Solving ill posed problems (in fluid mechanics)

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Prologue - Lax equivalence principle

Formulation for LINEAR problems

Peter D. Lax

- **Stability** - uniform bounds of approximate solutions
- **Consistency** - vanishing approximation error

$$\Longleftrightarrow$$

- **Convergence** - approximate solutions converge to exact solution

Euler system of gas dynamics

Leonhard Paul Euler
1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \ \mathbf{m} = \varrho \mathbf{u}$$

Momentum equation – Newton's second law

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0, \ p(\varrho) = a\varrho^\gamma$$

Impermeability and/or periodic boundary condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \Omega \subset R^d, \ \text{or } \Omega = \mathbb{T}^d$$

Initial state

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0$$

Classical solutions

- **Local existence.** Classical solutions exist locally in time as long as the initial data are regular and the initial density strictly positive
- **Finite time blow–up.** Classical solutions develop singularity (become discontinuous) in a *finite* time for a fairly generic class of initial data

Mythology concerning Euler equations in several dimensions

- **Existence.** The long time existence of (possibly weak) solutions is not known
- **Uniqueness.** The is no (known) selection criterion to identify a unique solution (semiflow)
- **Computation.** Oscillatory solutions cannot be visualized by numerical simulation (weak convergence)

Weak (distributional) solutions

Jacques Hadamard 1865–1963

Laurent Schwartz 1915–2002

Mass conservation

$$\int_B \Big[\varrho(t_2,\cdot) - \varrho(t_1,\cdot)\Big] \mathrm{d}x = - \int_{t_1}^{t_2} \int_{\partial B} \varrho \mathbf{u} \cdot \mathbf{n} \ \mathrm{dS}_x \mathrm{d}t$$

$$\left[\int_\Omega \varrho \varphi \ \mathrm{d}x \right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \Big[\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi \Big] \ \mathrm{d}x \mathrm{d}t, \ \ \mathbf{m} \equiv \varrho \mathbf{u}$$

Momentum balance

$$\int_B \Big[\mathbf{m}(t_2,\cdot) - \mathbf{m}(t_1,\cdot)\Big] \ \mathrm{d}x$$

$$= - \int_{t_1}^{t_2} \int_{\partial B} \Big[\mathbf{m} \otimes \mathbf{u} \cdot \mathbf{n} + p(\varrho) \mathbf{n} \Big] \ \mathrm{dS}_x \ \mathrm{d}t$$

$$\left[\int_\Omega \mathbf{m} \cdot \boldsymbol{\varphi} \ \mathrm{d}x \right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \mathrm{d}t$$

Time irreversibility – energy dissipation

Energy

$$\mathcal{E} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

$$p' \geq 0 \Rightarrow [\varrho, \mathbf{m}] \mapsto \left\{ \begin{array}{l} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0 \\ P(\varrho) \text{ if } |\mathbf{m}| = 0,\ \ \varrho \geq 0 \\ \infty \text{ otherwise} \end{array} \right. \quad \text{is convex l.s.c}$$

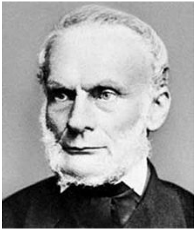
Energy balance (conservation)

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E} \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) = 0$$

Energy dissipation

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E} \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) \boxed{\leq} 0$$

$$E = \int_\Omega \mathcal{E}\ \mathrm{d}x,\ \ \partial_t E \leq 0,\ \ E(0+) = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right]\ \mathrm{d}x$$



Rudolf Clausius 1822–1888

Wild solutions?

Charles Hermite [1822-1901]

In a letter to Stieltjes

I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives

Known facts concerning global solvability

- Existence of infinitely many weak solution for any continuous initial data (Chiodaroli, DeLellis–Széhelyhidi, EF...)
- Existence of "many" initial data that give rise to infinitely many weak solutions satisfying the energy inequality (Chiodaroli, EF, Luo, Xie, Xin...)
- Existence of smooth initial data that ultimately give rise to infinitely many weak solutions satisfying the energy inequality (Kreml et al)
- Weak–strong uniqueness in the class of admissible weak solutions (Dafermos)

Ill posedness

Theorem [A.Abbatiello, EF 2019]

Anna Abbatiello (TU Berlin)

Let $d = 2, 3$. Let ϱ_0, $\mathbf{m}_0$ be given such that

$$\varrho_0 \in \mathcal{R},\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R},\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R},\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^{\infty} \subset (0, T)$ be an arbitrary (countable dense) set of times.
Then the Euler problem admits infinitely many weak solutions ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that

$$\varrho \in C_{\rm weak}([0, T]; L^{\gamma}(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

but

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)]\ \boxed{\text{is not}}\ \text{strongly continuous at any } \tau_i$$

FV numerical scheme

$$(\varrho_h^0, \mathbf{u}_h^0) = (\Pi_{\mathcal{T}} \varrho_0, \Pi_{\mathcal{T}} \mathbf{u}_0)$$

$$D_t \varrho_K^k + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} F_h(\varrho_h^k, \mathbf{u}_h^k) = 0$$

$$D_t (\varrho_h^k \mathbf{u}_h^k)_K + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \left(\mathbf{F}_h(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) + \overline{p(\rho_h^k)} \mathbf{n} - h^\beta \left[\left[\mathbf{u}_h^k \right]\right] \right) = 0.$$

Mária Lukáčová (Mainz)

Hana Mizerová (Bratislava)

Discrete time derivative

$$D_t r_K^k = \frac{r_K^k - r_K^{k-1}}{\Delta t}$$

Upwind, fluxes

$$\mathrm{Up}[r, \mathbf{v}] = \overline{r}\ \overline{\mathbf{v}} \cdot \mathbf{n} - \frac{1}{2} |\overline{\mathbf{v}} \cdot \mathbf{n}|\ [[r]]$$

$$F_h(r, \mathbf{v}) = \mathrm{Up}[r, \mathbf{v}] - h^\alpha\ [[r]]$$

Consistent approximation

Equation of continuity

$$\int_0^T \int_\Omega \left[\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi \right] \mathrm{d}x\mathrm{d}t = e_{1,n}[\varphi]$$

Momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \varphi + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t = e_{2,n}[\boldsymbol{\varphi}]$$

Stability - bounded energy

$$\mathcal{E}(\varrho_n, \mathbf{m}_n) \equiv \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] \mathrm{d}x \lesssim 1$$

Consistency

$$e_{1,n}[\varphi] \to 0, \ \ e_{2,n}[\boldsymbol{\varphi}] \to 0 \text{ as } n \to \infty$$

Weak vs strong convergence

Weak convergence

$$\varrho_n \to \varrho \text{ weakly-(*) } L^\infty(0,T;L^\gamma(\Omega))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ weakly-(*) } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d))$$

Strong convergence (Theorem EF, M.Hofmanová)

- Suppose

$$\Omega \subset R^d \text{ bounded}$$

$\varrho_n \to \varrho$, $\mathbf{m}_n \to \mathbf{m}$ strongly a.a. pointwise in $\mathcal{U}$ open, $\partial\Omega \subset \mathcal{U}$

- Then the following is equivalent:

$$\varrho, \mathbf{m} \text{ weak solution to the Euler system}$$

$$\Leftrightarrow$$

$$\varrho_n \to \varrho,\ \mathbf{m}_n \to \mathbf{m} \text{ strongly (pointwise) in } \Omega$$

Martina Hofmanová (Bielefeld)

Dissipative solutions – limits of numerical schemes

Dominic Breit (Edinburgh)

Martina Hofmanová (Bielefeld)

Equation of continuity

$$\partial_t \boxed{\varrho} + \mathrm{div}_x \mathbf{m} = 0$$

Momentum balance

$$\partial_t \boxed{\mathbf{m}} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \mathfrak{R}$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq 0, \ \ E(t) \leq E_0, \ \ E_0 = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$

$$\boxed{E} \equiv \left(\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x + d \int_{\overline{\Omega}} \mathrm{trace}[\mathfrak{R}] \right)$$

Reynolds stress

$$\mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\mathrm{sym}}))$$

Basic properties of dissipative solutions

Well posedness, weak strong uniqueness

- **Existence.** Dissipative solutions exist globally in time for any finite energy initial data
- **Limits of consistent approximations** Limits of consistent approximations are dissipative solutions, in particular limits of consistent numerical schemes.
- **Compatibility.** Any C^1 dissipative solution $[\varrho, \mathbf{m}]$, $\varrho > 0$ is a classical solution of the Euler system
- **Weak–strong uniqueness.** If $[\widetilde{\varrho}, \widetilde{\mathbf{m}}]$ is a classical solution and $[\varrho, \mathbf{m}]$ a dissipative solution starting from the same initial data, then $\mathfrak{R} = 0$ and $\varrho = \widetilde{\varrho}$, $\mathbf{m} = \widetilde{\mathbf{m}}$.
- **Maximal dissipation.** There exists a (possibly non–unique) dissipative solution that maximizes the energy dissipation rate $\approx$ minimizes the total energy. For this solution

$$\|\mathfrak{R}(t)\|_{\mathcal{M}} \to 0 \text{ as } t \to \infty.$$

Semiflow selection

Set of data

$$\mathcal{D} = \left\{ \varrho, \mathbf{m}, E \ \Big| \ \int_\Omega \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \, \mathrm{d}x \leq E \right\}$$

Set of trajectories

$$\mathcal{T} = \left\{ \varrho(t, \cdot), \mathbf{m}(t, \cdot), E(t-, \cdot) \Big| t \in (0, \infty) \right\}$$

Solution set

$$\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] = \Big\{ [\varrho, \mathbf{m}, E] \ \Big| [\varrho, \mathbf{m}, E] \text{ dissipative solution}$$

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0, \ E(0+) \leq E_0 \Big\}$$

Semiflow selection – semigroup

$$U[\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0], \ [\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{D}$$

$$U(t_1 + t_2)[\varrho_0, \mathbf{m}_0, E_0] = U(t_1) \circ \Big[U(t_2)[\varrho_0, \mathbf{m}_0, E_0] \Big], \ t_1, t_2 > 0$$

Andrej Markov (1856–1933)

N. V. Krylov

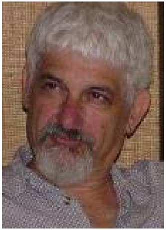
Strong instead of weak (numerics)

Komlos theorem (a variant of Strong Law of Large Numbers)

$$\{U_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q)$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} U_{n_k} \to \overline{U} \text{ a.a. in } Q \text{ as } N \to \infty$$



Janos Komlos (Rutgers Univ.)

Convergence of numerical solutions - EF, M.Lukáčová, H.Mizerová 2018

$$\frac{1}{N}\sum_{k=1}^{N} \varrho_{n_k} \to \varrho \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{m}_{n_k} \to \mathbf{m} \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \left[\frac{1}{2}\frac{|\mathbf{m}_{n,k}|^2}{\varrho_{n,k}} + P(\varrho_{n,k})\right] \to \overline{\mathcal{E}} \in L^1((0,T)\times\Omega) \text{ a.a. in } (0,T)\times\Omega$$

Computing defect – Young measure

Generating Young measure

$$\mathbf{U}_n = [\varrho_n, \mathbf{m}_n] \in R^{d+1} \text{ phase space}$$

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q; R^d) \approx \nu^n_{t,x} = \delta_{\mathbf{U}_n(t,x)}$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} \nu^{n_k}_{t,x} \to \nu_{t,x} \text{ narrowly } \boxed{a.a.} \text{ in } Q \text{ as } N \to \infty$$

Young measure

$$(t,x) \in Q \mapsto \nu_{t,x} \in \mathcal{P}[R^{d+1}] \text{ weakly-(*) measurable mapping}$$

Erich J. Balder (Utrecht)

$$\mathfrak{R} \approx \left\langle \nu; \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} \right\rangle - \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} \langle \nu; p(\varrho)\rangle - p(\varrho)$$

Computing defect numerically -EF, M.Lukáčová, B.She

Monge–Kantorowich (Wasserstein) distance

$$\left\| \mathrm{dist}\left(\frac{1}{N} \sum_{k=1}^{N} \nu_{t,x}^{n_k}; \nu_{t,x} \right) \right\|_{L^q(Q)} \to 0$$

for some $q > 1$

Mária Lukáčová (Mainz)

Convergence in the first variation

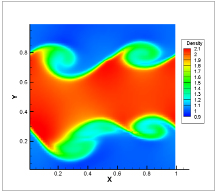
$$\frac{1}{N} \sum_{k=1}^{N} \left\langle \nu_{t,x}^{n_k}; \left| \widetilde{\mathbf{U}} - \frac{1}{N} \sum_{k=1}^{N} \mathbf{U}_n \right| \right\rangle \to \left\langle \nu_{t,x}; \left| \widetilde{\mathbf{U}} - \mathbf{U} \right| \right\rangle$$

in $L^1(Q)$

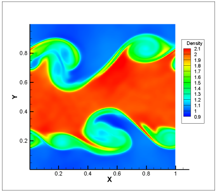
Bangwei She (CAS Praha)

Experiment I, density for Kelvin–Helmholtz problem (M. Lukáčová, Yue Wang)

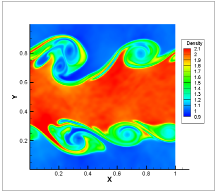
density ϱ
$n = 128, \ T = 2$



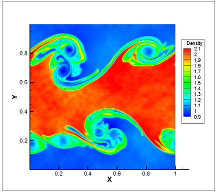
density ϱ
$n = 256, \ T = 2$



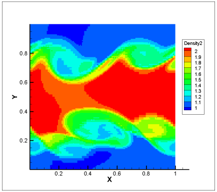
density ϱ
$n = 512, \ T = 2$



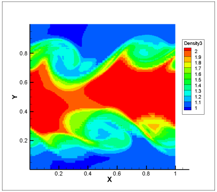
density ϱ
$n = 1024, \ T = 2$



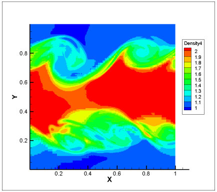
Cèsaro averages density ϱ
$n = 128, \ T = 2$



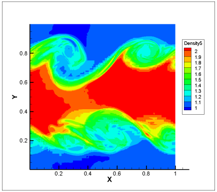
Cèsaro averages density ϱ
$n = 256, \ T = 2$



Cèsaro averages density ϱ
$n = 512, \ T = 2$

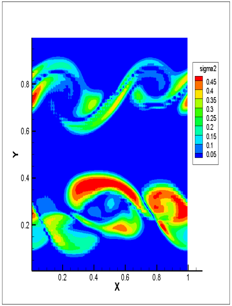


Cèsaro averages density ϱ
$n = 1024, \ T = 2$

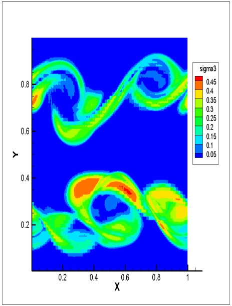


Experiment II, density variations for Kelvin–Helmholtz problem (M. Lukáčová, Yue Wang)

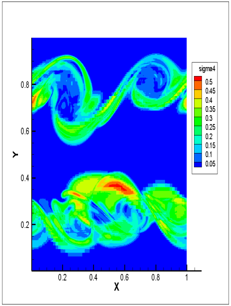
density variation
$n = 128, \ T = 2$



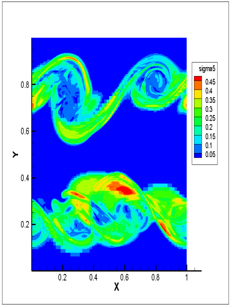
density variation
$n = 256, \ T = 2$



density variation
$n = 512, \ T = 2$



density variation
$n = 1024, \ T = 2$



Yue Wang, Mainz

Mária Lukáčová, Mainz

Consistent approximation of the Euler system

Leonhard Paul Euler 1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \ \mathbf{m} = \varrho \mathbf{u}$$

Momentum equation – Newton's second law

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0, \ p(\varrho) = a\varrho^\gamma$$

Far field /or periodic boundary condition

$$\Omega = R^d \ \varrho \to \overline{\varrho}, \ \mathbf{m} \to (\overline{\varrho}\ \overline{\mathbf{u}}) \text{ as } |x| \to \infty, \text{ or } \Omega = \mathbb{T}^d$$

Initial state

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0$$

Energy dissipation

Energy

$$E = E\left(\varrho, \mathbf{m} \middle| \overline{\varrho}, \overline{u}\right) = \frac{1}{2}\varrho \left| \frac{\mathbf{m}}{\varrho} - \overline{\mathbf{u}} \right|^2 + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho})$$

$$= \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] - \mathbf{m} \cdot \overline{\mathbf{u}} + \frac{1}{2}\varrho |\overline{\mathbf{u}}|^2 - P'(\overline{\varrho})\varrho + p(\overline{\varrho})$$

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

Energy dissipation

$$\mathcal{E} = \int_\Omega \mathcal{E} \ \mathrm{d}x, \ \partial_t E \leq 0, \ E(0+) = \int_\Omega E\left(\varrho_0, \mathbf{m}_0 \ \middle| \overline{\varrho}, \overline{\mathbf{u}}\right) \ \mathrm{d}x$$

Motivation

Incompressible Euler system

$$\mathrm{div}_x \mathbf{v} = 0, \ \partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

Result of Greengard and Thomann [1988]

There exists a sequence $\{\mathbf{v}_n\}_{n=1}^{\infty}$ of compactly supported (in the space variable R^3) of solutions to the incompressible Euler system converging *weakly* to zero.

Conclusion

Incompessible Euler system admits sequences of oscillatory spatially localized solutions converging weakly to another (weak) solution of the same problem

Stable consistent approximation

Equation of continuity

$$\int_0^T \int_{R^d} \left[\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi \right] \mathrm{d}x\mathrm{d}t = e_{1,n}[\varphi]$$

Momentum equation

$$\int_0^T \int_{R^d} \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \varphi + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t = e_{2,n}[\boldsymbol{\varphi}]$$

Stability - bounded energy

$$\mathcal{E}(\varrho_n, \mathbf{m}_n) \equiv \int_{R^d} \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) - P'(\overline{\varrho})(\varrho_n - \overline{\varrho}) - P(\overline{\varrho}) \right] \mathrm{d}x \leq \mathcal{E}_0$$

Consistency

$$e_{1,n}[\varphi] \to 0, \ \ e_{2,n}[\boldsymbol{\varphi}] \to 0 \text{ as } n \to \infty$$

Weak vs strong convergence

Weak convergence

$$p(\varrho) = a\varrho^{\gamma},\ P(\varrho) = \frac{a}{\gamma - 1}\varrho^{\gamma}$$

$$\varrho_n \to \varrho \text{ weakly-(*) } L^{\infty}(0, T; (L^{\gamma} + L^2)(R^d))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ weakly-(*) } L^{\infty}(0, T; (L^{\frac{2\gamma}{\gamma+1}} + L^2)(R^d; R^d))$$

Strong convergence ?

- Suppose

$$\int_0^T \int_B E\left(\varrho_n, \mathbf{m}_n \,\middle|\, \overline{\varrho}\right) \mathrm{d}x\mathrm{d}t \to \int_0^T \int_B E\left(\varrho, \mathbf{m} \,\middle|\, \overline{\varrho}\right) \mathrm{d}x\mathrm{d}t \text{ for any ball } B \subset R^d$$

-

$$\Rightarrow$$

$$\varrho_n \to \varrho \text{ in } L^{\gamma}_{\mathrm{loc}}([0, T] \times R^d),\ \mathbf{m}_n \to \mathbf{m} \text{ in } L^{\frac{2\gamma}{\gamma+1}}_{\mathrm{loc}}([0, T] \times R^d)$$

$$\Rightarrow$$

ϱ, $\mathbf{m}$ is a weak solution of the Euler system

Limit in the field equations

Equation of continuity

$$\int_0^T \int_{R^d} [\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi] \, \mathrm{d}x\mathrm{d}t = 0$$

Convective term

$$\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} + \left(p(\varrho_n) - p'(\overline{\varrho})(\varrho_n - \overline{\varrho}) - p(\overline{\varrho}) \right) \mathbb{I}$$

$$\to \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + \left(p(\varrho) - p'(\overline{\varrho})(\varrho - \overline{\varrho}) - p(\overline{\varrho}) \right) \mathbb{I}}$$

weakly-(*) in $L^\infty(0, T; \mathcal{M}(R^d; R^{d \times d}_{\rm sym})) \approx \left[L^1(0, T; C_c(R^d; R^{d \times d}_{\rm sym})) \right]^*$

Momentum equation and Reynolds defect

Reynolds defect

$$\mathfrak{R} = \overline{\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + \left(p(\varrho) - p'(\overline{\varrho})(\varrho - \overline{\varrho}) - p(\overline{\varrho})\right)\mathbb{I}}$$
$$- \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + \left(p(\varrho) - p'(\overline{\varrho})(\varrho - \overline{\varrho}) - p(\overline{\varrho})\right)\mathbb{I}\right)$$
$$= \overline{\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + p(\varrho)\mathbb{I}} - \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + p(\varrho)\mathbb{I}\right)$$

Momentum equation

$$\int_0^T \int_{R^d} \left[\mathbf{m}\cdot\partial_t\boldsymbol{\varphi} + \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} : \nabla_x\varphi + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}\right]\mathrm{d}x\mathrm{d}t$$
$$= -\int_0^T \int_{R^d} \nabla_x\boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}\,\mathrm{d}t$$

Positivity of Reynolds defect

Reynolds defect

$$\mathfrak{R} = \overline{\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + (p(\varrho) - p'(\overline{\varrho})(\varrho - \overline{\varrho}) - p(\overline{\varrho}))\,\mathbb{I}} - \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + (p(\varrho) - p'(\overline{\varrho})(\varrho - \overline{\varrho}) - p(\overline{\varrho}))\,\mathbb{I} \right)$$

$$\int_{R^d} g(\xi\otimes\xi) : \mathrm{d}\mathfrak{R} \geq 0 \text{ for any } \xi \in R^d,\ g \in C_c(R^d),\ g \geq 0$$

Convexity

$$[\varrho, \mathbf{m}] \mapsto \left(\frac{1}{2}\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + (p(\varrho) - p'(\overline{\varrho})(\varrho - \overline{\varrho}) - p(\overline{\varrho}))\,\mathbb{I} \right) : (\xi\otimes\xi)$$
$$= \left(\frac{1}{2}\frac{|\mathbf{m}\cdot\xi|^2}{\varrho} + (p(\varrho) - p'(\overline{\varrho})(\varrho - \overline{\varrho}) - p(\overline{\varrho}))\,|\xi|^2 \right)$$

is convex for any $\xi \in R^d$

Liouville type result for the defect

Weak formulation

$$\int_{R^d} \nabla_x \varphi : \mathrm{d}\mathfrak{R} = 0 \text{ for any } \varphi \in C_c^\infty(R^d),$$

$$\mathfrak{R} \in \mathcal{M}^+(R^d; R^{d\times d}_{\mathrm{sym}}) \text{ -- a finite tensor--valued measure}$$

Cut-off

$$0 \le \psi_R \le 1,\ \psi_R \in C_c^\infty(R^d)$$

$$\psi_R(Y) = 1 \text{ if } |Y| < r,\ \psi_R(Y) = 0 \text{ if } |Y| > 2r,\ |\nabla_x \psi_R| \le \frac{2}{R}$$

C^1 test functions

$$0 = \int_{R^d} \nabla_x(\psi_R \varphi) : \mathrm{d}\mathfrak{R} = \int_{R^d} \psi_R \nabla_x \varphi : \mathrm{d}\mathfrak{R} + \int_{R^d} (\nabla_x \psi_R \otimes \varphi) : \mathrm{d}\mathfrak{R}$$

$$= \int_{|x|<R} \nabla_x \varphi : \mathrm{d}\mathfrak{R} + \int_{|x|\ge R} [\psi_R \nabla_x \varphi + (\nabla_x \psi_R \otimes \varphi)] : \mathrm{d}\mathfrak{R}$$

Conclusion

Extending the class of text functions

$$\int_{R^d} \nabla_x \varphi : \mathrm{d}\mathfrak{R} = 0$$

for any $\varphi \in C^\infty(R^d)$, $|\nabla_x \varphi| \leq c$

Special test function

$$\varphi, \ \varphi_i = \sum_{j=1}^{N} \xi_i \xi_j x_j$$

Conclusion

$$\int_{R^d} (\xi \otimes \xi) : \mathrm{d}\mathfrak{R} = 0 \Rightarrow (\xi \otimes \xi) : \mathfrak{R} = 0 \Rightarrow \mathfrak{R} = 0$$

Dissipative solutions – limits of numerical schemes

Dominic Breit (Edinburgh)

Martina Hofmanová (Bielefeld)

Equation of continuity

$$\partial_t \boxed{\varrho} + \mathrm{div}_x \mathbf{m} = 0$$

Momentum balance

$$\partial_t \boxed{\mathbf{m}} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \mathfrak{R}$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(t) \leq 0, \ \mathcal{E}(t) \leq \mathcal{E}_0, \ \mathcal{E}_0 = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$

$$\boxed{\mathcal{E}} \equiv \left(\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x + d \int_{\overline{\Omega}} \mathrm{trace}[\mathfrak{R}] \right)$$

Reynolds stress

$$\mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\mathrm{sym}}))$$

Basic properties of dissipative solutions

Well posedness, weak strong uniqueness

- **Existence.** Dissipative solutions exist globally in time for any finite energy initial data
- **Limits of consistent approximations** Limits of consistent approximations are dissipative solutions, in particular limits of consistent numerical schemes.
- **Compatibility.** Any C^1 dissipative solution $[\varrho, \mathbf{m}]$, $\varrho > 0$ is a classical solution of the Euler system
- **Weak–strong uniqueness.** If $[\widetilde{\varrho}, \widetilde{\mathbf{m}}]$ is a classical solution and $[\varrho, \mathbf{m}]$ a dissipative solution starting from the same initial data, then $\mathfrak{R} = 0$ and $\varrho = \widetilde{\varrho}$, $\mathbf{m} = \widetilde{\mathbf{m}}$.
- **Maximal dissipation.** There exists a (possibly non–unique) dissipative solution that maximizes the energy dissipation rate $\approx$ minimizes the total energy. For this solution

$$\|\mathfrak{R}(t)\|_{\mathcal{M}} \to 0 \text{ as } t \to \infty.$$

Relative energy

Relative energy

$$E = E\left(\varrho, \mathbf{m} \middle| r, \mathbf{U}\right) = \frac{1}{2}\varrho\left|\frac{\mathbf{m}}{\varrho} - \mathbf{U}\right|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r)$$
$$= \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right] - \mathbf{m}\cdot\mathbf{U} + \frac{1}{2}\varrho|\mathbf{U}|^2 - P'(r)\varrho + p(r)$$

$$\mathcal{E}\left(\varrho, \mathbf{m} \middle| r, \mathbf{U}\right) = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right] \, \mathrm{d}x + d\int_\Omega \mathrm{dtr}[\mathfrak{R}] \, \mathrm{d}x$$
$$- \int_\Omega \mathbf{m}\cdot\mathbf{U} \, \mathrm{d}x + \frac{1}{2}\int_\Omega \varrho|\mathbf{U}|^2 \, \mathrm{d}x - \int_\Omega P'(r)\varrho \, \mathrm{d}x + \int_\Omega p(r) \, \mathrm{d}x$$

$$\mathfrak{R}(0) = 0$$

Relative energy inequality

Relative energy inequality

$$\mathcal{E}\left(\varrho,\mathbf{m}\Big|r,\mathbf{U}\right)(\tau)\mathrm{d}x \leq \mathcal{E}\left(\varrho,\mathbf{m}\Big|r,\mathbf{U}\right)(s)$$

$$-\int_s^\tau\int_\Omega\left[\varrho\left(\mathbf{U}-\frac{\mathbf{m}}{\varrho}\right)\cdot\nabla_x\mathbf{U}\cdot\left(\mathbf{U}-\frac{\mathbf{m}}{\varrho}\right)\right]\ \mathrm{d}x$$

$$-\int_s^\tau\int_\Omega\Big(p(\varrho)-p'(r)(\varrho-r)-p(r)\Big)\mathrm{div}_x\mathbf{U}\ \mathrm{d}x\mathrm{d}t$$

$$+\int_s^\tau\int_\Omega\Big[\partial_t(r\mathbf{U})+\mathrm{div}_x\left(r\mathbf{U}\otimes\mathbf{U}\right)+\nabla_x p(r)\Big]\cdot\frac{1}{r}\Big(\varrho\mathbf{U}-\mathbf{m}\Big)\ \mathrm{d}x\mathrm{d}t$$

$$+\int_s^\tau\int_\Omega\Big[\partial_t r+\mathrm{div}_x(r\mathbf{U})\Big]\left[\left(1-\frac{\varrho}{r}\right)p'(r)+\frac{1}{r}\mathbf{U}\cdot(\mathbf{m}-\varrho\mathbf{U})\right]\ \mathrm{d}x\mathrm{d}t$$

$$-\int_s^\tau\int_\Omega\nabla_x\mathbf{U}:\mathrm{d}\mathfrak{R}(t)\mathrm{d}t$$

$$r,\mathbf{U}\in C^1,\ \ r>0$$

Equations of general viscous fluids

Basic system of equations

- **conservation of mass**

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

- **balance of linear momentum**

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u})$$

- **viscous rheological law**

$$\mathbb{S} \in \partial F(\mathbb{D}_x \mathbf{u}),\ \mathbb{D}_x \mathbf{u} = \frac{\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t}{2}$$

Boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \mathbf{u}_B,\ \ \varrho|_{\Gamma_{in}} = \varrho_B,\ \ \Gamma_{in} = \left\{ x \in \partial\Omega \ \middle|\ \mathbf{u}_B \cdot \mathbf{n} < 0 \right\},$$

Viscous potential and energy balance

Viscous potential

$$F : R^{d\times d}_{\rm sym} \to [0,\infty] \text{ convex, l.s.c.}$$

Fenchel–Young inequality

$$\mathbb{D} : \mathbb{S} \leq F(\mathbb{D}) + F^*(\mathbb{S})$$

$$\mathbb{D} : \mathbb{S} = F(\mathbb{D}) + F^*(\mathbb{S}) \ \Leftrightarrow\ \mathbb{S} \in \partial F(\mathbb{D}_x \mathbf{u}) \ \Leftrightarrow\ \mathbb{D}_x \mathbf{u} \in \partial F^*(\mathbb{S})$$

Energy inequality

$$\frac{\rm d}{{\rm d}t} \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_B|^2 + P(\varrho) \right] \ {\rm d}x$$
$$+ \int_\Omega \mathbb{S} : \mathbb{D}_x \mathbf{u} \ {\rm d}x + \int_{\partial\Omega} P(\varrho) \mathbf{u}_B \cdot \mathbf{n} \ {\rm d}S_x$$
$$\leq - \int_\Omega \Big[p(\varrho)\mathbb{I} + \varrho \mathbf{u} \otimes \mathbf{u} \Big] : \nabla_x \mathbf{u}_B \ {\rm d}x - \int_\Omega \varrho \mathbf{u} \cdot \Big(\mathbf{u}_B \cdot \nabla_x \mathbf{u}_B \Big) \ {\rm d}x$$
$$+ \int_\Omega \mathbb{S} : \nabla_x \mathbf{u}_B {\rm d}x.$$

Dissipative solutions

Weak formulation

- **conservation of mass**

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

- **balance of linear momentum**

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S} - \boxed{\mathrm{div}_x \mathfrak{R}}$$

- **energy inequality**

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \left[\frac{1}{2}\varrho |\mathbf{u} - \mathbf{u}_B|^2 + P(\varrho) \right] \ \mathrm{d}x + d \boxed{\int_\Omega \mathrm{tr}[\mathfrak{R}]} \right)$$

$$+ \boxed{\int_\Omega \left(F(\mathbb{D}_x \mathbf{u}) + F^*(\mathbb{S}) \right) \ \mathrm{d}x} + \int_{\partial\Omega} P(\varrho)\mathbf{u}_B \cdot \mathbf{n} \ \mathrm{d}S_x$$

$$\leq - \int_\Omega \left[p(\varrho)\mathbb{I} + \varrho \mathbf{u} \otimes \mathbf{u} \right] : \nabla_x \mathbf{u}_B \ \mathrm{d}x - \int_\Omega \varrho \mathbf{u} \cdot \left(\mathbf{u}_B \cdot \nabla_x \mathbf{u}_B \right) \ \mathrm{d}x$$

$$+ \int_{\overline{\Omega}} \mathbb{S} : \mathbb{D}_x \mathbf{u}_B \mathrm{d}x - \boxed{\int_\Omega \mathbb{D}_x \mathbf{u}_B : \mathrm{d}\mathfrak{R}}$$

Dissipative solutions for Euler system revisited

Dominic Breit (Edinburgh)

Martina Hofmanová (Bielefeld)

Equation of continuity

$$\partial_t \boxed{\varrho} + \mathrm{div}_x \mathbf{m} = 0$$

Momentum balance

$$\partial_t \boxed{\mathbf{m}} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \mathfrak{R}$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(t) \leq 0, \ \mathcal{E}(t) \leq \mathcal{E}_0, \ \mathcal{E}_0 = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$

$$\boxed{\mathcal{E}} \equiv \left(\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x + d \int_{\overline{\Omega}} \mathrm{trace}[\mathfrak{R}] \right)$$

càglàd modification, $\mathcal{E}(t) = \mathcal{E}(t-)$

Reynolds stress

$$\mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\mathrm{sym}}))$$

Basic properties of dissipative solutions

Well posedness, weak strong uniqueness

- **Existence.** Dissipative solutions exist globally in time for any finite energy initial data
- **Limits of consistent approximations** Limits of consistent approximations are dissipative solutions, in particular limits of consistent numerical schemes.
- **Compatibility.** Any C^1 dissipative solution $[\varrho, \mathbf{m}]$, $\varrho > 0$ is a classical solution of the Euler system
- **Weak–strong uniqueness.** If $[\widetilde{\varrho}, \widetilde{\mathbf{m}}]$ is a classical solution and $[\varrho, \mathbf{m}]$ a dissipative solution starting from the same initial data, then $\mathfrak{R} = 0$ and $\varrho = \widetilde{\varrho}$, $\mathbf{m} = \widetilde{\mathbf{m}}$.
- **Maximal dissipation.** There exists a (possibly non–unique) dissipative solution that maximizes the energy dissipation rate $\approx$ minimizes the total energy. For this solution

$$\|\mathfrak{R}(t)\|_{\mathcal{M}} \to 0 \text{ as } t \to \infty.$$

Semiflow selection

Set of data

$$\mathcal{D} = \left\{ \varrho, \mathbf{m}, \mathcal{E} \ \Big| \ \int_\Omega \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \ \mathrm{d}x \leq \mathcal{E} \right\}$$

Set of trajectories

$$\mathcal{T} = \left\{ \varrho(t, \cdot), \mathbf{m}(t, \cdot), \mathcal{E}(t) \Big| t \in (0, \infty) \right\}$$

Solution set

$$\mathcal{U}[\varrho_0, \mathbf{m}_0, \mathcal{E}_0] = \left\{ [\varrho, \mathbf{m}, \mathcal{E}] \ \Big| [\varrho, \mathbf{m}, \mathcal{E}] \text{ dissipative solution} \right.$$

$$\left. \varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0, \ \mathcal{E}(0) = \mathcal{E}(0-) \equiv \mathcal{E}_0 \right\}$$

Semiflow selection – semigroup

$$U(t)[\varrho_0, \mathbf{m}_0, \mathcal{E}_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, \mathcal{E}_0], \ [\varrho_0, \mathbf{m}_0, \mathcal{E}_0] \in \mathcal{D}$$

$$U(t_1 + t_2)[\varrho_0, \mathbf{m}_0, \mathcal{E}_0] = U(t_1) \circ \Big[U(t_2)[\varrho_0, \mathbf{m}_0, \mathcal{E}_0] \Big], \ t_1, t_2 > 0$$

Andrej Markov (1856–1933)

N. V. Krylov

Abstract scheme

Data space:

$$\mathcal{D} = \left\{ \mathbf{U}_0 \ \middle| \ \mathbf{U}_0 \in X - \text{ Polish space} \right\}$$

Trajectory space:

$$\mathcal{T} = \left\{ \mathbf{U} : [0,\infty) \to X \subset X \ \middle| \ \mathbf{U} \text{ càglàd mapping}, \ Y \text{ Polish space} \right\}$$

Multi–valued solution mapping:

$$\mathcal{U} : \mathbf{U}_0 \mapsto \mathbf{U} \in 2^{\mathcal{T}}$$

Time shift:

$$S_T : [S_T \circ \mathbf{U}](t) = \mathbf{U}(T + t), \ t \geq 0.$$

Continuation

$$\mathbf{U}_1 \cup_T \mathbf{U}_2(\tau) = \begin{cases} \mathbf{U}_1(\tau) \text{ for } 0 \leq \tau \leq T, \\ \\ \mathbf{U}_2(\tau - T) \text{ for } \tau > T. \end{cases}$$

Basic ansatz

- **(A1) Compactness:** For any $\mathbf{U}_0 \in D$, the set $\mathcal{U}[\mathbf{U}_0]$ is a non–empty compact subset of $\mathcal{T}$
- **(A2) measurability:** The mapping

$$D \ni \mathbf{U}_0 \mapsto \mathcal{U}[\mathbf{U}_0] \in 2^{\mathcal{T}}$$

  is **Borel measurable**, where the range of $\mathcal{U}$ is endowed with the Hausdorff metric on the subspace of compact sets in $2^{\mathcal{T}}$
- **(A3) Shift invariance:** For any

$$\mathbf{U} \in \mathcal{U}[\varrho_0, \mathbf{m}_0, S_0],$$

  we have

$$S_T \circ \mathbf{U} \in \mathcal{U}[\mathbf{U}(T)] \text{ for any } T > 0$$

- **(A4) Continuation:** If $T > 0$, and

$$\mathbf{U}^1 \in \mathcal{U}[\mathbf{U}_0],\ \mathbf{U}^2 \in \mathcal{U}[\mathbf{U}^1(T)],$$

  then

$$\mathbf{U}^1 \cup_T \mathbf{U}^2 \in \mathcal{U}[\mathbf{U}_0]$$

Semiflow selection

Jorge E. Cardona (Darmstadt)

Lev Kapitanski (Florida)

System of functionals:

$$I_{\lambda,F}[\mathbf{U}] = \int_0^\infty \exp(-\lambda t) F(\mathbf{U}(t))\, \mathrm{d}t,\ \lambda > 0$$

where F is a bounded and continuous functional

Semiflow reduction:

$$I_{\lambda,F} \circ \mathcal{U}[\mathbf{U}_0] \\ = \left\{ \mathbf{U} \in \mathcal{U}[\mathbf{U}_0] \ \middle|\ I_{\lambda,F}[\mathbf{U}] \leq I_{\lambda,F}[\widetilde{\mathbf{U}}] \text{ for all } \widetilde{\mathbf{U}} \in \mathcal{U}[\mathbf{U}_0] \right\}$$

Induction argument:

$$\mathcal{U} \text{ satisfies (A1) - (A4)} \ \Rightarrow I_{\lambda,F} \circ \mathcal{U} \text{ satisfies (A1) - (A4)}$$

Maximal solutions of the Euler system

Comparison relation

$$(\varrho_1, \mathbf{m}_1, \mathcal{E}_1) \prec (\varrho_1, \mathbf{m}_1, \mathcal{E}_1) \ \Leftrightarrow \ E_1(t) \leq E_2(t) \text{ for all } t > 0$$

Maximal dissipative solution

A dissipative solution is *maximal* if it is minimal with respect to "$\prec$"

Existence of maximal solutions

$$\min \int_0^\infty \exp(-\lambda t) \beta\left(\mathcal{E}(t)\right) \ \mathrm{d}t$$

$$\beta : R \to R \text{ strictly increasing bounded}$$

Asymptotic regularity of maximal solutions

$$(\varrho, \mathbf{m}, \mathcal{E}) \text{ maximal } \ \Rightarrow \ \|\mathfrak{R}(t)\|_{\mathcal{M}^+(\Omega; R^d)} \to 0 \text{ as } t \to \infty$$

Euler system of gas dynamics

Leonhard Paul Euler
1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \ \ \mathbf{m} = \varrho \mathbf{u}$$

Momentum equation – Newton's second law

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0, \ \ p(\varrho) = a\varrho^\gamma$$

Impermeability and/or periodic boundary condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \ \Omega \subset R^d, \ \text{ or } \Omega = \mathbb{T}^d$$

Initial state

$$\varrho(0, \cdot) = \varrho_0, \ \ \mathbf{m}(0, \cdot) = \mathbf{m}_0$$

FV numerical scheme

$$(\varrho_h^0, \mathbf{u}_h^0) = (\Pi_{\mathcal{T}} \varrho_0, \Pi_{\mathcal{T}} \mathbf{u}_0)$$

$$D_t \varrho_K^k + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} F_h(\varrho_h^k, \mathbf{u}_h^k) = 0$$

$$D_t (\varrho_h^k \mathbf{u}_h^k)_K + \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \left(\mathbf{F}_h(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) + \overline{p(\rho_h^k)} \mathbf{n} - h^\beta \left[\left[\mathbf{u}_h^k \right]\right] \right) = 0.$$

Mária Lukáčová (Mainz)

Hana Mizerová (Bratislava)

Discrete time derivative

$$D_t r_K^k = \frac{r_K^k - r_K^{k-1}}{\Delta t}$$

Upwind, fluxes

$$\mathrm{Up}[r, \mathbf{v}] = \overline{r}\ \overline{\mathbf{v}} \cdot \mathbf{n} - \frac{1}{2} |\overline{\mathbf{v}} \cdot \mathbf{n}|\ [[r]]$$

$$F_h(r, \mathbf{v}) = \mathrm{Up}[r, \mathbf{v}] - h^\alpha\ [[r]]$$

Stable consistent approximation

Equation of continuity

$$\int_0^T \int_\Omega [\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi] \, \mathrm{d}x\mathrm{d}t = e_{1,n}[\varphi]$$

Momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \varphi + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t = e_{2,n}[\boldsymbol{\varphi}]$$

Stability - bounded energy

$$\mathcal{E}(\varrho_n, \mathbf{m}_n) \equiv \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] \mathrm{d}x \lesssim 1$$

Consistency

$$e_{1,n}[\varphi] \to 0, \ \ e_{2,n}[\boldsymbol{\varphi}] \to 0 \text{ as } n \to \infty$$

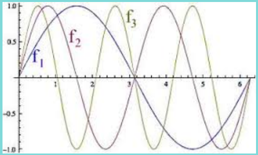
Oscillations vs. nonlinearity

Oscillatory solutions - velocity

$$U(x) \approx \sin(nx), \;\; U \to 0 \text{ in the sense of avarages (weakly)}$$

Oscillatory solutions - kinetic energy

$$\frac{1}{2}|U|^2(x) \approx \frac{1}{2}\sin^2(nx) \to \frac{1}{4} \neq \frac{1}{2}0^2 \text{ in the sense of avarages (weakly)}$$

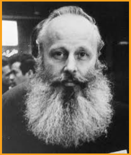


Statistical descrition – Young measure

Young measures

$$U(t,x) \approx \nu_{t,x}[U]$$

$\nu(B)$, $B \subset R^3$ probability that **U** belongs to the set B



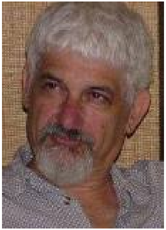
Laurence Chisholm Young
[1905-2000]

Siddhartha Mishra

Numerical results

Certain numerical solutions of the Euler system exhibit scheme independent oscillatory behavior

Strong instead of weak (numerics)



Janos Komlos (Rutgers Univ.)

Komlos theorem (a variant of Strong Law of Large Numbers)

$$\{U_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q)$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} U_{n_k} \to \overline{U} \text{ a.a. in } Q \text{ as } N \to \infty$$

Convergence of numerical solutions - EF, M.Lukáčová, H.Mizerová 2018

$$\frac{1}{N}\sum_{k=1}^{N} \varrho_{n_k} \to \varrho \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{m}_{n_k} \to \mathbf{m} \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \left[\frac{1}{2}\frac{|\mathbf{m}_{n,k}|^2}{\varrho_{n,k}} + P(\varrho_{n,k})\right] \to \overline{\mathcal{E}} \in L^1((0,T)\times\Omega) \text{ a.a. in } (0,T)\times\Omega$$

Computing defect – Young measure

Generating Young measure

$$\mathbf{U}_n = [\varrho_n, \mathbf{m}_n] \in R^{d+1} \text{ phase space}$$

$$\{\mathbf{U}_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q; R^d) \approx \nu^n_{t,x} = \delta_{\mathbf{U}_n(t,x)}$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} \nu^{n_k}_{t,x} \to \nu_{t,x} \text{ narrowly } \boxed{a.a.} \text{ in } Q \text{ as } N \to \infty$$

Erich J. Balder (Utrecht)

Young measure

$$(t,x) \in Q \mapsto \nu_{t,x} \in \mathcal{P}[R^{d+1}] \text{ weakly-(*) measurable mapping}$$

$$\mathfrak{R} \approx \left\langle \nu; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle - \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \langle \nu; p(\varrho) \rangle - p(\varrho)$$

Computing defect numerically -EF, M.Lukáčová, B.She

Monge–Kantorowich (Wasserstein) distance

$$\left\| \mathrm{dist} \left(\frac{1}{N} \sum_{k=1}^{N} \nu_{t,x}^{n_k} ; \nu_{t,x} \right) \right\|_{L^q(Q)} \to 0$$

for some $q > 1$

Convergence in the first variation

$$\frac{1}{N} \sum_{k=1}^{N} \left\langle \nu_{t,x}^{n_k} ; \left| \widetilde{\mathbf{U}} - \frac{1}{N} \sum_{k=1}^{N} \mathbf{U}_n \right| \right\rangle \to \left\langle \nu_{t,x} ; \left| \widetilde{\mathbf{U}} - \mathbf{U} \right| \right\rangle$$

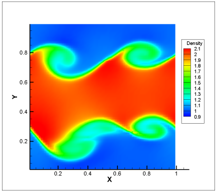
in $L^1(Q)$

Mária Lukáčová (Mainz)

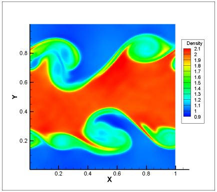
Bangwei She (CAS Praha)

Experiment I, density for Kelvin–Helmholtz problem (M. Lukáčová, Yue Wang)

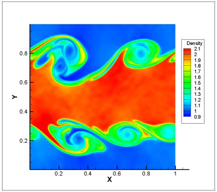
density ϱ
$n = 128, \ T = 2$



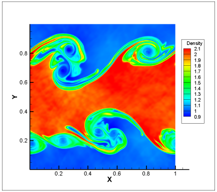
density ϱ
$n = 256, \ T = 2$



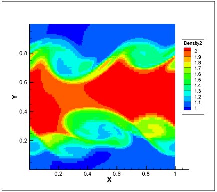
density ϱ
$n = 512, \ T = 2$



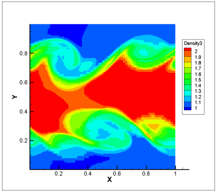
density ϱ
$n = 1024, \ T = 2$



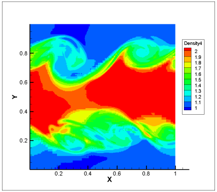
Cèsaro averages density ϱ
$n = 128, \ T = 2$



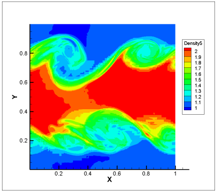
Cèsaro averages density ϱ
$n = 256, \ T = 2$



Cèsaro averages density ϱ
$n = 512, \ T = 2$

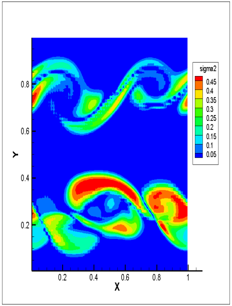


Cèsaro averages density ϱ
$n = 1024, \ T = 2$

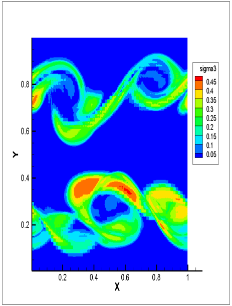


Experiment II, density variations for Kelvin–Helmholtz problem (M. Lukáčová, Yue Wang)

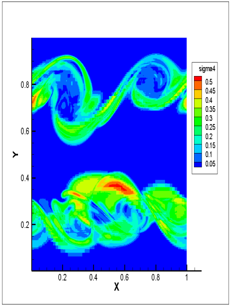
density variation
$n = 128, \; T = 2$



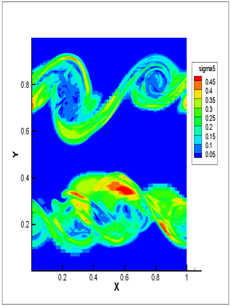
density variation
$n = 256, \; T = 2$



density variation
$n = 512, \; T = 2$



density variation
$n = 1024, \; T = 2$



Yue Wang, Mainz

Mária Lukáčová, Mainz