Lecture I: Fluid equations in continuum mechanics

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Basic principles

Physical domain: $\Omega \subset R^d$, d=1,2,3

Eulerian variables: time $t \in I = (0, T)$, $x \in \Omega$

Fields: $\mathbf{U} = \mathbf{U}(t, x) : I \times \Omega \to R^m$

- **volume densities of observables:** d = d(t,x)
- **fluxes:** $\mathbf{F} = \mathbf{F}$ (t,x)
- **sources:** s = s(t,x)

Field equations - balance laws:

$$\left[\int_B d(t,x)\mathrm{d}x\right]_{t=t_1}^{t=t_2} = \int_{t_1}^{t_2}\int_{\partial B} \mathbf{F}(t,x)\cdot\mathbf{n}\ \mathrm{d}S_x \mathrm{d}t + \int_{t_1}^{t_2}\int_B s(t,x)\ \mathrm{d}x\mathrm{d}t$$

for any

$$B \subset \Omega,\ t_1 < t_2$$

Balance laws as PDEs

Balance law (strong form):

- **scalar:**
$$\partial_t d(t,x) + \mathrm{div}_x \mathbf{F}(t,x) = s(t,x) \text{ for any } t, x$$
- **vectorial:**
$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F} = \mathbf{s}(t,x)$$

Weak (distributional) form:

$$\int_I \int_\Omega \Big[d(t,x)\partial_t \varphi(t,x) + \mathbf{F}(t,x) \cdot \nabla_x \varphi(t,x) \Big] \, \mathrm{d}x\mathrm{d}t = - \int_I \int_\Omega s(t,x)\varphi(t,x) \, \mathrm{d}x\mathrm{d}t$$

for any $\varphi \in C^1_c(I \times \Omega)$

Initial and/or boundary conditions:

$$\int_{t_1}^{t_2} \int_\Omega \Big[d\partial_t \varphi + \mathbf{F} \cdot \nabla_x \varphi \Big] \, \mathrm{d}x\mathrm{d}t$$
$$= \left[\int_\Omega d\varphi \, \mathrm{d}x \right]_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \int_{\partial\Omega} \mathbf{F}_b \cdot \mathbf{n}\varphi \, \mathrm{d}S_x \mathrm{d}t - \int_{t_1}^{t_2} \int_\Omega s\varphi \, \mathrm{d}x\mathrm{d}t$$

Physical principles

Mass conservation – equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

ϱ .... mass density
$\mathbf{u}$ .... fluid (bulk) velocity

Momentum conservation – Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x \mathbb{T} + \varrho \mathbf{g}$$

$\mathbb{T}$ .... Cauchy stress
$\mathbf{g}$ .... external volume force

Energy balance – First law of thermodynamics

$$\partial_t E + \mathrm{div}_x(E\mathbf{u}) = \mathrm{div}_x \Big(\mathbb{T} \cdot \mathbf{u} - \mathbf{q} \Big) + \varrho \mathbf{g} \cdot \mathbf{u}$$

E .... total energy
$\mathbf{q}$ .... (internal) energy flux

Energy balance

Total energy

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e$$

Kinetic energy balance

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 \right) + \mathrm{div}_x \left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u} \right) = \mathrm{div}_x(\mathbb{T}\cdot\mathbf{u}) - \mathbb{T} : \nabla_x\mathbf{u} + \varrho\mathbf{g}\cdot\mathbf{u}$$

Internal energy balance

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e\mathbf{u}) + \mathrm{div}_x\mathbf{q} = \mathbb{T} : \nabla_x\mathbf{u}$$

e .... internal energy

Constitutive relations

Stokes' law (mathematical definition of fluid)

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}$$

$\mathbb{S}$ .... viscous stress
p .... pressure

Gibbs' law – entropy

$$\vartheta D s = D e + p D \left(\frac{1}{\varrho} \right)$$

s .... entropy
ϑ .... (absolute) temperature

Second law of thermodynamics – entropy equation

Internal energy balance

$$\varrho \partial_t e + \varrho \mathbf{u} \cdot \nabla_x e + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u} \qquad | \frac{1}{\vartheta}$$

$$\text{Gibbs' law} \Rightarrow \frac{1}{\vartheta} \varrho \partial_t e = \varrho \partial_t s + \frac{1}{\vartheta \varrho} p \partial_t \varrho = \varrho \partial_t s - \frac{1}{\vartheta \varrho} p \mathrm{div}_x (\varrho \mathbf{u})$$

$$= \varrho \partial_t s - \frac{1}{\vartheta \varrho} p \nabla_x \varrho \cdot \mathbf{u} - \frac{1}{\vartheta} p \mathrm{div}_x \mathbf{u}$$

$$\frac{1}{\vartheta} \varrho \mathbf{u} \cdot \nabla_x e = \varrho \mathbf{u} \cdot \nabla_x s + \frac{1}{\vartheta \varrho} p \mathbf{u} \cdot \nabla_x \varrho$$

Entropy balance

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Entropy production rate

$$\frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \geq 0$$

Newtonian (linearly viscous) fluids

Newton's rheological law

$$\mathbb{S}(\mathbb{D}_x\mathbf{u}) = \mu\left(\nabla_x\mathbf{u} + \nabla_x\mathbf{u}^t - \frac{2}{d}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta\mathrm{div}_x\mathbf{u}\mathbb{I},\ \mathbb{D}_x \equiv \frac{1}{2}\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u}\right)$$

μ .... shear viscosity coefficient

η .... bulk viscosity coefficient

Fourier's law

$$\mathbf{q} = -\kappa\nabla_x\vartheta$$

κ .... heat conductivity coefficient

Entropy production rate

$$\frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right) \geq 0$$

$$\Rightarrow$$

$$\mu \geq 0,\ \eta \geq 0$$

Perfect fluids

$$\mathbb{S} \equiv 0, \ \mathbf{q} \equiv 0$$

$$\mathbf{m} \equiv \varrho \mathbf{u}, \ E = \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e$$

Leonhard Paul Euler 1707–1783

Euler system of gas dynamics

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p = 0$$

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0$$

(Incomplete) equation of state (gases)

$$p = (\gamma - 1) \varrho e$$

γ .... adiabatic coefficient

Perfect fluids–entropy

Entropy transport

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) = 0 \ \Rightarrow \ \partial_t s + \mathbf{u} \cdot \nabla_x s = 0$$

$$s = \bar{s} - \text{constant} \ \Rightarrow \ \text{isentropic EOS } p = p(\varrho)$$

Barotropic (isentropic) Euler system

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \ \partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + p(\varrho) = 0$$

"Nobody is perfect"

Entropy inequality

$$\partial_t(\varrho s) + \mathrm{div}_x(s\mathbf{m}) \geq 0$$

Renormalized entropy inequality

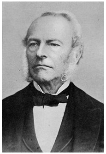
$$\partial_t(\varrho F(s)) + \mathrm{div}_x(F(s)\mathbf{m}) \geq 0$$

$$F' \geq 0$$

Navier–Stokes–Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u})$$



$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) - \mathrm{div}_x(\kappa \nabla_x \vartheta) = \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u}$$

Entropy balance equation

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) - \mathrm{div}_x \left(\frac{\kappa \nabla_x \vartheta}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S}(\mathbb{D}_x \mathbf{u}) : \nabla_x \mathbf{u} + \frac{\kappa |\nabla_x \vartheta|^2}{\vartheta} \right)$$

Entropies for viscous and perfect fluids

Renormalized entropy equation for the Euler system

$$\partial_t(\varrho F(s)) + \mathrm{div}_x(F(s)\mathbf{m}) = (\geq)0$$

$$\mathbf{m}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

Entropy minimum principle

$$s(t,x) \geq \inf_{y\in\Omega} s(0,y) \ \ t \geq 0$$

Entropy minimum principle for weak solutions

$$\int_\Omega \varrho F_n(s)(t,\cdot)\ \mathrm{d}x \geq \int_\Omega \varrho F_n(s)(0,\cdot)\ \mathrm{d}x$$

$$F_n(s) = n\min\{s - \underline{s}, 0\}\,,\ \ \underline{s} = \min_{y\in\Omega} s(0,y)$$

Absence of a large class of entropies for the Navier–Stokes–Fourier system
The composition $F(s)$ is in general not and entropy for the Navier–Stokes–Fourier system. The minimum principle may not hold either.

Boundary conditions for closed systems

Perfect fluid – impermeability of the boundary

$$\mathbf{m}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

Real fluid

- **Impermeability of the boundary, no heat flux**

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0$$

- **no slip or complete slip**

$$\mathbf{u}|_{\partial\Omega} = 0 \text{ or } (\mathbb{S}\cdot\mathbf{n})\times\mathbf{n}|_{\partial\Omega} = 0$$

Periodic boundary conditions

$$\Omega = \mathbb{T}^d,\ \mathbb{T}^d = \left([-1,1]|_{\{-1;1\}}\right)^d - \text{flat torus}$$

Boundary conditions for open systems – real fluids

Boundary velocity

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B$$

Boundary decomposition

$$\partial\Omega = \Gamma_{\rm in} \cup \Gamma_{\rm out} \cup \Gamma_{\rm wall}$$

$$\mathbf{u}_B \cdot \mathbf{n} < 0 \text{ on } \Gamma_{\rm in},\ \mathbf{u}_B \cdot \mathbf{n} > 0 \text{ on } \Gamma_{\rm out},\ \mathbf{u}_B \cdot \mathbf{n} = 0 \text{ on } \Gamma_{\rm wall}$$

Flux conditions

- **Mass flux**

$$\varrho|_{\Gamma_{\rm in}} = \varrho_B$$

- **Heat flux**

$$\left[\varrho_B e \mathbf{u}_B + \mathbf{q}\right] \cdot \mathbf{n}|_{\Gamma_{\rm in}} = F_B \quad \text{(Robin boundary conditions)}$$

$$\mathbf{q} \cdot \mathbf{n}|_{\Gamma_{\rm out} \cup \Gamma_{\rm wall}} = 0$$

Local well posedness for smooth data

Initial data

- **Smoothness in Sobolev class**

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0,\ \vartheta(0,\cdot) = \vartheta_0,\ \varrho_0, \mathbf{u}_0, \vartheta_0 \in W^{k,2},\ k > d$$

- **Smoothness of the boundary data**

$$\partial\Omega \text{ of class } C^{2+\nu}$$

$$\mathbf{u}_B,\ \varrho_B, F_B \in C^k,\ k \geq 2$$

- **Absence of vacuum**

$$\varrho_0,\ \varrho_B > 0,\ \vartheta_0 > 0$$

- **Compatibility conditions**
  Equations and their derivatives satisfied up to $t = 0$

Local existence
Both Euler and Navier–Stokes system admit smooth solutions existing on a maximal time interval $[0, T_{\max})$, $T_{\max} > 0$

Blow up for 1-d Euler system

Isentropic Euler system (constant entropy)

$$\partial_t \varrho + \partial_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \partial_x(\varrho \mathbf{u}^2) + a\partial_x \varrho^\gamma = 0$$

Step 1: Lagrange mass coordinates

$$t = t,\ y(t,x) = \int_{-\infty}^{x} \varrho(t,z)\ \mathrm{d}z$$

$$\partial_t V - \partial_y w = 0,\ \partial_t w + a\partial_y V^{-\gamma} = 0$$

$$V = \varrho^{-1} \text{ (specific volume)},\ w\left(t, \int_{-\infty}^{x} \varrho(t,z)\ \mathrm{d}z\right) = u(t,x)$$

Blow up for 1-d Euler system, continuation

Step 2: P–system

$$\partial_t V - \partial_y w = 0,\ \partial_t w - \partial_y P(V) = 0,\ P'(V) > 0$$

$$\sqrt{P'(V)}\partial_t V - \sqrt{P'(V)}\partial_y w = 0,\ \partial_t w - \sqrt{P'(V)}\sqrt{P'(V)}\partial_y V = 0$$

Step 3: Riemann invariants

$$\partial_t Z - A(Z)\partial_y w = 0,\ \partial_t w - A(Z)\partial_y Z = 0$$

$$Z(V) = \int_0^V \sqrt{P'(z)}\,\mathrm{d}z,\ A(Z) = \sqrt{P'(V(Z))}$$

Solution:

$$Z = \pm w$$

Blow up for 1-d Euler system, continuation

Step 3:

$$Z = -w,\ \partial_t Z + A(Z)\partial_x Z = 0$$

Burger's equation

$$\partial_t U + U\partial_y U = 0,\ A(Z) = U$$

Solutions of Burger's equation

$$U(t, y + tU_0(y)) = U_0(y),\ U(0, y) = U_0(y)$$

$$U_0(y_1) > U_0(y_2),\ y_2 > y_1 \ \Rightarrow \ \text{singularity at } \tau = \frac{y_2 - y_1}{U_0(y_1) - U_0(y_2)} > 0$$

Weak solutions to Euler system

Periodic boundary conditions

$$\Omega = \mathbb{T}^d$$

$$\int_0^T \int_{\mathbb{T}^d} \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \, \mathrm{d}x = - \int_{\mathbb{T}^d} \varrho_0 \varphi \, \mathrm{d}x$$

$$\varphi \in C_c^1([0, T) \times \mathbb{T}^d)$$

$$\int_0^T \int_{\mathbb{T}^d} \left[\mathbf{m} \cdot \partial \boldsymbol{\varphi} + 1_{\varrho > 0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \mathrm{d}t = - \int_{\mathbb{T}^d} \mathbf{m}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x$$

$$\boldsymbol{\varphi} \in C_c^1([0, T) \times \mathbb{T}^d; R^d)$$

$$\int_0^T \int_{\mathbb{T}^d} \left[E \partial_t \varphi + (E + p) \frac{\mathbf{m}}{\varrho} \cdot \nabla_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \mathrm{d}t = - \int_{\mathbb{T}^d} E_0 \varphi \, \mathrm{d}x$$

$$\varphi \in C_c^1([0, T) \times \mathbb{T}^d)$$

Entropy admissibility condition

$$\int_0^T \int_{\mathbb{T}^d} \left[\varrho s \partial_t \varphi + s \mathbf{m} \cdot \nabla_x \varphi \right] \, \mathrm{d}x \mathrm{d}t \leq - \int_{\mathbb{T}^d} \varrho_0 s_0 \varphi \, \mathrm{d}x, \ \varphi \geq 0$$

Ill–posedness of Euler system

Theorem:
Let $d = 2, 3$. Let $\varrho_0 > 0$, $\vartheta_0 > 0$ be piecewise constant, arbitrary functions. Then there exists $\mathbf{u}_0 \in L^\infty$ such that the Euler system admits infinitely many *admissible* weak solutions in $(0, T) \times \mathbb{T}^d$ with the initial data $[\varrho_0, \vartheta_0, \mathbf{u}_0]$.

Remarks:

- There are examples of Lipschitz initial data for which the Euler system admits infinitely many admissible weak solutions
- The result can be extended to the Euler system driven by stochastic forcing

Weak solutions to barotropic Euler system

Periodic boundary conditions

$$\Omega = \mathbb{T}^d$$

$$\int_0^T \int_{\mathbb{T}^d} \Big[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \Big] \ \mathrm{d}x = - \int_{\mathbb{T}^d} \varrho_0 \varphi \ \mathrm{d}x$$

$$\varphi \in C^1_c([0,T) \times \mathbb{T}^d)$$

$$\int_0^T \int_{\mathbb{T}^d} \left[\mathbf{m} \cdot \partial \boldsymbol{\varphi} + 1_{\varrho > 0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t = - \int_{\mathbb{T}^d} \mathbf{m}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x$$

$$\boldsymbol{\varphi} \in C^1_c([0,T) \times \mathbb{T}^d; R^d)$$

Energy inequality

$$\int_0^T \partial_t \psi \int_{\mathbb{T}^d} E(t) \ \mathrm{d}x\mathrm{d}t \geq - \int_{\mathbb{T}^d} E_0 \varphi \ \mathrm{d}x, \ \psi \in C^1_c[0,T), \ \psi \geq 0$$

$$E \equiv \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho), \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

Ill posedness for barotropic Euler system

Riemann integrable functions
$\mathcal{R}(Q)$ – the class of functions on Q that are Riemann integrable $\Leftrightarrow$ the functions are continuous at a.a. Lebesgue point

Theorem: Let $d = 2, 3$. Suppose that the initial data belong to the class

- $$\varrho_0 \in \mathcal{R}(\mathbb{T}^d),\ 0 < \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho}$$
- $$\mathbf{m}_0 \in \mathcal{R}(\mathbb{T}^d; R^d),\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R}(\mathbb{T}^d)$$

Let $\mathcal{E} \in \mathcal{R}(0, T)$ be given, $0 \leq \mathcal{E} \leq \overline{\mathcal{E}}$.
Then there exists a constant $\mathcal{E}_\infty \geq 0$ such that the barotropic Euler system admits infinitely many weak solutions with the initial data $[\varrho_0, \mathbf{m}_0]$ and the energy profile

$$\int_{\mathbb{T}^d} \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (t, \cdot)\ \mathrm{d}x = \mathcal{E} + \mathcal{E}_\infty$$

for a.a. $t \in (0, T)$.

Ill posedness for barotropic Euler system

Weak continuity vs. strong continuity

$$\varrho \in C_{\rm weak}([0,T]; L^\gamma(\mathbb{T}^d)),\ \mathbf{m} \in C_{\rm weak}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d; R^d))$$

Theorem: Let $d = 2, 3$. Let $\{\tau_j\}_{j=1}^\infty \subset (0, T)$ be a countable (possible dense) set of times. Suppose that the initial data belong to the class

- $$\varrho_0 \in \mathcal{R}(\mathbb{T}^d),\ 0 < \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho}$$
- $$\mathbf{m}_0 \in \mathcal{R}(\mathbb{T}^d; R^d),\ {\rm div}_x \mathbf{m}_0 \in \mathcal{R}(\mathbb{T}^d)$$

Then the barotropic Euler system admits infinitely many weak solutions $[\varrho, \mathbf{m}]$ with strictly decreasing total energy and such that

$$t \mapsto [\varrho(t,\cdot), \mathbf{m}(t,\cdot)] \text{ is NOT strongly continuous at any } \tau_j,\ j = 1, 2, \ldots$$

Lecture II: Stability and approximation

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Prologue - Lax equivalence principle

Formulation for LINEAR problems

Peter D. Lax

- **Stability** - uniform bounds of approximate solutions
- **Consistency** - vanishing approximation error

$$\Longleftrightarrow$$

- **Convergence** - approximate solutions converge to exact solution

Euler system of gas dynamics

Leonhard Paul Euler 1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \ \ \mathbf{m} = \varrho \mathbf{u}$$

Momentum equation – Newton's second law

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0, \ \ p(\varrho) = a \varrho^\gamma$$

Impermeability and/or periodic boundary condition

$$\mathbf{m} \cdot \mathbf{n}|_{\partial \Omega} = 0, \ \ \Omega \subset R^d, \ \text{or } \Omega = \mathbb{T}^d$$

Initial state

$$\varrho(0, \cdot) = \varrho_0, \ \ \mathbf{m}(0, \cdot) = \mathbf{m}_0$$

Weak (distributional) solutions

Jacques Hadamard 1865–1963

Laurent Schwartz 1915–2002

Mass conservation

$$\int_B \Big[\varrho(t_2,\cdot) - \varrho(t_1,\cdot)\Big]\mathrm{d}x = -\int_{t_1}^{t_2}\int_{\partial B} \varrho\mathbf{u}\cdot\mathbf{n}\ \mathrm{dS}_x\mathrm{d}t$$

$$\left[\int_\Omega \varrho\varphi\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big[\varrho\partial_t\varphi + \mathbf{m}\cdot\nabla_x\varphi\Big]\ \mathrm{d}x\mathrm{d}t,\ \ \mathbf{m}\equiv\varrho\mathbf{u}$$

Momentum balance

$$\int_B \Big[\mathbf{m}(t_2,\cdot) - \mathbf{m}(t_1,\cdot)\Big]\ \mathrm{d}x$$

$$= -\int_{t_1}^{t_2}\int_{\partial B}\Big[\mathbf{m}\otimes\mathbf{u}\cdot\mathbf{n} + p(\varrho)\mathbf{n}\Big]\ \mathrm{dS}_x\ \mathrm{d}t$$

$$\left[\int_\Omega \mathbf{m}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau\int_\Omega\left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}\right]\ \mathrm{d}x\mathrm{d}t$$

Time irreversibility – energy dissipation

Energy

$$E = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

$$p' \geq 0 \Rightarrow [\varrho, \mathbf{m}] \mapsto \left\{ \begin{array}{l} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0 \\ P(\varrho) \text{ if } |\mathbf{m}| = 0,\ \ \varrho \geq 0 \\ \infty \text{ otherwise} \end{array} \right. \quad \text{is convex l.s.c}$$

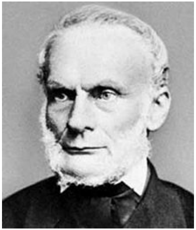
Energy balance (conservation)

$$\partial_t E + \mathrm{div}_x \left(E \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) = 0$$

Energy dissipation

$$\partial_t E + \mathrm{div}_x \left(E \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) \boxed{\leq} 0$$

$$\mathcal{E} = \int_\Omega E \ \mathrm{d}x,\ \ \partial_t \mathcal{E} \leq 0,\ \ \mathcal{E}(0+) = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$



Rudolf Clausius 1822–1888

Consistent approximation

Continuity equation

$$\int_0^T \int_\Omega [\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi] \, \mathrm{d}x\mathrm{d}t = - \int_\Omega \varrho_{0,n} \varphi(0, \cdot) \ \mathrm{d}x + e_{1,n}[\varphi]$$

$$\text{for any } \varphi \in C^1_c([0,T) \times \overline{\Omega})$$

Momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + 1_{\varrho_n > 0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \boldsymbol{\varphi} + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_{0,n} \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x + e_{2,n}[\boldsymbol{\varphi}]$$

$$\text{for any } \boldsymbol{\varphi} \in C^1_c([0,T) \times \overline{\Omega}; R^d) \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Energy dissipation

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] (\tau, \cdot) \mathrm{d}x \leq \mathcal{E}_{0,n}$$

Stability and Consistency

Stability

$$\limsup_{n\to\infty} \mathcal{E}_{0,n} < \infty$$

Data compatibility

$$\int_\Omega \varrho_{0,n}\varphi \ \mathrm{d}x \to \int_\Omega \varrho_0\varphi \ \mathrm{d}x \text{ for any } \varphi \in C_c^\infty(\Omega)$$

$$\int_\Omega \mathbf{m}_{0,n} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \to \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi} \ \mathrm{d}x \text{ for any } \boldsymbol{\varphi} \in C_c^\infty(\Omega; R^d)$$

$$\limsup_{n\to\infty} \mathcal{E}_{0,n} \le \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \mathrm{d}x$$

Vanishing approximation error

$$e_{1,n}[\varphi] \to 0 \text{ as } n \to \infty \text{ for any } \phi \in C_c^\infty([0,T) \times \overline{\Omega})$$

$$e_{2,n}[\boldsymbol{\varphi}] \to 0 \text{ as } n \to \infty \text{ for any } \boldsymbol{\varphi} \in C_c^\infty([0,T) \times \overline{\Omega}; R^d),\ \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Identifying the limit system, weak convergence

Isentropic pressure

$$p(\varrho) = a\varrho^{\gamma},\ \gamma > 1$$

Energy bounds

$$\varrho_n \text{ bounded in } L^{\infty}(0,T;L^{\gamma}(\Omega)),\ \mathbf{m}_n \text{ bounded in } L^{\infty}(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d))$$

Convergence (up to a subsequence)

$$\varrho_{n_k} \to \ \varrho \text{ weakly-(*) in } L^{\infty}(0,T;L^{\gamma}(\Omega))$$

$$\mathbf{m}_{n_k} \to \ \mathbf{m} \text{ weakly-(*) in } L^{\infty}(0,T;L^{\gamma}(\Omega;R^d))$$

$$E_{n_k} = \frac{1}{2}\frac{|\mathbf{m}_{n_k}|^2}{\varrho_{n_k}} + P(\varrho_{n_k}) \to \overline{\left(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right)} \text{ weakly-(*) in } L^{\infty}(0,T;\mathcal{M}^+(\overline{\Omega}))$$

$$1_{\varrho_{n_k}>0}\frac{\mathbf{m}_{n_k}\otimes\mathbf{m}_{n_k}}{\varrho_{n_k}} \to \overline{\left(1_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right)} \text{ weakly-(*) in } L^{\infty}(0,T;\mathcal{M}^+(\overline{\Omega};R^{d\times d}_{\mathrm{sym}}))$$

$$p(\varrho_{n_k}) \to \overline{p(\varrho)} \text{ weakly-(*) in } L^{\infty}(0,T;\mathcal{M}^+(\overline{\Omega}))$$

Convergence via Young Measures

Identification

$$(\varrho_n, \mathbf{m}_n)(t,x) \approx \delta_{\varrho_{n(t,x)}, \mathbf{m}_{n(t,x)}} = \mathcal{V}_n, \ \mathcal{V}_n : (0,T) \times \Omega \mapsto \mathfrak{P}(R^{d+1})$$

$$\mathcal{V}_n \in L^\infty_{\rm weak-(*)}((0,T)\times\Omega; \mathcal{M}^+(R^{d+1}))$$

$$\mathcal{V}_{n_k} \to \mathcal{V} \text{ weakly-(*) in } L^\infty_{\rm weak-(*)}((0,T)\times\Omega; \mathcal{M}^+(R^{d+1}))$$

$$\Leftrightarrow$$

Young measure

$$b(\varrho_{n_k}, \mathbf{m}_{n_k}) \to \overline{b(\varrho, \mathbf{m})} \text{ weakly-(*) in } L^\infty((0,T)\times\Omega) \text{ for any } b \in C_c(R^{d+1})$$

$$\langle \mathcal{V}_{t,x}; b(\tilde\varrho, \tilde{\mathbf{m}})\rangle = \overline{b(\varrho, \mathbf{m})}(t,x) \text{ for a.a. } (t,x) \in (0,T)\times\Omega$$

Basic properties:

$$\mathcal{V}_{t,x} \in \mathfrak{P}(R^{d+1}) \text{ for a.a. } (t,x) \in (0,T)\times\Omega$$

$$\mathcal{V}_{t,x} \text{ admits finite first moments and barycenter } \varrho = \langle \mathcal{V}; \tilde\varrho\rangle, \ \mathbf{m} = \langle \mathcal{V}; \tilde{\mathbf{m}}\rangle$$

Limit problem, I

Continuity equation

$$\int_0^T \int_\Omega \Big[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \Big] \ \mathrm{d}x = - \int_\Omega \varrho_0 \varphi(0, \cdot) \ \mathrm{d}x$$

$$\int_0^T \int_\Omega \Big[\langle \mathcal{V}_{t,x}; \tilde{\varrho} \rangle \, \partial_t \varphi + \langle \mathcal{V}_{t,x}; \tilde{\mathbf{m}} \rangle \cdot \nabla_x \varphi \Big] \ \mathrm{d}x = - \int_\Omega \varrho_0 \varphi(0, \cdot) \ \mathrm{d}x$$

for any $\varphi \in C^1_c([0,T) \times \overline{\Omega})$

Energy inequality

$$\int_{\overline{\Omega}} \mathrm{d} \overline{\left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right)} (\tau) \leq \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \ \mathrm{d}x \text{ for a.a. } \tau \geq 0$$

$$\int_\Omega \left\langle \mathcal{V}_{\tau,x}; \left(\frac{1}{2} \frac{|\tilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) \right) \right\rangle \ \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) \leq \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \ \mathrm{d}x$$

Energy concentration defect

$$\mathfrak{E}_{\rm conc} = \overline{\left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right)} - \left\langle \mathcal{V}_{\tau,x}; \left(\frac{1}{2} \frac{|\tilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) \right) \right\rangle \geq 0$$

Limit problem, II

Momentum equation

$$\int_0^T \int_\Omega \left[\langle \mathcal{V}; \tilde{\mathbf{m}} \rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}; 1_{\tilde{\varrho}>0} \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\varrho}} \right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}; p(\tilde{\varrho}) \rangle \, \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi} \ \mathrm{d}x - \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}(t) \mathrm{d}t$$

Reynolds concentration defect

$$\mathfrak{R}_{\rm conc} = \overline{\left(1_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\tilde{\varrho}} \right)} - \left\langle \mathcal{V}; 1_{\tilde{\varrho}>0} \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\varrho}} \right\rangle + \left(\overline{p(\varrho)} - \langle \mathcal{V}; p(\tilde{\varrho}) \rangle \right) \mathbb{I}$$

$$\mathfrak{R}_{\rm conc} : (\xi \otimes \xi)$$

$$= \overline{\left(1_{\varrho>0} \frac{|\mathbf{m} \cdot \xi|^2}{\tilde{\varrho}} \right)} - \left\langle \mathcal{V}; 1_{\tilde{\varrho}>0} \frac{|\tilde{\mathbf{m}} \cdot \xi|^2}{\tilde{\varrho}} \right\rangle + \left(\overline{p(\varrho)} - \langle \mathcal{V}; p(\tilde{\varrho}) \rangle \right) |\xi|^2 \geq 0$$

$$\Rightarrow \ \mathfrak{R}_{\rm conc} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\rm sym}))$$

Defect compatibility

$$\min \left\{ \gamma - 1; \frac{1}{2} \right\} \mathfrak{E}_{\rm conc} \leq \mathrm{trace}[\mathfrak{R}_{\rm conc}] \leq \max\{\gamma - 1; 2\} \mathfrak{E}_{\rm conc}$$

Dissipative measure–valued (DMV) solutions

Continuity equation

$$\int_0^T \int_\Omega \Big[\langle \mathcal{V}; \tilde{\varrho} \rangle \, \partial_t \varphi + \langle \mathcal{V}; \tilde{\mathbf{m}} \rangle \cdot \nabla_x \varphi \Big] \, \mathrm{d}x = - \int_\Omega \mathbf{m}_0 \varphi(0, \cdot) \, \mathrm{d}x$$

Momentum equation

$$\int_0^T \int_\Omega \left[\langle \mathcal{V}; \tilde{\mathbf{m}} \rangle \cdot \partial_t \boldsymbol{\varphi} + \left\langle \mathcal{V}; 1_{\tilde{\varrho} > 0} \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\varrho}} \right\rangle : \nabla_x \boldsymbol{\varphi} + \langle \mathcal{V}; p(\tilde{\varrho}) \rangle \, \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x - \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}_{\mathrm{conc}}(t) \mathrm{d}t$$

Energy inequality

$$\int_\Omega \left\langle \mathcal{V}_{\tau,x}; \left(\frac{1}{2} \frac{|\tilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) \right) \right\rangle \, \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}_{\mathrm{conc}}(\tau) \leq \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \, \mathrm{d}x$$

Defect compatibility

$$\underline{d} \mathfrak{E}_{\mathrm{conc}} \leq \mathrm{trace}[\mathfrak{R}_{\mathrm{conc}}] \leq \overline{d} \mathfrak{E}_{\mathrm{conc}}$$

Oscillation defect

Energy oscillation defect

$$\mathfrak{E}_{\rm osc} = \left\langle \mathcal{V}; \left(\frac{1}{2} \frac{|\tilde{\mathbf{m}}|^2}{\tilde{\varrho}} + P(\tilde{\varrho}) \right) \right\rangle - \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \geq 0 \in L^\infty(0, T; L^1(\Omega))$$

Reynolds oscillation defect

$$\mathfrak{R}_{\rm osc} = \left\langle \mathcal{V}; 1_{\tilde{\varrho}>0} \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\varrho}} \right\rangle - 1_{\varrho>0} \frac{\mathbf{m} \otimes \mathbf{m}}{\tilde{\varrho}} + \left(\langle \mathcal{V}; p(\tilde{\varrho}) \rangle - p(\varrho) \right) \mathbb{I}$$

Convexity:

$$\mathfrak{R}_{\rm osc} : (\xi \otimes \xi) = \left\langle \mathcal{V}; 1_{\tilde{\varrho}>0} \frac{|\tilde{\mathbf{m}} \cdot \xi|^2}{\tilde{\varrho}} \right\rangle - 1_{\varrho>0} \frac{|\mathbf{m} \cdot \xi|^2}{\tilde{\varrho}}$$

$$+ \left(\langle \mathcal{V}; p(\tilde{\varrho}) \rangle - p(\varrho) \right) |\xi|^2 \geq 0$$

Defect compatibility

$$\min \left\{ \gamma - 1; \frac{1}{2} \right\} \mathfrak{E}_{\rm osc} \leq {\rm trace}[\mathfrak{R}_{\rm osc}] \leq \max\{\gamma - 1; 2\} \mathfrak{E}_{\rm osc}$$

Dissipative solutions

Continuity equation

$$\int_0^T \int_\Omega \left[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \right] \, \mathrm{d}x = - \int_\Omega \varrho_0 \varphi(0, \cdot) \, \mathrm{d}x$$

Momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + 1_{\varrho > 0} \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x \mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x - \int_0^T \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}(t) \mathrm{d}t$$

Energy inequality

$$\int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right) \, \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) \leq \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right) \, \mathrm{d}x$$

Defect compatibility

$$\underline{d} \mathfrak{E} \leq \mathrm{trace}[\mathfrak{R}] \leq \overline{d} \mathfrak{E}$$

(DMV)/dissipative solutions – summary

- Any stable consistent approximation generates (up to a subsequence) a DMV solution
- Any convex combination of DMV solutions (with the same initial data) is a DMV solutions
- Barycenter of any DMV solution is a dissipative solution

Defects

$$\mathfrak{R} = \mathfrak{R}_{\rm conc} + \mathfrak{R}_{\rm osc} \ \mathfrak{E} = \mathfrak{E}_{\rm conc} + \mathfrak{E}_{\rm osc}$$

$$\mathfrak{E}_{\rm conc} = 0 \Rightarrow \mathfrak{R}_{\rm conc} = 0$$

$\Rightarrow$ total energy of the generating sequence is equi–integrable

$$\mathfrak{E}_{\rm osc} = 0 \Rightarrow \mathfrak{R}_{\rm osc} = 0$$

$\Rightarrow$ the generating sequence converges strongly in $L^1((0,T) \times \Omega)$

Bregman distance – relative energy

Bregman distance

$$E \text{ convex: } d_B(\mathbf{U};\mathbf{V}) = E(\mathbf{U}) - \partial E(\mathbf{V})(\mathbf{U}-\mathbf{V}) - E(\mathbf{V}) \geq 0$$

Relative energy

$$E\left(\varrho,\mathbf{m}\,\middle|\,r,\mathbf{M}\right) = 1_{\varrho>0}\frac{1}{2}\varrho\left|\frac{\mathbf{m}}{\varrho}-\frac{\mathbf{M}}{r}\right|^2 + P(\varrho) - P'(r)(\varrho-r) - P(r)$$

$$E\left(\varrho,\mathbf{m}\,\middle|\,r,\mathbf{U}\right) = 1_{\varrho>0}\frac{1}{2}\varrho\left|\frac{\mathbf{m}}{\varrho}-\mathbf{U}\right|^2 + P(\varrho) - P'(r)(\varrho-r) - P(r),\ \ r\mathbf{U}=\mathbf{M}$$

$$E\left(\varrho,\mathbf{m}\,\middle|\,r,\mathbf{U}\right) = \underbrace{\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)}_{\text{energy}} + \underbrace{\left(\frac{1}{2}|\mathbf{U}|^2 - P'(r)\right)\varrho}_{\text{computable from continuity equation}}$$

$$- \underbrace{\mathbf{m}\cdot\mathbf{U}}_{\text{computable from momentum equation}} + \underbrace{p(r)}_{\text{reference pressure}}$$

Relative energy inequality

$$\int_{\Omega} E\left(\varrho, \mathbf{m} \ \middle| r, \mathbf{U}\right)(\tau, \cdot) \ \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}(\tau) \leq \int_{\Omega} E\left(\varrho_0, \mathbf{m}_0 \ \middle| r(0,\cdot), \mathbf{U}(0,\cdot)\right) \ \mathrm{d}x$$

$$- \int_0^\tau \int_{\Omega} \varrho \left(\mathbf{U} - \frac{\mathbf{m}}{\varrho}\right) \cdot \mathbb{D}_x \mathbf{U} \cdot \left(\mathbf{U} - \frac{\mathbf{m}}{\varrho}\right) \ \mathrm{d}x\mathrm{d}t$$

$$- \int_0^\tau \int_{\Omega} \Big(p(\varrho) - p'(r)(\varrho - r) - p(r)\Big) \mathrm{div}_x \mathbf{U} \ \mathrm{d}x\mathrm{d}t$$

$$+ \int_s^\tau \int_{\Omega} \Big[\partial_t (r\mathbf{U}) + \mathrm{div}_x \left(r\mathbf{U} \otimes \mathbf{U}\right) + \nabla_x p(r)\Big] \cdot \frac{1}{r}\Big(\varrho\mathbf{U} - \mathbf{m}\Big) \ \mathrm{d}x\mathrm{d}t$$

$$+ \int_s^\tau \int_{\Omega} \Big[\partial_t r + \mathrm{div}_x (r\mathbf{U})\Big] \left[\left(1 - \frac{\varrho}{r}\right) p'(r) + \frac{1}{r}\mathbf{U} \cdot (\mathbf{m} - \varrho\mathbf{U})\right] \ \mathrm{d}x\mathrm{d}t$$

$$- \int_0^\tau \int_{\overline{\Omega}} \mathbb{D}_x \mathbf{U} : \mathrm{d}\mathfrak{R}(t)\mathrm{d}t$$

Weak–strong uniqueness and other applications

- **Weak–strong uniqueness.** Suppose that the Euler system admits a C^1(Lipschitz) solution $[\varrho, \mathbf{m}]$ in $[0, T) \times \overline{\Omega}$. Then $\mathfrak{E} = 0$, $\mathfrak{R} = 0$, and

$$\mathcal{V} = \delta_{[\varrho, \mathbf{m}]}$$

  for any DMV solution starting from the same initial data.

- **Compatibility.** Let $\mathcal{V}$ be a DMV solution. Suppose its barycenter $[\varrho, \mathbf{m}] \in C^1[0, T) \times \overline{\Omega}$. Then $[\varrho, \mathbf{m}]$ is a classical solution of the Euler system and $\mathfrak{E} = 0$, $\mathfrak{R} = 0$.

- **Lax equivalence principle for the Euler system**
  Suppose that the Euler system admits a C^1-solution. Then any stable consistent approximation converges strongly, specifically

$$\varrho_n \to \varrho \text{ in } L^q(0, T; L^\gamma(\Omega))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ in } L^q(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)) \text{ for any } 1 \leq q < \infty.$$

  There is no need to extract a subsequence.

- The above results can be extended to weak solutions satisfying a "one sided Lipschitz condition" $\mathbb{D}_x \mathbf{U} > -C$

Convergence to weak solution, I

Spatial domain

$$\Omega = R^d$$

Consistency

$$\int_0^T \int_{R^d} \left[\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi \right] \mathrm{d}x \mathrm{d}t = e_{1,n}[\varphi]$$

$$e_{1,n}[\varphi] \to 0 \text{ for any } \varphi \in C^1_c((0,T) \times R^d)$$

$$\int_0^T \int_{R^d} \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + 1_{\varrho_n > 0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \boldsymbol{\varphi} + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \mathrm{d}t = e_{2,n}[\boldsymbol{\varphi}]$$

$$e_{2,n}[\boldsymbol{\varphi}] \to 0 \text{ for any } \boldsymbol{\varphi} \in C^1_c((0,T) \times R^d; R^d)$$

Stability

$$\sup_{n>0} \int_{R^d} E\left(\varrho_n, \mathbf{m}_n \;\middle|\; \varrho_\infty, \mathbf{m}_\infty \right) \mathrm{d}x < \infty$$

$\varrho_\infty,\ \mathbf{m}_\infty$ far field conditions (not necessarily constant)

Convergence to weak solution, II

Weak convergence

$$\varrho_n \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma_{\rm loc}(R^d))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}_{\rm loc}(R^d;R^d))$$

Weak solution

$$0 = \int_0^T \int_{R^d} \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + 1_{\varrho>0}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} : \nabla_x\boldsymbol{\varphi} + p(\varrho){\rm div}_x\boldsymbol{\varphi}\right] {\rm d}x{\rm d}t$$
$$= -\int_0^T \int_{R^d} \nabla_x\boldsymbol{\varphi} : {\rm d}\mathfrak{R}(t){\rm d}t$$

$$\begin{aligned}
{\rm trace}[\mathfrak{R}] &\approx \overline{E(\varrho,\mathbf{m})} - E(\varrho,\mathbf{m}) \\
&= \overline{E(\varrho,\mathbf{m})} - \lim_{n\to\infty} \partial E[\varrho_\infty,\mathbf{m}_\infty]\cdot(\varrho_n-\varrho,\mathbf{m}_n-\mathbf{m}) - E(\varrho,\mathbf{m}) \\
&= \overline{E(\varrho,\mathbf{m})} - \lim_{n\to\infty} \partial E[\varrho_\infty,\mathbf{m}_\infty]\cdot(\varrho_n-\varrho_\infty,\mathbf{m}_n-\mathbf{m}_\infty) - E(\varrho_\infty,\mathbf{m}_\infty) \\
&\quad - \left[E(\varrho,\mathbf{m}) - \lim_{n\to\infty} \partial E[\varrho_\infty,\mathbf{m}_\infty]\cdot(\varrho-\varrho_\infty,\mathbf{m}-\mathbf{m}_\infty) - E(\varrho_\infty,\mathbf{m}_\infty)\right] \\
&= \overline{E\left(\varrho,\mathbf{m}\middle|\varrho_\infty,\mathbf{m}_\infty\right)} - E\left(\varrho,\mathbf{m}\middle|\varrho_\infty,\mathbf{m}_\infty\right) \geq 0
\end{aligned}$$

Convergence to weak solution, II

Limit problem

$$\mathrm{div}_x \mathfrak{R} = 0 \text{ in } \mathcal{D}'(R^d),\ \|\mathrm{trace}[\mathfrak{R}]\|_{\mathcal{M}(R^d)} < \infty,\ \mathfrak{R} \in \mathcal{M}^+(R^d; R^{d\times d}_{\mathrm{sym}})$$

$$\Rightarrow$$

$$\mathfrak{R} \equiv 0$$

Conclusion

If the limit is a weak solution of the Euler system, then the convergence is strong,

$$\varrho_n \to \varrho \text{ in } L^q(0,T; L^{\gamma}_{\mathrm{loc}}(R^d))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ in } L^q(0,T; L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}(R^d; R^d))$$

for any $1 \leq q < \infty$

FV numerical scheme

$$(\varrho_h^0, \mathbf{u}_h^0) = (\Pi_{\mathcal{T}} \varrho_0, \Pi_{\mathcal{T}} \mathbf{u}_0)$$

$$D_t \varrho_K^k + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} F_h(\varrho_h^k, \mathbf{u}_h^k) = 0$$

$$D_t(\varrho_h^k \mathbf{u}_h^k)_K + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \left(\mathbf{F}_h(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) + \overline{p(\rho_h^k)} \mathbf{n} - h^\beta \left[\left[\mathbf{u}_h^k \right]\right] \right) = 0.$$

Mária Lukáčová (Mainz)

Hana Mizerová (Bratislava)

Discrete time derivative

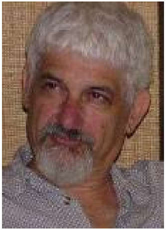
$$D_t r_K^k = \frac{r_K^k - r_K^{k-1}}{\Delta t}$$

Upwind, fluxes

$$\mathrm{Up}[r, \mathbf{v}] = \overline{r}\ \overline{\mathbf{v}} \cdot \mathbf{n} - \frac{1}{2} |\overline{\mathbf{v}} \cdot \mathbf{n}|\, [[r]]$$

$$F_h(r, \mathbf{v}) = \mathrm{Up}[r, \mathbf{v}] - h^\alpha\, [[r]]$$

Strong instead of weak (numerics)



Janos Komlos (Rutgers Univ.)

Komlos theorem (a variant of Strong Law of Large Numbers)

$$\{U_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q)$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} U_{n_k} \to \overline{U} \text{ a.a. in } Q \text{ as } N \to \infty$$

Convergence of numerical solutions - EF, M.Lukáčová, H.Mizerová 2018

$$\frac{1}{N}\sum_{k=1}^{N} \varrho_{n_k} \to \varrho \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{m}_{n_k} \to \mathbf{m} \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \left[\frac{1}{2}\frac{|\mathbf{m}_{n,k}|^2}{\varrho_{n,k}} + P(\varrho_{n,k})\right] \to \overline{E} \in L^1((0,T)\times\Omega) \text{ a.a. in } (0,T)\times\Omega$$

Lecture III: Methods of averaging

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Strong Law of Large Numbers

Strong Law of Large Numbers

$$\{U_n\}_{n=1}^{\infty}\ \boxed{\textit{independent}}\ \text{random variables},\ \mathbb{E}(U_n) = \mu$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{n=1}^{N} U_n \to \mu \text{ as } N \to \infty \text{a.s.}$$

Subsequence principle – Banach–Saks Theorem

$$\int_{\Omega} |U_n|^q \, \mathrm{d}x \le c \text{ uniformly for } n \to \infty,\ q > 1$$

$$\Rightarrow$$

there is a subsequence $\{U_{n_k}\}_{k=1}^{\infty}$ such that

$$\frac{1}{N}\sum_{l=1}^{N} U_{n_l} \to U \text{ in } L^q(\Omega) \text{ as } N \to \infty$$

for any subsequence $\{n_l\} \subset \{n_k\}$

Komlós theorem

$$\int_\Omega |U_n| \, \mathrm{d}x \leq c \text{ uniformly for } n \to \infty$$

$$\Rightarrow$$

there is a subsequence $\{U_{n_k}\}_{k=1}^\infty$ such that

$$\frac{1}{N} \sum_{l=1}^N U_{n_l} \to U \in L^1(\Omega) \text{ as } N \to \infty \text{ a.a. in } \Omega$$

for any subsequence $\{n_l\} \subset \{n_k\}$

Elementary proof of Banach–Saks Theorem in L^2:

U_n an orthonormal basis, $U_n \to 0$ weakly in L^2

$$\int_Q \left(\sum_{n=1}^N U_n \right)^2 \mathrm{d}y = \sum_{n=1}^N \int_Q |U_n|^2 \, \mathrm{d}y = N$$

$$\Rightarrow$$

$$\left\| \frac{1}{N} \sum_{n=1}^N U_n \right\|_{L^2}^2 = \frac{N}{N^2} = \frac{1}{N}$$

Example : Ergodic hypothesis

Asymptotic behavior of dynamical systems

$$t \in [0, \infty) \mapsto \mathbf{U}(t) \in X,$$

ω-limit set

$$\omega[\mathbf{U}] = \left\{ \mathbf{u} \in X \;\middle|\; \text{there exists } t_n \to \infty \ \mathbf{U}(t_n) \to \mathbf{u} \right\}$$

Ergodic hypothesis

$$\frac{1}{T} \int_0^T F(\mathbf{U}(t)) \mathrm{d}t \to \overline{F} \text{ as } T \to \infty \text{ for any Borel } F \in \mathcal{B}(X; R)$$

Birkhoff–Khinchin ergodic theorem

$$\mathbf{U}(t) : R \to X \text{ stationary process } \Rightarrow \frac{1}{T} \int_0^T F(\mathbf{U}(t)) \mathrm{d}t \to \overline{F} \text{ a.s.}$$

Euler system of gas dynamics

Leonhard Paul Euler
1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \ \mathbf{m} = \varrho \mathbf{u}$$

Momentum equation – Newton's second law

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0, \ p(\varrho) = a\varrho^\gamma$$

Impermeability and/or periodic boundary condition

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \Omega \subset R^d, \ \text{or } \Omega = \mathbb{T}^d$$

Initial state

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0$$

Consistent approximation

Continuity equation

$$\int_0^T \int_\Omega \left[\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi \right] \mathrm{d}x\mathrm{d}t = - \int_\Omega \varrho_{0,n} \varphi(0, \cdot) \ \mathrm{d}x + e_{1,n}[\varphi]$$

$$\text{for any } \varphi \in C^1_c([0,T) \times \overline{\Omega})$$

Momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + 1_{\varrho_n > 0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \boldsymbol{\varphi} + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_{0,n} \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x + e_{2,n}[\boldsymbol{\varphi}]$$

$$\text{for any } \boldsymbol{\varphi} \in C^1_c([0,T) \times \overline{\Omega}; R^d) \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Energy dissipation

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] (\tau, \cdot) \mathrm{d}x \leq \mathcal{E}_{0,n}$$

Stability and Consistency

Stability

$$\limsup_{n\to\infty} \mathcal{E}_{0,n} < \infty$$

Data compatibility

$$\int_\Omega \varrho_{0,n}\varphi \, \mathrm{d}x \to \int_\Omega \varrho_0\varphi \, \mathrm{d}x \text{ for any } \varphi \in C_c^\infty(\Omega)$$

$$\int_\Omega \mathbf{m}_{0,n} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \to \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x \text{ for any } \boldsymbol{\varphi} \in C_c^\infty(\Omega; R^d)$$

$$\limsup_{n\to\infty} \mathcal{E}_{0,n} \leq \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \mathrm{d}x$$

Vanishing approximation error

$$e_{1,n}[\varphi] \to 0 \text{ as } n \to \infty \text{ for any } \phi \in C_c^\infty([0,T) \times \overline{\Omega})$$

$$e_{2,n}[\boldsymbol{\varphi}] \to 0 \text{ as } n \to \infty \text{ for any } \boldsymbol{\varphi} \in C_c^\infty([0,T) \times \overline{\Omega}; R^d), \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Weak convergence of consistent approximation

Weak convergence
If consistent approximation DOES NOT converge strongly, the following must be satisfied:

- the limit Euler system does not admit a strong solution
- the limit $[\varrho, \mathbf{m}]$ is not C^1 smooth
- the limit $[\varrho, \mathbf{m}]$ IS NOT a weak solution of the Euler system (slightly vague statement)

Visualization of weak convergence?

- **Oscillations.** Weakly converging sequence may develop oscillations. Example:

$$\sin(nx) \to 0 \text{ weakly as } n \to \infty$$

- **Concentrations.**

$$n\theta(nx) \to \delta_0 \text{ weakly-(*) in } \mathcal{M}(R)$$

if

$$\theta \in C_c^\infty(R),\ \theta \geq 0, \int_R \theta = 1$$

Statistical description of oscillations – Young measures

Laurence Chisholm Young 1905–2000

Young measure

$$b(\varrho_n, \mathbf{m}_n) \to \overline{b(\varrho, \mathbf{m})} \text{ weakly-(*) in } L^\infty((0,T)\times\Omega)$$

(**up to a subsequence**) for any $b \in C_c(R^{d+1})$

Young measure $\mathcal{V}$ – a parametrized family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ on the phase space R^{d+1}:

$$\overline{b(\varrho, \mathbf{m})}(t,x) = \langle \mathcal{V}_{t,x}; b(\widetilde{\varrho}, \widetilde{\mathbf{m}}) \rangle \text{ for a.a. } (t,x)$$

Visualizing Young measure

visualizing Young measure $\Leftrightarrow$ computing $\overline{b(\varrho, \mathbf{m})}$

Problems

- $b(\varrho_n, \mathbf{m}_n)$ converge only weakly
- extracting subsequences
- only statistical properties relevant $\Rightarrow$ knowledge of the "tail" of the sequence of approximate solutions absolutely necessary

(S) - convergence, basic idea

Trivial example of oscillatory sequence

$$U_n = \begin{cases} 1 \text{ for } n \text{ odd} \\ -1 \text{ for } n \text{ even} \end{cases}$$

Convergence via Young measure approach

Convergence up to a $\boxed{\textit{subsequence}}$

$$U_n \approx \delta_{U_n},\ \ U_{n_k} \to \begin{cases} \delta_1 \text{ as } k \to \infty,\ \ n_k \text{ odd} \\ \delta_{-1} \text{ as } k \to \infty,\ \ n_k \text{ even} \end{cases}$$

Convergence via averaging

$$U_n \approx \delta_{U_n},\ \ \frac{1}{N}\sum_{n=1}^{N} U_n \to \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$$

$$\frac{1}{w_N}\sum_{n=1}^{N} w\left(\frac{n}{N}\right) U_n \to \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1,\ \ w_N \equiv \sum_{n=1}^{N} w\left(\frac{n}{N}\right)$$

(S)–convergence

(S)–convergent sequence

A sequence $\{\mathbf{U}_n\}_{n=1}^{\infty}$ is $\boxed{(S) - convergent}$ if for any $b \in C_c(R^D)$:

- **Correlation limit**

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m) \mathrm{d}y \text{ exists for any fixed } m$$

- **Correlation disintegration**

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{n,m=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m)\ \mathrm{d}y$$

$$= \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{N} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m)\ \mathrm{d}y \right)$$

Basic properties of (S)–convergence, I

Equivalence to convergence of ergodic (Cesàro means)

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \text{ (S)–convergent } \Leftrightarrow \frac{1}{N}\sum_{n=1}^{N} b(\mathbf{U}_n) \to \overline{b(\mathbf{U})} \boxed{\textit{strongly}} \text{ in } L^1(Q)$$

(S)– limit (parametrized measure)

$$\mathbf{U}_n \xrightarrow{(S)} \mathcal{V},\ \{\mathcal{V}_y\}_{y\in Q},\ \mathcal{V}_y \in \mathfrak{P}(R^D),\ \left\langle \mathcal{V}_y; b(\widetilde{U}) \right\rangle = \overline{b(\mathbf{U})}(y)$$

Relation to Young measure

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \text{ generates Young measure } \nu \Rightarrow\ \nu = \mathcal{V} \text{ a.a.}$$

Convergence in Wasserstein distance

$$\int_Q |\mathbf{U}_n|^p \ \mathrm{d}y \leq c \text{ uniformly for } n = 1, 2, \dots,\ p > 1$$

$$\mathbf{U}_n \xrightarrow{(S)} \mathcal{V}\ \Rightarrow\ \int_Q \left| d_{W_s}\left[\frac{1}{N}\sum_{n=1}^{N} \delta_{\mathbf{U}_n(y)}; \mathcal{V}_y \right] \right|^s \ \mathrm{d}y \to 0 \text{ as } N \to \infty,\ s < p$$

Basic properties of (S)–convergence, II

Statistically equivalent sequences

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \overset{(S)}{\approx} \{\mathbf{V}_n\}_{n=1}^{\infty},$$

$$\Leftrightarrow \text{ for any } \varepsilon > 0$$

$$\frac{\#\left\{ n \leq N \;\middle|\; \int_Q |\mathbf{U}_n - \mathbf{V}_n| \, \mathrm{d}y > \varepsilon \right\}}{N} \to 0 \text{ as } N \to \infty.$$

Robustness

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \overset{(S)}{\approx} \{\mathbf{V}_n\}_{n=1}^{\infty} \;\Rightarrow\; \mathbf{U}_n \overset{(S)}{\longrightarrow} \mathcal{V} \;\Leftrightarrow \mathbf{V}_n \overset{(S)}{\longrightarrow} \mathcal{V}$$

Corollary

$$\mathbf{U}_n \to \mathbf{U} \text{ in } L^1(Q) \;\Rightarrow\; \mathbf{U}_n \overset{(S)}{\longrightarrow} \delta_{\mathbf{U}(y)}$$

Basic properties of (S)–convergence III

Stationarity

$$\int_Q B(\mathbf{U}_{k_1}, \ldots, \mathbf{U}_{k_j})\mathrm{d}y = \int_Q B(\mathbf{U}_{k_1+n}, \ldots, \mathbf{U}_{k_j+n})\mathrm{d}y$$

Birkhoff–Khinchin Theorem

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \text{ stationary, } b \in \mathcal{B}(R^d) \text{ Borel measurable } \int_Q b(\mathbf{U}_0) < \infty$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{n=1}^{N} b(\mathbf{U}_n) \text{ converges for a.a. } y \in Q$$

$$\Rightarrow$$

$\mathbf{U}_n$ is (S)–convergent

Asymptotically stationary consistent approximation

Asymptotically stationary sequence

$\{\mathbf{U}_n\}_{n=1}^{\infty}$ is *asymptotically stationary* if for any $b \in C_c(R^D)$ there holds:

- **Correlation limit**

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\int_Q b(\mathbf{U}_n)b(\mathbf{U}_m)\;\mathrm{d}y \text{ exists}$$

  for any fixed m

- **Asymptotic correlation stationarity**

$$\left|\int_Q \Big[b(\mathbf{U}_{k_1})b(\mathbf{U}_{k_2}) - b(\mathbf{U}_{k_1+n})b(\mathbf{U}_{k_2+n})\Big]\mathrm{d}y\right| \le \omega(b,k)$$

  for any $1 \le k \le k_1 \le k_2$, and any $n \ge 0$

$$\omega(b,k) \to 0 \text{ as } k \to \infty$$

Sufficient conditions for (S)–convergence

Asymptotically stationary sequence

$\{\mathbf{U}_n\}_{n=1}^{\infty}$ *asymptotically stationary* $\Rightarrow$ $\{\mathbf{U}_n\}_{n=1}^{\infty}$ (S)–convergent

Subsequence principle [Balder]

$$\int_Q F(|\mathbf{U}_n|)\ \mathrm{d}y \leq 1 \text{ uniformly for } n \to \infty,$$

$$F : [0,\infty) \to [0,\infty) \text{ continuous}, \ \lim_{r\to\infty} F(r) = \infty$$

$$\Rightarrow$$

there is an (S)–convergent **subsequence** $\{\mathbf{U}_{n_k}\}_{k=1}^{\infty}$

Application to consistent approximation of the Euler system

(S)–convergent consistent approximation

$$\mathbf{U}_n = [\varrho_n, \mathbf{m}_n] \ \ Q = (0, T) \times \Omega$$

$$\mathbf{U}_n \xrightarrow{(S)} \mathcal{V}$$

DMV solution

$\mathcal{V}$ is a dissipative measure valued solutions of the Euler system

Convergence in Wasserstein distance

$$\int_0^T \int_\Omega \left| d_{W_s} \left[\frac{1}{N} \sum_{n=1}^N \delta_{\mathbf{U}_n(y)}; \mathcal{V}_y \right] \right|^s \ \mathrm{d}x \ \mathrm{d}t \to 0 \text{ as } N \to \infty$$

$$1 \leq s < \frac{2\gamma}{\gamma + 1}$$

Deterministic convergence

Strong solution

Euler system admits strong solution $\Rightarrow \mathcal{V}_{(t,x)} = \delta_{[\varrho,\mathbf{m}](t,x)}$

Regular limit

$$[\varrho = \langle \mathcal{V}; \tilde{\varrho} \rangle \,, \ \mathbf{m} = \langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle] \in C^1$$

$$\Rightarrow$$

$$[\varrho, \mathbf{m}] \text{ strong solution of Euler}, \ \mathcal{V}_{(t,x)} = \delta_{[\varrho,\mathbf{m}](t,x)}$$

Convergence to weak solution

$$[\varrho = \langle \mathcal{V}; \tilde{\varrho} \rangle \,, \ \mathbf{m} = \langle \mathcal{V}; \widetilde{\mathbf{m}} \rangle] \text{ weak solution to Euler system}$$

convergence is strong "near" the boundary $\partial\Omega$

$$\Rightarrow$$

$$\mathcal{V}_{(t,x)} = \delta_{[\varrho,\mathbf{m}](t,x)}$$

Improving convergence criteria

Statistical significant set

$$\mathcal{S} \subset N \times N$$

$$\frac{\#\left\{n, m \leq N \middle| \ (n,m) \in S\right\}}{N^2} \to 1 \text{ as } N \to \infty$$

Asymptotic stationarity

$$\|U_n\|_{L^\infty(Q)} \leq C$$

$$\text{for any } \varepsilon > 0 \text{ there is } k, L \text{ such that}$$

$$\left| \int_Q (U_{n+l} - U)(U_n - U) - (U_{k+l} - U)(U_k - U) \ \mathrm{d}y \right| < \varepsilon$$

$$\text{for all } n \geq k, \ \ l \geq L, \ \ (n+l, n) \in \mathcal{S}$$

$$\Rightarrow$$

$$\frac{1}{N} \sum_{n=1}^{N} U_n \to U \text{ in } L^1(Q) \text{ as } N \to \infty$$

General summation method

Weighted averages

$$w \in C[0,1],\ \ w \geq 0,\ \ w_N = \sum_{n=1}^{N} w\left(\frac{n}{N}\right)$$

Regular summation method

$$\{s_{n,N}\}_{n=1,N=1}^{\infty}$$

$$0 \leq s_{n,N} \leq \overline{s} \text{ for any } n, N$$

$$s_{n,N} = 0 \text{ whenever } n > N$$

$$\sum_{n=1}^{N} s_{n,N} = N \text{ for any } N = 1, 2, \ldots$$

$$\frac{1}{w_N} \sum_{n=1}^{N} w\left(\frac{n}{N}\right) \mathbf{U}_n,\ \text{ or }\ \frac{1}{N} \sum_{n=1}^{N} s_{n,N} \mathbf{U}_n$$

Summary

Computations of averages

- **Strong convergence** to the limit measure in terms of Wasserstein distance. Efficient computation and visualization of oscillatory solutions - Young measures
- Possibly slow convergence – remedy in more efficient summation method
- The result identical to the "standard" methods in the case of regular solutions of the limit problem

Computing defect numerically -EF, M.Lukáčová, B.She

Monge–Kantorowich (Wasserstein) distance

$$\left\| \mathrm{dist} \left(\frac{1}{N} \sum_{k=1}^{N} \nu_{t,x}^{n_k}; \nu_{t,x} \right) \right\|_{L^q(Q)} \to 0$$

for some $q > 1$

Convergence in the first variation

$$\frac{1}{N} \sum_{k=1}^{N} \left\langle \nu_{t,x}^{n_k}; \left| \widetilde{\mathbf{U}} - \frac{1}{N} \sum_{k=1}^{N} \mathbf{U}_n \right| \right\rangle \to \left\langle \nu_{t,x}; \left| \widetilde{\mathbf{U}} - \mathbf{U} \right| \right\rangle$$

in $L^1(Q)$

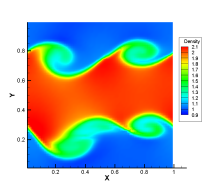
Mária Lukáčová (Mainz)

Bangwei She (CAS Praha)

Experiment I, density for Kelvin–Helmholtz problem (M. Lukáčová, Yue Wang)

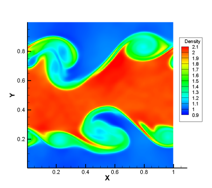
density ϱ

$n = 128,\ T = 2$



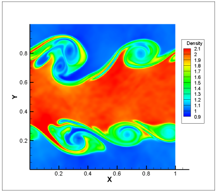
density ϱ

$n = 256,\ T = 2$



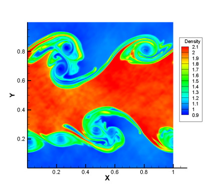
density ϱ

$n = 512,\ T = 2$



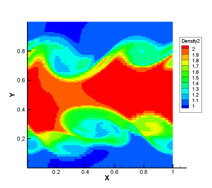
density ϱ

$n = 1024,\ T = 2$



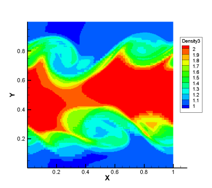
Cèsaro averages density ϱ

$n = 128,\ T = 2$



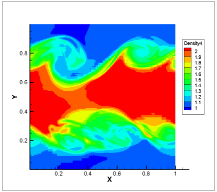
Cèsaro averages density ϱ

$n = 256,\ T = 2$



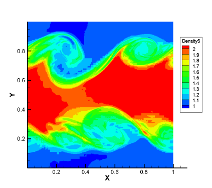
Cèsaro averages density ϱ

$n = 512,\ T = 2$



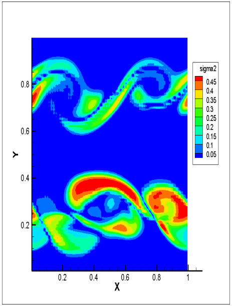
Cèsaro averages density ϱ

$n = 1024,\ T = 2$

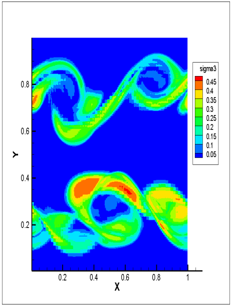


Experiment II, density variations for Kelvin–Helmholtz problem (M. Lukáčová, Yue Wang)

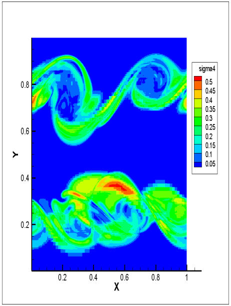
density variation
$n = 128, \ T = 2$



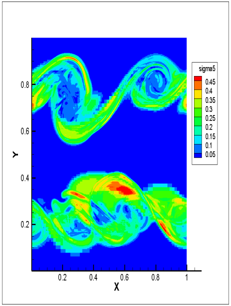
density variation
$n = 256, \ T = 2$



density variation
$n = 512, \ T = 2$



density variation
$n = 1024, \ T = 2$



Yue Wang, Mainz

Mária Lukáčová, Mainz

Lecture IV: Large time behavior, turbulence

Eduard Feireisl

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague
Technische Universität Berlin

TU Bielefeld, November 2020

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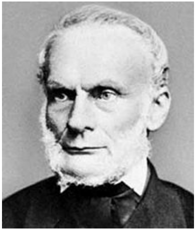
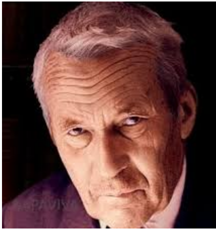
Motivation



Rudolf Clasius
1822–1888

Basic principles of thermodynamics of closed systems
The energy of the world is constant; its entropy tends to a maximum

Turbulence - ergodic hypothesis
Time averages along trajectories of the flow converge, for large enough times, to an ensemble average given by a certain probability measure



Andrey Nikolaevich Kolmogorov
1903–1987

Dynamical systems

Dynamical system

$$\mathbf{U}(t,\cdot) : [0,\infty) \times X \to X$$

- **Closed system:** $\mathbf{U}(t, X_0) \to \mathbf{U}_\infty$ equilibrium solution as $t \to \infty$
- **Open system:** $$\frac{1}{T}\int_0^T F(\mathbf{U}(t,X_0))\mathrm{d}t \to \int_X F(X)\ \mathrm{d}\mu,\ T \to \infty$$

μ a.s. in X_0

Principal mathematical problems:

- **Low regularity of global in time solutions**
  Global in time solutions necessary. For many problems in fluid dynamics – Navier–Stokes or Euler system – only weak solutions available
- **Lack of uniqueness**
  Solutions do not, or at least are not known to, depend uniquely on the initial data. Spaces of trajectories: Sell, Nečas, Temam and others
- **Propagation of oscillations**
  Realistic systems are partly hyperbolic: propagation of oscillations "from the past", singularities

Abstract setting

George Roger Sell 1937–2015

Space of entire trajectories

$$\mathcal{T} = C_{\rm loc}(R; X),\ \ t \in (-\infty, \infty)$$

ω–**limit set**

$$\omega[\mathbf{U}(\cdot, X_0)] \subset \mathcal{T}$$

$$\omega[\mathbf{U}(\cdot, X_0)] = \left\{ \mathbf{V} \in \mathcal{T} \ \middle| \ \mathbf{U}(\cdot + t_n, X_0) \to \mathbf{V} \text{ in } \mathcal{T} \text{ as } t_n \to \infty \right\}$$

Necessary ingredients

- **Dissipativity** – ultimate boundedness of trajectories
- **Compactness** – in appropriate spaces

Strong and weak ergodic hypothesis

Krylov – Bogolyubov construction

$$T \mapsto \frac{1}{T}\int_0^T \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t \text{ – a family of probability measures on } \mathcal{T}$$

$$\text{tightness in } \mathcal{T} \ \Rightarrow\ T_n \mapsto \frac{1}{T_n}\int_0^{T_n} \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

$$\textbf{Ergodic hypothesis} \ \Leftrightarrow\ \mu \text{ is unique} \ \Rightarrow\ T \mapsto \frac{1}{T}\int_0^T \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t \to \mu$$

unique $\approx$ unique on $\omega[\mathbf{U}(\cdot, X_0)]$

Weak ergodic hypothesis

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \delta_{\mathbf{U}(\cdot+t,X_0)}\mathrm{d}t = \mu \text{ exists in the narrow sense in } \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

Barotropic Navier–Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \mathbf{g}$$

Constitutive equations

- barotropic (isentropic) pressure–density EOS $p = p(\varrho)$ ($p = a\varrho^\gamma$)
- Newton's rheological law

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0$$

- Gravitational external force

$$\mathbf{g} = \nabla_x F, \ F = F(x)$$

Energy

$$E(\varrho, \mathbf{m}) \equiv \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - \varrho F, \ P'(\varrho)\varrho - P(\varrho) = p(\varrho), \ \mathbf{m} = \varrho \mathbf{u}$$

Energetically insulated system

Conservative boundary conditions

$\Omega \subset R^d$ bounded (sufficiently regular) domain

• **impermeability** $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$

• **no–slip** $[\mathbf{u}]_{\rm tan}|_{\partial\Omega} = 0$

Long–time behavior – Clausius scenario

- Total mass conserved

$$\int_\Omega \varrho(t, \cdot) \,{\rm d}x = M_0$$

- Total energy – Lyapunov function

$$\frac{\rm d}{{\rm d}t} \int_\Omega E(\varrho, \mathbf{m}) \,{\rm d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \,{\rm d}x = (\leq) 0, \ \int_\Omega E(\varrho, \mathbf{m}) \,{\rm d}x \searrow \mathcal{E}_\infty$$

- Stationary solution

$$\mathbf{m}_\infty = 0, \ \nabla_x p(\varrho_\infty) = \varrho_\infty \nabla_x F, \ \int_\Omega \varrho_\infty \,{\rm d}x = M_0, \int_\Omega E(\varrho_\infty, 0) \,{\rm d}x = \mathcal{E}_\infty$$

Energetically open system

In/out flow boundary conditions

$$\mathbf{u} = \mathbf{u}_b \text{ on } \partial\Omega$$

$$\Gamma_{\rm in} = \left\{ x \in \partial\Omega \ \Big| \ \mathbf{u}_b(x) \cdot \mathbf{n}(x) < 0 \right\}, \ \Gamma_{\rm out} = \left\{ x \in \partial\Omega \ \Big| \ \mathbf{u}_b(x) \cdot \mathbf{n}(x) \geq 0 \right\}$$

Density (pressure) on the inflow boundary

$$\varrho = \varrho_b \text{ on } \Gamma_{\rm in}$$

Energy balance

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_b|^2 + P(\varrho) \ \mathrm{d}x + \int_\Omega \mathbb{S} : \nabla_x \mathbf{u} \ \mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\Gamma_{\rm in}} P(\varrho_b) \mathbf{u}_b \cdot \mathbf{n} \ \mathrm{d}S_x + \int_{\Gamma_{\rm out}} P(\varrho) \mathbf{u}_b \cdot \mathbf{n} \ \mathrm{d}S_x \\
&\quad = (\leq) - \int_\Omega \left[\varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I} \right] : \nabla_x \mathbf{u}_b \ \mathrm{d}x + \frac{1}{2} \int_\Omega \varrho \mathbf{u} \cdot \nabla_x |\mathbf{u}_b|^2 \ \mathrm{d}x\mathrm{d}t \\
&\quad + \int_\Omega \mathbb{S} : \nabla_x \mathbf{u}_b \ \mathrm{d}x\mathrm{d}t + \int_\Omega \varrho \nabla_x F \cdot (\mathbf{u} - \mathbf{u}_b) \ \mathrm{d}x
\end{aligned}$$

Global bounded trajectories

Global in time weak solutions
$\mathbf{U} = [\varrho, \mathbf{m} = \varrho \mathbf{u}]$ – weak solution of the Navier–Stokes system satisfying energy inequality and defined for $t > T_0$

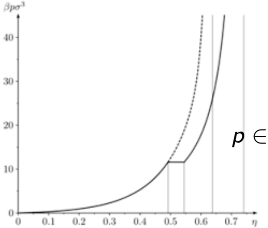
Bounded energy

$$\limsup_{t \to \infty} \int_\Omega E(\varrho, \mathbf{m}) \, \mathrm{d}x \leq \mathcal{E}_\infty$$

Available results

- **Existence:** T. Chang, B. J. Jin, and A. Novotný, *SIAM J. Math. Anal.*, **51**(2):1238–1278, 2019
  H. J. Choe, A. Novotný, and M. Yang *J. Differential Equations*, **266**(6):3066–3099, 2019
- **Globally bounded solutions:** F. Fanelli, E. F., and M. Hofmanová **arxiv preprint No. 2006.02278**, 2020
  J. Březina, E. F., and A. Novotný, Communications in PDE's 2020

Hard sphere pressure EOS



$$p \in C[0, \overline{\varrho}) \cap C^1(0, \overline{\varrho}), \ p'(\varrho) > 0 \text{ for } \varrho > 0, \ \lim_{\varrho \to \overline{\varrho}-} p(\varrho) = \infty$$

Ultimate boundedness of trajectories – bounded absorbing set

$$\limsup_{t \to \infty} \int_{\Omega} E(\varrho, \mathbf{m}) \ \mathrm{d}x \leq \mathcal{E}_\infty$$

$\mathcal{E}_\infty$ – universal constant

ω – limit sets

$$p \approx a\varrho^\gamma,\ \gamma > \frac{d}{2} \text{ or hard sphere EOS}$$

Trajectory space

$$X = \left\{ \varrho, \mathbf{m} \ \middle|\ \varrho(t,\cdot) \in L^\gamma(\Omega),\ \mathbf{m}(t,\cdot) \in L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d) \hookrightarrow W^{-k,2} \right\}$$

$$\mathcal{T} = C_{\rm loc}(R; L^1 \times W^{-k,2})$$

Fundamental result on compactness [Fanelli, EF, Hofmanová, 2020]

The ω–limit set $\omega[\varrho, \mathbf{m}]$ of each global in time trajectory with globally bounded energy is:

- *non − empty*
- *compact* in $\mathcal{T}$
- time shift invariant
- consists of entire (defined for all $t \in R$) weak solutions of the Navier–Stokes system

Propagation of oscillations

Equation of continuity

$$\partial_t \varrho + \mathbf{u} \cdot \nabla_x \varrho = -\varrho \mathrm{div}_x \mathbf{u}$$

Renormalized equation of continuity

$$\partial_t b(\varrho) + \mathrm{div}_x (b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

Weak convergence

$$b(\varrho_n) \to \overline{b(\varrho)} \text{ weakly in } L^1$$

$$\partial_t \Big[\overline{b(\varrho)} - b(\varrho) \Big] + \mathrm{div}_x \Big(\overline{b(\varrho)\mathbf{u}} - b(\varrho)\mathbf{u} \Big)$$

$$= \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} - \overline{\Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u}}$$

$$\Big[\overline{b(\varrho)} - b(\varrho) \Big] (0, \cdot) = 0 \text{ is needed!}$$

Vanishing oscillation defect, I

Compactness of densities:

$$\varrho_n \equiv \varrho(\cdot + T_n) \to \varrho \text{ in } C_{\rm weak,loc}(R; L^\gamma(\Omega))$$

$$\varrho_n \log(\varrho_n) \to \overline{\varrho \log(\varrho)} \geq \varrho \log(\varrho)$$

$$\textbf{oscillation defect: } D(t) \equiv \int_\Omega \overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \ {\rm d}x \geq 0$$

Renormalized equation:

$$\frac{\rm d}{{\rm d}t} D + \int_\Omega \left[\overline{\varrho {\rm div}_x \mathbf{u}} - \varrho {\rm div}_x \mathbf{u} \right] \ {\rm d}x = 0, \ 0 \leq D \leq \overline{D}, \ t \in R$$

Lions' identity

$$\overline{\varrho {\rm div}_x \mathbf{u}} - \varrho {\rm div}_x \mathbf{u} = \overline{p(\varrho)\varrho} - \overline{p(\varrho)} \ \varrho \geq 0$$

Vanishing oscillation defect, II

Crucial differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}D + \Psi(D) \leq 0,\ 0 \leq D \leq \overline{D},\ t \in R$$

$$\Psi \in C(R),\ \Psi(0) = 0,\ \Psi(Z)Z > 0 \text{ for } Z \neq 0$$

$$\Rightarrow$$

$$D \equiv 0$$

Statistical stationary solutions

Application of Krylov – Bogolyubov method

$$\frac{1}{T_n}\int_0^{T_n} \delta_{\varrho(\cdot+t,\cdot),\mathbf{m}(\cdot+t,\cdot)}\ \mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}] \text{ narrowly}$$

$[\mathcal{T},\mu]$ (canonical representation) – statististical stationary solution

$\mu(t)|_X$ (marginal) independent of $t \in R$

Application of Birkhoff – Khinchin ergodic theorem

$$\frac{1}{T}\int_0^{T} F(\varrho(t,\cdot),\mathbf{m}(t,\cdot))\mathrm{d}t \to \overline{F} \text{ as } T \to \infty$$

F bounded Borel measurable on X for μ – a.a. $(\varrho,\mathbf{m}) \in \omega$

Related results for incompressible Navier–Stokes system with conservative boundary conditions

F.Flandoli and D. Gatarek, F.Flandoli and M.Romito (stochastic forcing),
P. Constantin and I. Procaccia, C. Foiaș, O. Manley, R. Rosa, and R. Temam,
M. Vishik and A. Fursikov etc (deterministic forcing)

Back to ergodic hypothesis – conclusion

Ergodicity

$$\mu \text{ ergodic } \Leftrightarrow \ \mathcal{B} \subset \omega[\varrho, \mathbf{m}] \text{ shift invariant } \Rightarrow \ \mu[\mathcal{B}] = 1 \text{ or } \mu[\mathcal{B}] = 0$$

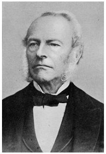
$$\mu \in \text{conv}\Big\{ \text{ergodic measures on } \omega[\varrho, \mathbf{m}] \Big\}$$

State of the art for compressible Navier–Stokes system

- Each bounded energy global trajectory generates a stationary statistical solution – a shift invariant measure μ – sitting on its $\omega-$limit set $\omega[\varrho, \mathbf{m}]$
- The weak ergodic hypothesis (the existence of limits of ergodic averages for any Borel measurable F) holds on $\omega[\varrho, \mathbf{m}]$ $\mu-$a.s.
- The (strong) ergodic hypothesis definitely holds for enegetically isolated systems and a class of potential forces F, where all solutions tend to equilibrium

Complete Navier–Stokes–Fourier system

Claude Louis Marie Henri Navier [1785-1836]



George Gabriel Stokes [1819-1903]

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f}$$

Entropy production

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

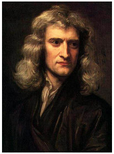
$$\sigma = (\geq) \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Constitutive relations

Joseph Fourier [1768-1830]

Fourier's law

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x \vartheta$$



Isaac Newton [1643-1727]

Newton's rheological law

$$\mathbb{S} = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}$$

Boundary conditions

Impermeability

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

No-slip

$$\mathbf{u}_{\tan}|_{\partial\Omega} = 0$$

No-stick

$$[\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Thermal insulation

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Long–time behavior

Dichotomy for the closed/open system

$$\mathbf{f} = \mathbf{f}(x)$$

Either

$$\mathbf{f} = \nabla_x F \;\Rightarrow\; \text{all solutions tend to a single equilibrium}$$

or

$$\mathbf{f} \neq \nabla_x F \;\Rightarrow\; \int_\Omega E(t,\cdot)\,\mathrm{d}x \to \infty \text{ as } t \to \infty$$

Lecture V: General systems of fluid mechanics

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TU Bielefeld, November 2020

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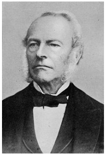
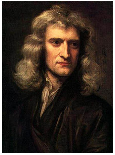
Navier–Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u}), \ \mathbf{u}|_{\partial\Omega} = 0$$





Newton's law

$$\mathbb{S}(\mathbb{D}_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega E \, \mathrm{d}x + \int_\Omega \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} \, \mathrm{d}x \leq 0, \ E(\varrho, \mathbf{u}) = \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho), \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

"Implicit" rheological law

Fenchel–Young inequality

$$\boxed{\mathbb{S} : \mathbb{D}} \leq F(\mathbb{D}) + F^*(\mathbb{S})$$

$$\mathbb{S} : \mathbb{D} = F(\mathbb{D}) + F^*(\mathbb{S}) \ \Leftrightarrow \ \mathbb{S} \in \partial F(\mathbb{D}) \ \Leftrightarrow \ \mathbb{D} \in \partial F^*(\mathbb{S})$$

Reformulation of the Navier–Stokes system

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \ \mathrm{d}x + \int_\Omega \Big(F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}) \Big) \ \mathrm{d}x \leq 0$$

Dissipation potential

$$F : R^{d \times d}_{\mathrm{sym}} \to [0, \infty] \text{ convex l.s.c.}$$

Example - isothermal case

Constitutive relations

$$\boxed{\text{linear pressure}}\ p(\varrho) = a\varrho,\ \ P(\varrho) = a\varrho\log(\varrho)$$

Dissipation potential

$$F(|\mathbb{D}|) \approx |\mathbb{D}|^q,\ \ q > d$$

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + a\nabla_x \varrho = \mathrm{div}_x \mathbb{S}$$

$$\left[\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + a\varrho\log(\varrho) \right)\ \mathrm{d}x \right]_{t=0}^{\tau} + \int_0^\tau \int_\Omega \Big(F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}) \Big)\ \mathrm{d}x\ \mathrm{d}t \leq 0$$

Dissipative solutions

Basic properties of generalized solutions

- **Existence.** Generalized solutions exist and represent limits of *consistent* approximations
- **Compatibility.** Smooth generalized solutions are classical solutions
- **Weak–strong uniqueness.** Generalized solution coincides with a smooth solution emanating from the same initial data as long as the latter solution exists
- **Semigroup selection.** The class of generalized solution admits a (Borel measurable) semigroup selection
- **Statistical solution.** Semigroup selection $\Rightarrow$ existence of statistical solutions. Markovian a.a. semigroup:

$$M_t : \mathfrak{P}[\mathrm{data}] \to \mathfrak{P}[\mathrm{data}],\ M_{t+s} = M_t \circ M_s \text{ for a.a. } s \geq 0$$

$\mathfrak{P}[\mathrm{data}]$ - the set of Borel probability measures on the set of initial/boundary data

Problem formulation

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}$$

$$\mathbb{S} \in \partial F(\mathbb{D}_x \mathbf{u}),\ F : R^{d \times d}_{\mathrm{sym}} \to [0, \infty] \text{ convex l.s.c.}$$

Boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B,\ \Gamma_{\mathrm{in}} = \left\{ x \in \partial\Omega \ \middle|\ \mathbf{u}_B(x) \cdot \mathbf{n} < 0 \right\}$$

$$\varrho|_{\Gamma_{\mathrm{in}}} = \varrho_B$$

Initial conditions

$$\varrho(0, \cdot) = \varrho_0,\ (\varrho \mathbf{u})(0, \cdot) = \mathbf{m}_0$$

Dissipative solutions

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S} - \boxed{\mathrm{div}_x \mathfrak{R}} \equiv \mathrm{div}_x \mathbb{S}_{\mathrm{eff}}$$

Velocity boundary condition and Reynolds stress

$$(\mathbf{u} - \mathbf{u}_B) \in W_0^{1,q}(\Omega; R^d)$$

$$\boxed{\mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\mathrm{sym}}))}$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_B|^2 + P(\varrho) + d \, \boxed{\mathrm{tr}[\mathfrak{R}]} \right] \ \mathrm{d}x + \int_\Omega \Big[F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}) \Big] \ \mathrm{d}x$$

$$+ \int_{\partial\Omega} P(\varrho) \mathbf{u}_B \cdot \mathbf{n} \ \mathrm{d}S_x \leq - \int_\Omega \Big[p(\varrho) \mathbb{I} + \varrho \mathbf{u} \otimes \mathbf{u} \Big] : \mathbb{D}_x \mathbf{u}_B \ \mathrm{d}x$$

$$- \int_\Omega \varrho \mathbf{u} \cdot \Big(\mathbf{u}_B \cdot \nabla_x \mathbf{u}_B \Big) \ \mathrm{d}x + \int_\Omega \mathbb{S} : \mathbb{D}_x \mathbf{u}_B \ \mathrm{d}x - \int_{\overline{\Omega}} \boxed{\mathfrak{R}} : \mathbb{D}_x \mathbf{u}_B \ \mathrm{d}x$$

Relative energy

Relative energy

$$E\left(\varrho, \mathbf{u} \Big| \tilde{\varrho}, \tilde{\mathbf{u}}\right) = \left[\frac{1}{2}\varrho|\mathbf{u} - \tilde{\mathbf{u}}|^2 + P(\varrho) - P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - P(\tilde{\varrho})\right]$$

Integrated relative energy

$$\mathcal{E}\left(\varrho, \mathbf{u} \Big| \tilde{\varrho}, \tilde{\mathbf{u}}\right) = \int_{\Omega} \left[\frac{1}{2}\varrho|\mathbf{u} - \tilde{\mathbf{u}}|^2 + P(\varrho) - P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - P(\tilde{\varrho})\right] \mathrm{d}x$$

Augmented relative energy

$$\mathcal{E}\left(\varrho, \mathbf{u} \Big| \tilde{\varrho}, \tilde{\mathbf{u}}\right) + d \int_{\overline{\Omega}} \mathrm{d}\ \mathrm{tr}[\mathfrak{R}]$$

Basic tool

Relative energy inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega E\left(\varrho,\mathbf{u}\;\middle|\;\tilde\varrho,\tilde{\mathbf{u}}\right)\,\mathrm{d}x + d\int_{\overline\Omega}\mathrm{tr}[\mathfrak{R}]\right) + \int_\Omega\Big[F(\mathbb{D}_x\mathbf{u}) + F^*(\mathbb{S})\Big]\,\mathrm{d}x\\
&-\int_\Omega \mathbb{S}:\nabla_x\tilde{\mathbf{u}}\,\mathrm{d}x + \int_{\Gamma_{\rm out}}\big[P(\varrho) - P'(\tilde\varrho)(\varrho-\tilde\varrho) - P(\tilde\varrho)\big]\,\mathbf{u}_B\cdot\mathbf{n}\,\mathrm{d}S_x\\
&+\int_{\Gamma_{\rm in}}\big[P(\varrho_B) - P'(\tilde\varrho)(\varrho_B-\tilde\varrho) - P(\tilde\varrho)\big]\,\mathbf{u}_B\cdot\mathbf{n}\,\mathrm{d}S_x\\
\le&-\int_\Omega \varrho(\tilde{\mathbf{u}}-\mathbf{u})\cdot(\tilde{\mathbf{u}}-\mathbf{u})\cdot\nabla_x\tilde{\mathbf{u}}\,\mathrm{d}x\\
&-\int_\Omega\Big[p(\varrho) - p'(\tilde\varrho)(\varrho-\tilde\varrho) - p(\tilde\varrho)\Big]\mathrm{div}_x\tilde{\mathbf{u}}\,\mathrm{d}x\\
&+\int_\Omega\frac{\varrho}{\tilde\varrho}(\tilde{\mathbf{u}}-\mathbf{u})\cdot\Big[\partial_t(\tilde\varrho\tilde{\mathbf{u}}) + \mathrm{div}_x(\tilde\varrho\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}}) + \nabla_x p(\tilde\varrho)\Big]\,\mathrm{d}x\\
&+\int_\Omega\left(\frac{\varrho}{\tilde\varrho}(\mathbf{u}-\tilde{\mathbf{u}})\cdot\tilde{\mathbf{u}} + p'(\tilde\varrho)\left(1-\frac{\varrho}{\tilde\varrho}\right)\right)\Big[\partial_t\tilde\varrho + \mathrm{div}_x(\tilde\varrho\tilde{\mathbf{u}})\Big]\,\mathrm{d}x\\
&-\int_{\overline\Omega}\nabla_x\tilde{\mathbf{u}}:\mathrm{d}\,\mathfrak{R}
\end{aligned}
$$

Maximal solutions

Comparison relation

$$\mathcal{E} \equiv \int_{\Omega} \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_B|^2 + P(\varrho) + d \ \mathrm{tr}[\mathfrak{R}] \right] \ \mathrm{d}x$$

$$[\varrho_1, \mathbf{u}_1, \mathcal{E}_1] \prec [\varrho_2, \mathbf{u}_2, \mathcal{E}_2] \ \Leftrightarrow \ \mathcal{E}_1 \leq \mathcal{E}_2 \text{ for all } t > 0$$

Maximal solutions

A solution $[\varrho, \mathbf{u}, \mathcal{E}]$ is maximal if it is minimal with respect to $\prec$

Asymptotic behavior of maximal solutions

Suppose $\mathbf{u}_B = 0$ - no-slip boundary conditions.
If $[\varrho, \mathbf{u}]$ is maximal, then

$$\|\mathfrak{R}(t)\|_{\mathcal{M}^+} \to 0 \text{ as } t \to \infty$$

Semigroup selection

Set of data

$$\mathcal{D} = \left\{ \varrho, \mathbf{m}, E \ \Big| \ \int_\Omega \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \ \mathrm{d}x \leq \mathcal{E} \right\}, \ \mathcal{E} \text{ càglàd}$$

Set of trajectories

$$\mathcal{T} = \left\{ \varrho(t, \cdot), \mathbf{m}(t, \cdot), \mathcal{E}(t, \cdot) \Big| t \in (0, \infty) \right\}$$

Solution set

$$\mathcal{U}[\varrho_0, \mathbf{m}_0, \mathcal{E}_0] = \Big\{ [\varrho, \mathbf{m}, \mathcal{E}] \ \Big| [\varrho, \mathbf{m}, \mathcal{E}] \text{ dissipative solution}$$

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0, \ \mathcal{E}(0+) \leq \mathcal{E}_0 \equiv \mathcal{E}(0-) \Big\}$$

Semiflow selection – semigroup

$$U[\varrho_0, \mathbf{m}_0, \mathcal{E}_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, \mathcal{E}_0], \ [\varrho_0, \mathbf{m}_0, \mathcal{E}_0] \in \mathcal{D}$$

$$U(t_1 + t_2)[\varrho_0, \mathbf{m}_0, \mathcal{E}_0] = U(t_1) \circ \Big[U(t_2)[\varrho_0, \mathbf{m}_0, \mathcal{E}_0] \Big], \ t_1, t_2 > 0$$

Andrej Markov (1856–1933)

N. V. Krylov

Convergence to equilibria

Hypotheses

$$p(\varrho) \approx \varrho^{\gamma},\ F(\mathbb{D}) \approx |\mathbb{D}|^{q}, \mathbf{u}_B = 0$$

$$\frac{1}{\gamma} + \frac{1}{q} \leq 1 \text{ if } q > \frac{d}{2},\ \frac{\gamma+1}{2\gamma} + \frac{d-q}{dq} < 1 \text{ if } q \leq \frac{d}{2}.$$

Long time behavior

$$[\varrho, \mathbf{m}] \text{ maximal}$$

$$\Rightarrow$$

$$\varrho(t, \cdot) \to \varrho_S \text{ in } L^{\gamma}(\Omega),\ \mathbf{m} \to 0\ L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d) \text{ as } t \to \infty$$

Stability of rarefaction waves, I

1-D Euler system

$$\partial_t \tilde{\varrho} + \partial_{x_1}(\tilde{\varrho}\widetilde{u}) = 0$$
$$\partial_t(\tilde{\varrho}\widetilde{u}) + \partial_{x_1}(\tilde{\varrho}\widetilde{u}^2) + \partial_{x_1} p(\tilde{\varrho}) = 0$$

Riemann data

$$[\tilde{\varrho}(0,x_1), \widetilde{u}(0,x_1)] = [\varrho_0, \mathbf{u}_0] := \left\{ \begin{array}{l} [\tilde{\varrho}_L, \widetilde{u}_L] \text{ if } x_1 < 0, \\ [\tilde{\varrho}_R, \widetilde{u}_R] \text{ if } x_1 \geq 0, \end{array} \right.$$

Rarefaction waves

$$\left| \int_{\tilde{\varrho}_L}^{\tilde{\varrho}_R} \frac{p'(z)}{z} \ \mathrm{d}z \right| \leq \widetilde{u}_R - \widetilde{u}_L \leq \int_0^{\tilde{\varrho}_L} \frac{p'(z)}{z} \ \mathrm{d}z + \int_0^{\tilde{\varrho}_R} \frac{p'(z)}{z} \ \mathrm{d}z,$$

$$\tilde{\varrho} = \tilde{\varrho}\left(\frac{x_1}{t}\right), \ \widetilde{u} = \widetilde{u}\left(\frac{x_1}{t}\right), \ \text{Lipschitz for } \boxed{t > 0}$$

Stability of rarefaction waves, II

Spatial domain, relative energy

$$\Omega := \left\{ [x_1, \dots, x_d] \ \middle| \ x_1 \in (-L, L), \ [x_2, \dots, x_d] \in \mathcal{T}^{d-1} \right\}, \ d = 2, 3,$$

$$E\left(\varrho, \mathbf{m} \ \middle| \tilde{\varrho}, \tilde{\mathbf{u}} \right) = \frac{1}{2} \varrho \left| \frac{\mathbf{m}}{\varrho} - \tilde{\mathbf{u}} \right|^2 + P(\varrho) - P'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - P(\tilde{\varrho}).$$

Vanishing viscosity limit

$$F_\varepsilon(\mathbb{D}) \approx \varepsilon F(\mathbb{D})$$

Stability of planar rarefaction wave

$$\tilde{\varrho}, \ \tilde{\mathbf{u}} \text{ planar rarefaction wave profile}$$

$$\int_\Omega E\left(\varrho_{0,\varepsilon}, \mathbf{m}_{0,\varepsilon} \ \middle| \tilde{\varrho}_0, \tilde{\mathbf{u}}_0 \right) \ \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0$$

$$\Rightarrow$$

$$\int_\Omega E\left(\varrho_\varepsilon(\tau, \cdot), \mathbf{m}_\varepsilon(\tau, \cdot) \ \middle| \tilde{\varrho}(\tau, \cdot), \tilde{\mathbf{u}}(\tau, \cdot) \right) \ \mathrm{d}x \to 0 \text{ as } \varepsilon \to 0, \ \tau > 0$$