sup_{0 \leq t \leq T} \int_\Omega E(\varrho, \mathbf{m} | \varrho_\infty, \mathbf{u}_\infty) \ \mathrm{d}x \right] < \overline{E}$$

$$\Rightarrow \sup_{0 \leq t \leq T} \left(\|\varrho\|_{L^\gamma(M)} + \|\varrho \mathbf{u}\|_{L^{\frac{2\gamma}{\gamma+1}}(M;R^d)} \right) \stackrel{<}{\sim} \overline{E}(M) \text{ for any compact } M \subset \Omega$$

Tightness

$$\mathbb{E} \left[\varepsilon_N \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x\mathrm{d}t \right] < \overline{D}$$

$$\Leftrightarrow \ \int_Q \left[\varepsilon_N \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x\mathrm{d}t \right] \mathrm{d}\nu < \overline{D}$$

$$\Rightarrow \ \nu \left\{ \left[\varepsilon_N \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x\mathrm{d}t \right] \geq K \right\} \leq \frac{D}{K} \to 0 \text{ as } K \to \infty$$

Tightness, II

Tightness, energy bounds

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\int_\Omega E(\varrho,\mathbf{m}|\varrho_\infty,\mathbf{u}_\infty)\ \mathrm{d}x\right] < \overline{E}$$

$$\Leftrightarrow\ \int_Q\left[\sup_{0\leq t\leq T}\int_\Omega E(\varrho,\mathbf{m}|\varrho_\infty,\mathbf{u}_\infty)\ \mathrm{d}x\right]\mathrm{d}\nu < \overline{E}$$

$$\Rightarrow\ \nu\left\{\sup_{0\leq t\leq T}\int_\Omega E(\varrho,\mathbf{m}|\varrho_\infty,\mathbf{u}_\infty)\ \mathrm{d}x \geq K\right\} \leq \frac{E}{K} \to 0 \text{ as } K\to\infty$$

Compactness in the trajectory space

boundedness of energy and energy dissipation rate

$\Rightarrow$

compactness in the trajectory space $\mathcal{T}$

Tightness, III

Compactness in the space of weakly continuous functions

$$\left| \int_\Omega [\varrho(t_1,\cdot) - \varrho(t_2,\cdot)]\,\phi \ \mathrm{d}x \right| \leq \int_{t_1}^{t_2} \left| \int_\Omega \varrho \mathbf{u} \cdot \nabla_x \phi \ \mathrm{d}x \right| \mathrm{d}t$$

$$|t_1 - t_2| \sup \|\nabla_x \phi\| \sup_{0 \leq t \leq T} \|\varrho \mathbf{u}\|_{L^1(\Omega)}$$

$$\lesssim |t_1 - t_2| \sup \|\nabla_x \phi\| \sup_{0 \leq t \leq T} \int_\Omega E(\varrho, \mathbf{m} | \varrho_\infty, \mathbf{u}_\infty) \ \mathrm{d}x$$

Skorokhod representation theorem

Convergence in distribution (law)

$$(\varrho_N, \mathbf{m}_N, \varepsilon_N) \approx \mathcal{V}_N \in \mathfrak{P}(\mathcal{T} \times R) \text{ tight}$$

$$\text{Prokhorov theorem } \mathcal{V}_N \to \mathcal{V} \text{ narrowly } \mathcal{V} \in \mathfrak{P}(\mathcal{T} \times \{0\}) \approx \mathfrak{P}(\mathcal{T})$$

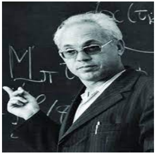
Skorokhod Theorem

$$\{\mathcal{V}_N\}_{N=1}^{\infty} \text{ tight } \Rightarrow \mathcal{V}_{N_k} \to \mathcal{V} \text{ weakly (narrowly) in } \mathfrak{P}[\mathcal{T} \times R]$$

$$\Rightarrow$$

$$\exists(\widetilde{\varrho}_N, \widetilde{\mathbf{m}}_N, \widetilde{\varepsilon}_N) \text{ with law } \mathcal{V}_N \Leftrightarrow (\widetilde{\varrho}_N, \widetilde{\mathbf{m}}_N, \widetilde{\varepsilon}_N) \approx (\varrho_N, \mathbf{m}_N, \varepsilon_N)$$

$$\widetilde{\varrho}_N \to \varrho, \ \widetilde{\mathbf{m}}_N \to \mathbf{m} \text{ in } \mathcal{T}, \widetilde{\varepsilon}_N \to 0 \ \boxed{a.s.}$$



A.V. Skorokhod 1930–2011

High Reynolds number limit revisited

Fluid domain and obstacle

$$\Omega = R^d \setminus B, \ d = 2, 3$$

B compact, convex

Navier–Stokes system

$$\partial_t \widetilde{\varrho}_N + \mathrm{div}_x \widetilde{\mathbf{m}}_N = 0$$

$$\partial_t \widetilde{\mathbf{m}}_N + \mathrm{div}_x \left(\frac{\widetilde{\mathbf{m}}_N \otimes \widetilde{\mathbf{m}}_N}{\widetilde{\varrho}_N} \right) + \nabla_x p(\widetilde{\varrho}_N) = \widetilde{\varepsilon}_N \mathrm{div}_x \widetilde{\mathbb{S}}_N$$

Boundary and far field conditions

$$\widetilde{\varrho}_N \tilde{\mathbf{u}}_N = \widetilde{\mathbf{m}}_N$$

$$\tilde{\mathbf{u}}_N|_{\partial\Omega} = 0, \ \widetilde{\varrho}_N \to \varrho_\infty, \ \tilde{\mathbf{u}}_N \to \mathbf{u}_\infty \text{ as } |x| \to \infty$$

Consergence a.s.

$$\widetilde{\varrho}_N \to \varrho, \ \widetilde{\mathbf{m}}_N \to \widetilde{\mathbf{m}} \text{ in } \mathcal{T}, \ \widetilde{\varepsilon}_N \to 0 \ \widetilde{\nu} - a.s. (\text{almost surely})$$

Limit problem

Fluid domain and obstacle

$$\Omega = R^d \setminus B, \ d = 2, 3$$

B compact, convex

Field equations

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho)\mathbb{I}} \right) = 0$$

in $\mathcal{D}'((0, T) \times \Omega)$ and ν *a.s.*

Boundary and far field conditions

$$\mathbb{E} \left[\int_\Omega E(\varrho, \mathbf{m} | \varrho_\infty, \mathbf{m}_\infty) \ \mathrm{d}x \right] < \overline{E}$$

Reynolds stress tensor

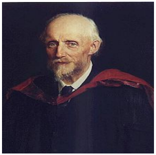
Weak limit of convective term

$$\frac{\widetilde{\mathbf{m}}_N \otimes \widetilde{\mathbf{m}}_N}{\widetilde{\varrho}_N} + p(\widetilde{\varrho}_N)\mathbb{I} \to \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho)\mathbb{I}}$$

weakly - (*) in $L^\infty(0,T;\mathcal{M}(\Omega;R^{d\times d}_{\rm sym}))$ a.s.

Reynolds stress

$$\mathfrak{R} \equiv \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho)\mathbb{I}} - \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho)\mathbb{I} \right)$$



Osborne Reynolds 1842–1912

Limit problem revisited

Fluid domain and obstacle

$$\Omega = R^d \setminus B, \ d = 2, 3$$

$$B \text{ compact, convex}$$

Perturbed Euler system

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \mathfrak{R}$$

$$\text{in } \mathcal{D}'((0, T) \times \Omega) \text{ and } \nu \ a.s.$$

Boundary and far field conditions

$$\mathbb{E} \left[\int_\Omega E(\varrho, \mathbf{m} | \varrho_\infty, \mathbf{m}_\infty) \ \mathrm{d}x \right] < \overline{E}$$

Properties of Reynolds stress

Positivity (semi) via convexity

$$\left[\overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho)\mathbb{I}} - \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho)\mathbb{I} \right) \right] : (\xi \otimes \xi)$$

$$= \left[\overline{\frac{|\mathbf{m} \cdot \xi|^2}{\varrho} + p(\varrho)|\xi|^2} - \left(\frac{|\mathbf{m} \cdot \xi|^2}{\varrho} + p(\varrho)|\xi| \right) \right] \geq 0$$

Integrability

$$\mathrm{trace}[\mathfrak{R}] \approx \overline{E(\varrho, \mathbf{m})} - E(\varrho, \mathbf{m})$$

$$= \overline{E(\varrho, \mathbf{m})} - \lim_{n \to \infty} \partial E[\varrho_\infty, \mathbf{m}_\infty] \cdot (\varrho_n - \varrho, \mathbf{m}_n - \mathbf{m}) - E(\varrho, \mathbf{m})$$

$$= \overline{E(\varrho, \mathbf{m})} - \lim_{n \to \infty} \partial E[\varrho_\infty, \mathbf{m}_\infty] \cdot (\varrho_n - \varrho_\infty, \mathbf{m}_n - \mathbf{m}_\infty) - E(\varrho_\infty, \mathbf{m}_\infty)$$

$$- \left[E(\varrho, \mathbf{m}) - \lim_{n \to \infty} \partial E[\varrho_\infty, \mathbf{m}_\infty] \cdot (\varrho - \varrho_\infty, \mathbf{m} - \mathbf{m}_\infty) - E(\varrho_\infty, \mathbf{m}_\infty) \right]$$

$$= \overline{E\left(\varrho, \mathbf{m} \middle| \varrho_\infty, \mathbf{m}_\infty \right)} - E\left(\varrho, \mathbf{m} \middle| \varrho_\infty, \mathbf{m}_\infty \right) \geq 0$$

Statistical equivalence

statistical equivalence $\Leftrightarrow$ identity in expectation of some quantities

$$(\varrho, \mathbf{m}) \text{ statistically equivalent to } (\widetilde{\varrho}, \widetilde{\mathbf{m}})$$

$$\Leftrightarrow$$

- **density and momentum**

$$\mathbb{E}\left[\int_D \varrho\right] = \mathbb{E}\left[\int_D \widetilde{\varrho}\right],\ \mathbb{E}\left[\int_D \mathbf{m}\right] = \mathbb{E}\left[\int_D \widetilde{\mathbf{m}}\right]$$

- **kinetic and internal energy**

$$\mathbb{E}\left[\int_D \frac{|\mathbf{m}|^2}{\varrho}\right] = \mathbb{E}\left[\int_D \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}}\right],\ \mathbb{E}\left[\int_D p(\varrho)\right] = \mathbb{E}\left[\int_D p(\widetilde{\varrho})\right]$$

- **angular energy**

$$\mathbb{E}\left[\int_D \frac{1}{\varrho}(\mathbb{J}_{x_0} \cdot \mathbf{m}) \cdot \mathbf{m}\right] = \mathbb{E}\left[\int_D \frac{1}{\widetilde{\varrho}}(\mathbb{J}_{x_0} \cdot \widetilde{\mathbf{m}}) \cdot \widetilde{\mathbf{m}}\right]$$

$$D \subset (0,T) \times Q,\ x_0 \in R^d,\ \mathbb{J}_{x_0}(x) \equiv |x - x_0|^2 \mathbb{I} - (x - x_0) \otimes (x - x_0)$$

Equivalence to a stochastically driven Euler system ?

Kiyosi Itô
1915–2008

Fluid domain and obstacle

$$\Omega = R^d \setminus B, \ d = 2, 3$$

B compact, convex

Limit Euler system

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \mathfrak{R}$$

Stochastically driven Euler system

$$\mathrm{d}\widetilde{\varrho} + \mathrm{div}_x \widetilde{\mathbf{m}} \ \mathrm{d}t = 0$$

$$\mathrm{d}\widetilde{\mathbf{m}} + \mathrm{div}_x \left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right) \ \mathrm{d}t + \nabla_x p(\widetilde{\varrho}) \ \mathrm{d}t = \sum_{i=1}^{\infty} F_i(\widetilde{\varrho}, \widetilde{\mathbf{m}}) \ \mathrm{d}W_i$$

W_i – cylindrical Wiener process, Itô integral

Equivalence of expected values

Martingale property

$$\mathbb{E}\left[\int_0^t \sum_{i=1}^{\infty} F_i(\widetilde{\varrho}, \widetilde{\mathbf{m}})\, \mathrm{d}W_i\right] = 0$$

Comparing the systems

$$\mathbb{E}\left[\mathrm{div}_x \mathfrak{R}\right] = \mathbb{E}\left[\mathrm{div}_x\left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}\right)\right]$$

in $\mathcal{D}'((0, T) \times Q)$

statistical equivalence of angular energies $\Rightarrow \mathbb{E}\left[\mathrm{div}_x\left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}\right)\right] = 0$

$$\mathbb{E}\left[\mathrm{div}_x \mathfrak{R}\right] \approx \mathrm{div}_x \mathbb{E}\left[\mathfrak{R}\right] = 0$$

Final problem

limit problem statistically equivalent to a stochastically driven Euler system

$$\Rightarrow$$

$$\mathfrak{R} \approx \mathbb{E}\left[\mathfrak{R}\right]$$

Equation for the Reynolds stress (for a.a. $t \in (0, T)$)

$$\mathrm{div}_x \mathfrak{R} = 0 \text{ in } \mathcal{D}'(R^d \setminus B), \ B \text{ compact convex}$$

$$\mathfrak{R} : [\xi \otimes \xi] \geq 0, \ \int_{R^d \setminus B} \mathrm{d} \ \mathrm{trace}[\mathfrak{R}] < \infty$$

Goal

Show $\mathfrak{R} \equiv 0$!

Final problem – solution, I

Assume

$$B = B_R - \text{ball of radius } R \text{ centered at } 0$$

- **Test functions.**

$$\phi_L(x) = \chi\left(\frac{|x|}{L}\right)\nabla_x F(|x|^2),\ \phi \in C^1_c(Q),\ L \geq 1$$

$$\chi \in C^\infty_c[0,\infty),\ \chi(Z) = 1 \text{ for } Z \leq 1,\ \chi(Z) = 0 \text{ for } Z \geq 2$$

$$F \text{ convex},\ F(Z) = 0 \text{ for } 0 \leq Z \leq R^2,\ 0 < F'(Z) \leq \overline{F} \text{ for } R^2 < Z < R^2 + 1$$

$$F'(Z) = \overline{F} \text{ if } Z \geq R^2 + 1,$$

- **Integral identity.**

$$\int_{R^d \setminus B_R} \nabla_x \phi_L : \mathrm{d}\mathfrak{R}\ \mathrm{d}x = 0$$

Final problem – solution, II

Step I

$$0 = \int_{R^d \setminus B_R} \chi\left(\frac{|x|}{L}\right) \nabla_x^2 F(|x|^2) : \mathrm{d}\mathfrak{R}$$
$$+ \frac{2}{L} \int_{L \le |x| \le 2L} \chi'\left(\frac{|x|}{L}\right) F'(|x|)|x| \left(\frac{x}{|x|} \otimes \frac{x}{|x|}\right) : \mathrm{d}\mathfrak{R}$$

Step II, integrability of $\mathfrak{R}$

$$\frac{2}{L} \left| \int_{L \le |x| \le 2L} \chi'\left(\frac{|x|}{L}\right) F'(|x|)|x| \left(\frac{x}{|x|} \otimes \frac{x}{|x|}\right) : \mathrm{d}\mathfrak{R} \right|$$
$$\le 4 \left| \int_{L \le |x| \le 2L} \chi'\left(\frac{|x|}{L}\right) F'(|x|) \left(\frac{x}{|x|} \otimes \frac{x}{|x|}\right) : \mathrm{d}\mathfrak{R} \right| \to 0 \text{ as } L \to \infty$$

Final problem – solution, III

Step III, convexity of F

$$\nabla_x^2 F(|x|^2) = 2\nabla_x \left(F'(|x|^2)x \right) = 4F''(|x|^2)(x \otimes x) + 2F'(|x|^2)\mathbb{I}$$

$$\Rightarrow$$

$$\liminf_{L\to\infty} \int_{R^d\setminus B_R} \chi\left(\frac{|x|}{L}\right) \nabla_x^2 F(|x|^2) : \mathrm{d}\mathfrak{R} \geq 2 \int_{R^d\setminus B_R} F'(|x|^2)\mathrm{d}\ \mathrm{trace}[\mathfrak{R}]$$

Step IV, general convex domain

$$R^d \setminus B = \cup\{R^d \setminus B_R \mid B_R \text{ contains } B\}$$

Stratonovich drift

Stochastic Euler system

$$\mathrm{d}\widetilde{\varrho} + \mathrm{div}_x \widetilde{\mathbf{m}} \mathrm{d}t = 0$$

$$\mathrm{d}\widetilde{\mathbf{m}} + \mathrm{div}_x \left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right) \mathrm{d}t + \nabla_x p(\widetilde{\varrho}) \mathrm{d}t = \boxed{(\sigma \cdot \nabla_x)\widetilde{\mathbf{m}} \circ \mathrm{d}W_1} + \mathbf{F}\, \mathrm{d}W_2$$

Additional hypotheses

- $Q = R^d$
- If $d = 2$, we need $\varrho_\infty = 0$; if $d = 3$, we need $\varrho_\infty = 0$, $\mathbf{u}_\infty = 0$, and $1 < \gamma \leq 3$

Ruslan Stratonovich 1930–1997

Similar type of noise used recently by Flandoli et al to produce a regularizig effect in the incompressible Navier–Stokes system

Conclusion

Hypothesis:

$(\varrho, \mathbf{m})$ statistically equivalent to a solution of the stochastic Euler system $(\widetilde{\varrho}, \widetilde{\mathbf{m}})$

Conclusion:

- **Noise inactive**
  $\mathfrak{R} = 0$, $(\varrho, \mathbf{m})$ is a statistical solution to a deterministic Euler system
- **S-convergence (up to a subsequence) to the limit system**

$$\frac{1}{N} \sum_{n=1}^{N} b(\varrho_n, \mathbf{m}_n) \to \mathbb{E}\left[b(\varrho, \mathbf{m})\right] \text{ strongly in } L^1_{\rm loc}((0,T) \times Q)$$

  for any $b \in C_c(R^{d+1})$, $\varphi \in C_c^\infty((0,T) \times Q)$
- **Conditional statistical convergence**

$$\text{barycenter } (\overline{\varrho}, \overline{\mathbf{m}}) \equiv \mathbb{E}\left[(\varrho, \mathbf{m})\right] \text{ solves the Euler system}$$

$$\Rightarrow$$

$$\frac{1}{N} \# \left\{ n \leq N \middle| \|\varrho_n - \overline{\varrho}\|_{L^\gamma(K)} + \|\mathbf{m}_n - \overline{\mathbf{m}}\|_{L^{\frac{2\gamma}{\gamma+1}}(K;R^d)} > \varepsilon \right\} \to 0 \text{ as } N \to \infty$$

  for any $\varepsilon > 0$, and any compact $K \subset [0,T] \times Q$

Perspectives?

- Stochastically driven Euler system irrelevant in the description of compressible turbulence (slightly extrapolated statement)

Possible scenarios:

- **Oscillatory limit.** The sequence $(\varrho_n, \mathbf{m}_n)$ generates a Young measure. Its barycenter (weak limit of $(\varrho_n, \mathbf{m}_n)$) is not a weak solution of the Euler system. Statistically, however, the limit is a single object. This scenario is compatible with the hypothesis that the limit is independent of the choice of $\varepsilon_n \searrow 0 \Rightarrow$ computable numerically.
- **Statistical limit.** The limit is a statistical solution of the Euler system. In agreement with Kolmogorov hypothesis concerning turbulent flow advocated in the compressible setting by Chen and Glimm. This scenario is not compatible with the hypothesis that the limit is independent of $\varepsilon_n \searrow 0$ ($\Rightarrow$ numerically problematic) unless the limit is a monoatomic measure in which case the convergence must be strong.