Computing and visualizing oscillatory solutions to the Euler system

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Euler system of gas dynamics

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Momentum balance – Newton's Second Law

$$\partial_t \mathbf{m} + \mathrm{div}_x \left[\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right] + \nabla_x p = 0$$

Energy balance – First Law of Thermodynamics

$$\partial_t E + \mathrm{div}_x \left[(E + p) \frac{\mathbf{m}}{\varrho} \right] = 0$$

Boundary conditions

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Constitutive relations – Second Law of Thermodynamics

Pressure, internal energy, entropy

$$E = \underbrace{\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho}}_{\text{kinetic energy}} + \underbrace{\varrho e}_{\text{internal energy}}, \quad \underbrace{p = (\gamma - 1)\varrho e}_{\text{EOS (incomplete)}}$$

Entropy

$$s = \mathcal{S}\left(\frac{p}{\varrho^\gamma}\right), \quad \underbrace{S = \varrho s}_{\text{total entropy}}$$

Boyle–Mariotte Law:

$$p = \varrho\vartheta, \ \ e = \frac{1}{\gamma - 1}\vartheta, \ \ s = \log\left(\frac{\vartheta^{\frac{1}{\gamma-1}}}{\varrho}\right)$$

Entropy balance (inequality) – Second Law of Thermodynamics

$$\partial_t S + \mathrm{div}_x\left[S\frac{\mathbf{m}}{\varrho}\right] = (\geq) 0$$

Thermodynamic stability

Conservative–entropy variables

density ϱ, momentum $\mathbf{m}$, total entropy S, $[\varrho, \mathbf{m}, S]$

Thermodynamic stability – energy

$$E = E(\varrho, \mathbf{m}, S) : R^{d+2} \to [0, \infty]$$

$$E = (\varrho, \mathbf{m}, S) = \infty \text{ if } \varrho < 0, \ E(0, \mathbf{m}, S) = \lim_{\varrho \to 0+} E(\varrho, \mathbf{m}, S)$$

convex, lower semi–continuous on R^{d+2}

Thermodynamic stability in standard variables

$$\text{positive compressibility } \frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0$$

$$\text{positive specific heat at constant volume } \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Known facts about solvability of Euler system

Classical solutions

Smooth initial state gives rise to smooth solution on a (generically) finite time interval $T_{\max}$, singularities (shocks) develop after $T_{\max}$

Weak solutions

Admissible (weak + entropy inequality) weak solutions exist globally in time. There is a “vast” class of initial data for which the problem admits infinitely many admissible weak solutions, the system is ill–posed in the class of admissible weak solutions

Generalized - oscillatory solutions

There are various concepts of generalized solutions: measure–valued solutions, dissipative measure–valued solutions, etc. They can be seen as limits of *consistent* approximations. They are “inseparable” from the process how they were obtained.

Consistent approximation

Approximate field equations (in the distributional sense)

$$\partial_t \varrho_n + \mathrm{div}_x \mathbf{m}_n = e_n^1$$

$$\partial_t \mathbf{m}_n + \mathrm{div}_x \left[\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} \right] + \nabla_x p(\varrho_n, S_n) = e_n^2$$

$$\partial_t E(\varrho_n, \mathbf{m}_n, S_n) + \mathrm{div}_x \left[(E + p) (\varrho_n, \mathbf{m}_n, S_n) \frac{\mathbf{m}_n}{\varrho_n} \right] = e_n^3$$

$$\partial_t S_n + \mathrm{div}_x \left[S_n \frac{\mathbf{m}_n}{\varrho_n} \right] \geq e_n^4$$

Vanishing consistency errors

$$e_n^1, \ e_n^2, \ e_n^4 \to 0 \text{ in the distributional sense}$$

$$\int_\Omega e_n^3 \ \mathrm{d}x \to 0 \text{ uniformly in time}$$

Stability:

$$\int_\Omega E(\varrho_n, \mathbf{m}_n, S_n) \ \mathrm{d}x \leq c, \ s_n = \frac{S_n}{\varrho_n} \geq -c \ \text{ uniformly in time}$$

Consistent approximation - basic properties

Examples of consistent approximations

- **Vanishing dissipation limit** from the Navier–Stokes–Fourier system to the Euler system
- **Limits of entropy (energy) preserving numerical schemes,** Lax–Friedrichs scheme, Rusanov scheme, Brenner model based scheme (EF, M.Lukáčová, H. Mizerová)

Convergence of consistent approximation

- $$\varrho_n \to \varrho, \ S_n \to S \text{ weakly-(*) in } L^\infty(0, T; L^\gamma(\Omega))$$
  $$\mathbf{m}_n \to \mathbf{m} \text{ weakly-(*) in } L^\infty(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$
- the limit $[\varrho, \mathbf{m}, S]$ is a generalized (dissipative) solution of the Euler system

up to a suitable subsequence!

Convergence of consistent approximation

Strong convergence

- **Strong convergence to strong solution**
  Euler system admits a smooth solution $\Rightarrow [\varrho, \mathbf{m}, S]$ is the unique smooth solution and convergence is strong and unconditional (no need for subsequence) in L^1
- **Strong convergence to smooth limit**
  The limit $[\varrho, \mathbf{m}, S]$ is of class $C^1 \Rightarrow$ the limit is the unique strong solution of the Euler system and convergence is strong and unconditional (no need for subsequence) in L^1
- **Strong convergence to weak solution**
  EF, M.Hofmanová (2019):

  The limit $[\varrho, \mathbf{m}, S]$ is a weak solution of the Euler system $\Rightarrow$ convergence is strong in L^1

Weak convergence of consistent approximation

Weak convergence

If consistent approximation DOES NOT converge strongly, the following must be satisfied:

- the limit Euler system does not admit a strong solution
- the limit $[\varrho, S, \mathbf{m}]$ is not C^1 smooth
- the limit $[\varrho, S, \mathbf{m}]$ IS NOT a weak solution of the Euler system

Visualization of weak convergence?

- **Oscillations.** Weakly converging sequence may develop oscillations. Example:

$$\sin(nx) \to 0 \text{ weakly as } n \to \infty$$

- **Concentrations.**

$$n\theta(nx) \to \delta_0 \text{ weakly-(*) in } \mathcal{M}(R)$$

if

$$\theta \in C_c^\infty(R),\ \theta \geq 0, \int_R \theta = 1$$

Statistical description of oscillations – Young measures

Young measure

$$b(\varrho_n, \mathbf{m}_n, S_n) \to \overline{b(\varrho, \mathbf{m}, S)} \text{ weakly-(*) in } L^\infty((0,T) \times \Omega)$$

(up to a subsequence) for any $b \in C_c(R^{d+2})$

Young measure $\mathcal{V}$ – a parametrized family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ on the phase space R^{d+2}:

$$\overline{b(\varrho, \mathbf{m}, S)}(t,x) = \left\langle \mathcal{V}_{t,x}; b(\widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{S}) \right\rangle \text{ for a.a. } (t,x)$$

Visualizing Young measure

visualizing Young measure $\Leftrightarrow$ computing $\overline{b(\varrho, \mathbf{m}, S)}$

Problems

- $b(\varrho_n, \mathbf{m}_n, S_n)$ converge only weakly
- extracting subsequences
- only statistical properties relevant $\Rightarrow$ knowledge of the "tail" of the sequence of approximate solutions absolutely necessary

Probability – Strong Law of Large Numbers

Cesàro averages

Banach–Saks Theorem, Komlos Theorem:

$$\frac{1}{N}\sum_{n=1}^{N} b(\varrho_n, \mathbf{m}_n, S_n) \to \overline{b(\varrho, \mathbf{m}, S)} \textbf{ strongly in } L^1 \text{ and } \textbf{a.a.}$$

(up to a subsequence) for any $b \in C_c(R^{d+2})$

Reformulation in terms of Young measure

$$\frac{1}{N}\sum_{k=1}^{N} \delta_{[\varrho_n, \mathbf{m}_n, S_n](t,x)} \to \mathcal{V}_{t,x} \text{ a.a. and in } L^1((0,T)\times\Omega;\mathcal{P}),$$

where $\mathcal{P}$ is the Polish space of probability measures on R^{d+1} and convergence is in the sense of Wasserstein (Monge–Kantorowich) distance

Breaking the curse of subsequence

Stationary approximation

$$[\varrho_n, \mathbf{m}_n, S_n] \approx \mathbf{U}_n,\ \ Q \equiv (0,T) \times \Omega,\ \ \{\mathbf{U}_n\}_{n=1}^{\infty} \text{ is stationary}$$

$$\Leftrightarrow$$

$$\int_Q B(\mathbf{U}_{k_1}, \dots, \mathbf{U}_{k_j}) = \int_Q B(\mathbf{U}_{k_1+n}, \dots, \mathbf{U}_{k_j+n})$$

for any $B \in C_c(R^{jd})$, and $[k_1, \dots, k_j]$, and any $n \geq 0$

Asymptotically stationary approximate sequences

An approximate sequence is asymptotically stationary if it approaches asymptotically (to be specified) a stationary sequence

Asymptotic properties of stationary processes

Birkhoff–Khinchin Theorem

$\{\mathbf{U}_n\}_{n=1}^{\infty}$ stationary, $b \in \mathcal{B}(R^d)$ (Borel measurable)

$$\int_Q b(\mathbf{U}_0) < \infty$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{n=1}^{N} b(\mathbf{U}_n) \text{ converges for a.a. } y \in Q$$

Asymptotically stationary approximation

Asymptotically stationary approximate sequence

An approximate sequence $\{\mathbf{U}_n\}_{n=1}^{\infty}$ is asymptotically stationary if for *any* b and any subsequence n_k the following limits exist:

- $$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} b(\mathbf{U}_{n_k}) \text{ in the sense of weak-(*) topology on } L^{\infty}(Q)$$
- $$\lim_{N\to\infty} \frac{1}{N^2} \int_Q \sum_{k=1,m=1}^{n} b(\mathbf{U}_{n_k}) B(\mathbf{U}_{m_k})$$

Note carefully

- the limits may be different for different subsequence (but in fact they are not)
- the first limit is in the weak-(*) not strong topology
- the second limit is limit of correlations

Conclusion

Generation of Young measure

Asymptotically stationary approximate sequences generate unconditionally (no need of subsequence) a unique Young measure

Convergence of Cesàro averages

If $[\varrho_n, \mathbf{m}_n, S_n]$ is an asymptotically stationary approximate sequence, then

$$\frac{1}{N}\sum_{n=1}^{N} b(\varrho_n, \mathbf{m}_n, S_n) \to \overline{b(\varrho, \mathbf{m}, S)} \text{ strongly in } L^1(Q)$$

Visualization of Young measure

The Young measure generated by asymptotically stationary approximate sequence can be computed (visualized) as a strong limit in $L^1(Q)$ of approximate distribution measures:

$$\frac{1}{N}\sum_{n=1}^{N} \delta_{[\varrho_n, \mathbf{m}_n, S_n](t,x)} \ \to\ \mathcal{V}_{t,x} \text{ in Monge–Kantorowich distance}$$