Ergodic theory for energetically open compressible fluid flows

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Motto

Energetically closed systems

Clausius:

The energy of the world is constant; its entropy tends to a maximum

Energetically open systems – ergodic hypothesis

Time averages along trajectories of the flow converge, for large enough times, to an ensemble average given by a certain probability measure.

Dynamical system

$$\mathbf{U}(t,\cdot) : [0,\infty) \times X \to X$$

- **Closed system:** $\mathbf{U}(t, X_0) \to \mathbf{U}_\infty$ equilibrium solution as $t \to \infty$
- **Open system:** $\frac{1}{T}\int_0^T F(\mathbf{U}(t, X_0))\mathrm{d}t \to \int_X F(X)\ \mathrm{d}\mu,\ \ T \to \infty$

μ a.s. in X_0

Abstract setting

Space of entire trajectories

$$\mathcal{T} = C_{\rm loc}(R; X), \ t \in (-\infty, \infty)$$

ω–limit set

$$\omega[\mathbf{U}(\cdot, X_0)] \subset \mathcal{T}$$

$$\omega[\mathbf{U}(\cdot, X_0)] = \left\{ \mathbf{V} \in \mathcal{T} \ \middle| \ \mathbf{U}(\cdot + t_n, X_0) \to \mathbf{V} \text{ in } \mathcal{T} \right\}$$

Stationary (statistical) solution

$\mathbf{V} : R \to X$ stationary process (law is time shift invariant)

$\mathbf{V} \in \omega[\mathbf{U}(\cdot, X_0)]$ a.s.

$\mathbf{V}$ solves the associated evolutionary equation a.s.

Strong and weak ergodic hypothesis

Krylov – Bogolyubov construction

$$T \mapsto \frac{1}{T}\int_0^T \delta_{\mathbf{U}(t,X_0)} \mathrm{d}t \text{ – a family of probability measures on } \mathcal{T}$$

$$\text{tightness in } \mathcal{T} \ \Rightarrow \ T_n \mapsto \frac{1}{T_n}\int_0^{T_n} \delta_{\mathbf{U}(t,X_0)} \mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

$$\textbf{Ergodic hypothesis} \ \Leftrightarrow \ \mu \text{ is unique} \ \Rightarrow \ T \mapsto \frac{1}{T}\int_0^T \delta_{\mathbf{U}(t,X_0)} \mathrm{d}t \to \mu$$

unique $\approx$ unique on $\omega[\mathbf{U}(\cdot, X_0)]$

Weak ergodic hypothesis

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \delta_{\mathbf{U}(t,X_0)} \mathrm{d}t = \mu \text{ exists in the narrow sense in } \mathcal{P}[\mathcal{T}]$$

$[\mathcal{T}, \mu]$ stationary statistical solution

Barotropic Navier–Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{g}$$

Constitutive equations

- barotropic (isentropic) pressure–density EOS $p = p(\varrho)$ ($p = a\varrho^\gamma$)
- Newton's rheological law

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0$$

- Gravitational external force

$$\mathbf{g} = \nabla_x F, \ F = f(x)$$

Energy

$$e(\varrho, \mathbf{m}) \equiv \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - \varrho F, \ P'(\varrho)\varrho - P(\varrho) = p(\varrho), \ \mathbf{m} = \varrho \mathbf{u}$$

Energetically insulated system

Conservative boundary conditions

$$\Omega \subset R^d \text{ bounded (sufficiently regular) domain}$$

- **impermeability** $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$
- **no–slip** $[\mathbf{u}]_{\tan}|_{\partial\Omega} = 0$

Long–time behavior – Clausius scenario

- Total mass conserved

$$\int_\Omega \varrho(t, \cdot) \, \mathrm{d}x = M_0$$

- Total energy – Lyapunov function

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega e(\varrho, \mathbf{m}) \, \mathrm{d}x + \int_\Omega \mathbb{S} : \nabla_x \mathbf{u} \, \mathrm{d}x = (\leq) 0, \ \int_\Omega e(\varrho, \mathbf{m}) \, \mathrm{d}x \searrow E_\infty$$

- Stationary solution

$$\mathbf{m}_\infty = 0, \ \nabla_x p(\varrho_\infty) = \varrho_\infty \nabla_x F, \ \int_\Omega \varrho_\infty \, \mathrm{d}x = M_0, \int_\Omega e(\varrho_\infty, 0) \, \mathrm{d}x = E_\infty$$

Navier–Stokes–Fourier system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$
$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{g}$$
$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u},$$

Conservative boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \ \ \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Dichotomy (EF, H. Petzeltová 2002)

Either

$$\mathbf{g} = \nabla_x F \ \Rightarrow \ \text{any solution converges to static equilibrium as } t \to \infty$$

or

$$\mathbf{g} \neq \nabla_x F \ \Rightarrow \ \text{the energy of the system becomes infinite as } t \to \infty$$

Stochastically driven Navier–Stokes–Fourier system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})\mathrm{d}t + \nabla_x p \mathrm{d}t = \mathrm{div}_x \mathbb{S} \mathrm{d}t + \varrho \mathbf{g} \mathrm{d}W$$

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u},$$

Conservative boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \ \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Non–stationarity (D. Breit, EF 2020)

There exists a (statistically) stationary solution

$$\Leftrightarrow$$

$$\mathbf{g} \equiv 0$$

Barotropic Navier–Stokes system revisited

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{g}$$

Constitutive equations

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- Gravitational external force

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Energy

$$e(\varrho, \mathbf{m}) \equiv \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - \varrho F, \ P'(\varrho)\varrho - P(\varrho) = p(\varrho), \ \mathbf{m} = \varrho \mathbf{u}$$

Energetically open system

In/out flow boundary conditions

$$\mathbf{u} = \mathbf{u}_b \text{ on } \partial\Omega$$

$$\Gamma_{\rm in} = \left\{ x \in \partial\Omega \ \Big| \ \mathbf{u}_b(x)\cdot\mathbf{n}(x) < 0 \right\},\ \Gamma_{\rm out} = \left\{ x \in \partial\Omega \ \Big| \ \mathbf{u}_b(x)\cdot\mathbf{n}(x) \geq 0 \right\}$$

Density (pressure) on the inflow boundary

$$\varrho = \varrho_b \text{ on } \Gamma_{\rm in}$$

Energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho|\mathbf{u}-\mathbf{u}_b|^2 + P(\varrho)\ \mathrm{d}x + \int_\Omega \mathbb{S} : \nabla_x \mathbf{u}\ \mathrm{d}x\mathrm{d}t$$

$$+ \int_{\Gamma_{\rm in}} P(\varrho_b)\mathbf{u}_b\cdot\mathbf{n}\ \mathrm{d}S_x + \int_{\Gamma_{\rm out}} P(\varrho)\mathbf{u}_b\cdot\mathbf{n}\ \mathrm{d}S_x$$

$$= (\leq) - \int_\Omega \left[\varrho\mathbf{u}\otimes\mathbf{u} + p(\varrho)\mathbb{I}\right] : \nabla_x\mathbf{u}_b\ \mathrm{d}x + \frac{1}{2}\int_\Omega \varrho\mathbf{u}\cdot\nabla_x|\mathbf{u}_b|^2\ \mathrm{d}x\mathrm{d}t$$

$$+ \int_\Omega \mathbb{S} : \nabla_x\mathbf{u}_b\ \mathrm{d}x\mathrm{d}t + \int_\Omega \varrho\nabla_x F\cdot(\mathbf{u}-\mathbf{u}_b)\ \mathrm{d}x$$

Global bounded trajectories

Global in time weak solutions

$\mathbf{U} = [\varrho, \mathbf{m} = \varrho \mathbf{u}]$ – weak solution of the Navier–Stokes system satisfying energy inequality and defined for $t > T_0$

Bounded energy

$$\limsup_{t \to \infty} \int_{\Omega} e(\varrho, \mathbf{m}) \, \mathrm{d}x \leq E_{\infty}$$

Available results

- **Existence:** T. Chang, B. J. Jin, and A. Novotný, *SIAM J. Math. Anal.*, **51**(2):1238–1278, 2019
  H. J. Choe, A. Novotný, and M. Yang *J. Differential Equations*, **266**(6):3066–3099, 2019
- **Globally bounded solutions:** F. Fanelli, E. F., and M. Hofmanová **arxiv preprint No. 2006.02278**, 2020
  J. Březina, E. F., and A. Novotný, preprint 2020

ω – limit sets

$$p \approx a\varrho^\gamma,\ \gamma > \frac{d}{2}$$

Trajectory space

$$X = \left\{ \varrho, \mathbf{m} \ \middle|\ \varrho(t,\cdot) \in L^\gamma(\Omega),\ \mathbf{m}(t,\cdot) \in L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d) \hookrightarrow W^{-k,2} \right\}$$

$$\mathcal{T} = C_{\rm loc}(R; L^\gamma \times W^{-k,2})$$

Fundamental result on compactness [Fanelli, EF, Hofmanová, 2020]

The ω–limit set $\omega[\varrho, \mathbf{m}]$ of each global in time trajectory with globally bounded energy is:

- non–empty
- compact in $\mathcal{T}$
- time shift invariant
- consists of entire (defined for all $t \in R$) weak solutions of the Navier–Stokes system

Vanishing oscillation defect

Compactness of densities

$$\varrho_n \equiv \varrho(\cdot + T_n) \to \varrho \text{ in } C_{\rm weak,loc}(R; L^\gamma(\Omega))$$

$$\varrho_n \log(\varrho_n) \to \overline{\varrho \log(\varrho)} \geq \varrho \log(\varrho)$$

$$D(t) \equiv \int_\Omega \overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \ {\rm d}x \geq 0$$

Problem: Unlike in the *existence* proof, there is no information on oscillations of "initial data"!

Crucial differential inequality

$$\frac{\rm d}{{\rm d}t} D + \Psi(D) \leq 0, \ 0 \leq D \leq \overline{D}, \ t \in R$$

$$\Psi \in C(R), \ \Psi(0) = 0, \ \Psi(Z)Z > 0 \text{ for } Z \neq 0$$

$$\Rightarrow$$

$$D \equiv 0$$

Statistical stationary solutions

Application of Krylov – Bogolyubov method

$$\frac{1}{T_n}\int_0^{T_n} \delta_{\varrho(t,\cdot),\mathbf{m}(t,\cdot)}\, \mathrm{d}t \to \mu \in \mathcal{P}[\mathcal{T}] \text{ narrowly}$$

$[\mathcal{T}, \mu]$ (canonical representation) – statististical stationary solution

$\mu(t)|_X$ (marginal) independent of $t \in R$

Application of Birkhoff – Khinchin ergodic theorem

$$\frac{1}{T}\int_0^{T} F(\varrho(t,\cdot),\mathbf{m}(t,\cdot))\mathrm{d}t \to \overline{F} \text{ as } T \to \infty$$

F bounded Borel measurable on X

for $\mu-$a.a. $(\varrho, \mathbf{m}) \in \omega$

Related results for incompressible Navier–Stokes system with conservative boundary conditions

F.Flandoli and D. Gatarek, F.Flandoli and M.Romito (stochastic forcing), P. Constantin and I. Procaccia, C. Foiaș, O. Manley, R. Rosa, and R. Temam, M. Vishik and A. Fursikov etc (deterministic forcing)

Back to ergodic hypothesis – conclusion

Ergodicity

$$\mu \text{ ergodic } \Leftrightarrow \mathcal{B} \subset \omega[\varrho, \mathbf{m}] \text{ shift invariant } \Rightarrow \mu[\mathcal{B}] = 1 \text{ or } \mu[\mathcal{B}] = 0$$

$$\mu \in \text{conv}\Big\{ \text{ergodic measures on } \omega[\varrho, \mathbf{m}] \Big\}$$

State of the art for compressible Navier–Stokes system

- Each bounded energy global trajectory generates a stationary statistical solution – a shift invariant measure μ – sitting on its $\omega-$limit set $\omega[\varrho, \mathbf{m}]$
- The weak ergodic hypothesis (the existence of limits of ergodic averages for any Borel measurable F) holds on $\omega[\varrho, \mathbf{m}]$ $\mu-$a.s.
- The (strong) ergodic hypothesis definitely holds for enegetically isolated systems and a class of potential forces F, where all solutions tend to equilibrium