$\mathcal{K}$–convergence and weak solution method in the analysis of numerical schemes

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Euler system for a barotropic inviscid fluid

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Momentum equation

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0, \ p(\varrho) = a\varrho^\gamma, \ a > 0, \ \gamma > 1$$

Impermeability boundary conditions or periodic boundary conditions

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \text{or } \Omega = \left([-1,1]|_{\{-1,1\}} \right)^d, \ d = 2,3$$

Initial conditions

$$\varrho(0,\cdot) = \varrho_0, \ \mathbf{m}(0,\cdot) = \mathbf{m}_0$$

Admissible solutions – energy dissipation

Energy

$$\mathcal{E} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

$$p' \geq 0 \Rightarrow [\varrho, \mathbf{m}] \mapsto \begin{cases} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0 \\ P(\varrho) \text{ if } |\mathbf{m}| = 0 \\ \infty \text{ if } \varrho = 0, |\mathbf{m}| \neq 0 \end{cases} \quad \text{is convex l.s.c}$$

Energy balance (conservation)

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E} \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) = 0$$

Energy dissipation

$$\partial_t \mathcal{E} + \mathrm{div}_x (\mathcal{E}\mathbf{u}) + \mathrm{div}_x (p\mathbf{u}) \boxed{\leq} 0$$

$$E = \int_\Omega \mathcal{E}\ \mathrm{d}x,\ \ \partial_t E \leq 0,\ \ E(0+) = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right]\ \mathrm{d}x$$

Known facts about Euler equations

Well/ill posedness

- Local in time existence of unique smooth solutions for smooth initial data
- Blow–up (shock wave) in a finite time for a generic class of initial data
- Existence of infinitely many weak solution for any continuous initial data
- Existence of "many" initial data that give rise to infinitely many weak solutions satisfying the energy inequality
- Existence of smooth initial data that give rise to infinitely many weak solutions satisfying the energy inequality
- Weak–strong uniqueness in the class of admissible weak solutions

Dissipative solutions

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \ \ \varrho(0,\cdot) = \varrho_0$$

Momentum balance

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \left(\mathfrak{R}_v + \mathfrak{R}_p \mathbb{I} \right), \ \ \mathbf{m}(0,\cdot) = \mathbf{m}_0$$

Energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq 0, \ \ E(t) \leq E_0, \ \ E_0 = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$

$$E \equiv \left(\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x + \int_{\overline{\Omega}} \mathrm{d} \frac{1}{2} \mathrm{trace}[\mathfrak{R}_v] + \int_{\overline{\Omega}} \mathrm{d} \frac{1}{\gamma - 1} \mathfrak{R}_p \right)$$

Turbulent defect measures

$$\mathfrak{R}_v \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\mathrm{sym}})), \ \ \mathfrak{R}_p \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}))$$

Basic properties of dissipative solutions

Well posedness, weak strong uniqueness

- **Existence.** Dissipative solutions exist globally in time for any finite energy initial data
- **Compatibility.** Any C^1 dissipative solution $[\varrho, \mathbf{m}]$, $\varrho > 0$ is a classical solution of the Euler system
- **Weak–strong uniqueness.** If $[\widetilde{\varrho}, \widetilde{\mathbf{m}}]$ is a classical solution and $[\varrho, \mathbf{m}]$ a dissipative solution starting from the same initial data, then $\mathfrak{R}_v = \mathfrak{R}_p = 0$ and $\varrho = \widetilde{\varrho}$, $\mathbf{m} = \widetilde{\mathbf{m}}$.
- **Semiflow selection.** There exists a measurable selection of dissipative solution that forms a semigroup

Stability properties

Stability - energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}E_n(t) \leq 0, \ E_n(t) \leq E_{0,n} + e_{1,n}, \ E_{0,n} = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_{0,n}|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) \right] \ \mathrm{d}x$$

Consistency - equation of continuity, momentum equation

$$\partial_t \varrho_n + \mathrm{div}_x \mathbf{m}_n = e_{2,n}, \ \varrho_n(0,\cdot) = \varrho_{0,n}$$

$$\partial_t \mathbf{m}_n + \mathrm{div}_x \left(\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} \right) + \nabla_x p(\varrho_n) = e_{3,n}, \ \mathbf{m}_n(0,\cdot) = \mathbf{m}_{0,n}$$

Convergence (up to a subsequence), $e_{1,n}, \ e_{2,n}, \ e_{3,n} \to 0 \ \Rightarrow$

$$\varrho_n \to \varrho \text{ weakly-(*) in } L^\infty(0,T; L^\gamma(\Omega))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ weakly-(*) in } L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

$$\mathcal{E}_n \equiv \left[\frac{1}{2}\frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] \to \mathcal{E} \text{ in } L^\infty_{\mathrm{weak}}(0,T; \mathcal{M}^+(\overline{\Omega}))$$

$$\varrho, \ \mathbf{m}, \ E = \int_{\overline{\Omega}} \mathrm{d}\mathcal{E} \text{ is a dissipative solution}$$

Komlos ($\mathcal{K}$) convergence

Komlos theorem

$$\{U_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q)$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} U_{n_k} \to \overline{U} \text{ a.a. in } Q \text{ as } N \to \infty$$

Conclusion for the approximate solutions

$$\frac{1}{N}\sum_{k=1}^{N} \varrho_{n_k} \to \varrho \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \mathbf{m}_{n_k} \to \mathbf{m} \text{ in } L^1((0,T)\times\Omega) \text{ as } N \to \infty$$

$$\frac{1}{N}\sum_{k=1}^{N} \left[\frac{1}{2}\frac{|\mathbf{m}_{n,k}|^2}{\varrho_{n,k}} + P(\varrho_{n,k})\right] \to \overline{\mathcal{E}} \in L^1((0,T)\times\Omega) \text{ a.a. in } (0,T)\times\Omega$$

$\mathcal{K}$–convergence of Young measures [Balder]

Young measure

$$\{U_n\}_{n=1}^{\infty} \text{ bounded in } L^1(Q) \approx \nu^n_{t,x} = \delta_{U_n(t,x)}$$

$$\Rightarrow$$

$$\frac{1}{N}\sum_{k=1}^{N} \nu^{n_k}_{t,x} \to \nu_{t,x} \text{ narrowly a.a. in } Q \text{ as } N \to \infty$$

Lévy–Prokhorov distance

$$\left\| r\left(\frac{1}{N}\sum_{k=1}^{N} \nu^{n_k}_{t,x}; \nu_{t,x} \right) \right\|_{L^q(Q)} \to 0$$

for any $1 \leq q < \infty$.

Energy defect

Energy defect

$$\left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \leq \overline{\mathcal{E}} \in L^1((0,T) \times \Omega) \leq \mathcal{E} \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}))$$

Oscillation defect

$$\overline{\mathcal{E}} - \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \in L^1((0,T) \times \Omega)$$

Concentration defect

$$\mathcal{E} - \overline{\mathcal{E}} \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}))$$

Total energy defect

$$\mathcal{E} - \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] = \frac{1}{2} \mathrm{trace}[\mathfrak{R}_v] + \frac{1}{\gamma - 1} \mathfrak{R}_p \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}))$$

Strong convergence

Strong convergence

$$\textbf{oscillation defect} = 0 \ \Rightarrow$$

$$\varrho_n \to \varrho \text{ in } L^1((0,T)\times\Omega),\ \mathbf{m}_n \to \mathbf{m} \text{ in } L^1((0,T)\times\Omega; R^d)$$

Equi–integrability

$$\textbf{concentration defect} = 0 \ \Rightarrow$$

$$\varrho_n^\gamma \text{ equi–integrable},\ \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} \text{ equi–integrable}$$

Convergence to a strong solution

$$\textbf{total energy defect} = 0 \ \Rightarrow$$

$$\varrho_n \to \varrho \text{ in } L^\gamma((0,T)\times\Omega),\ \mathbf{m}_n \to \mathbf{m} \text{ in } L^{\frac{2\gamma}{\gamma+1}}((0,T)\times\Omega; R^d)$$

$[\varrho, \mathbf{m}]$ is a strong solution of the Euler system

Weak convergence

Conditional result (EF, M.Hofmanová 2019)

total energy defect $= 0$ on a neighborhood of the boundary $\partial\Omega$

$[\varrho, \mathbf{m}]$ is a weak solution to the Euler system

$$\Rightarrow$$

total energy defect $= 0$ in $(0, T) \times \overline{\Omega}$

convergence is strong in $(0, T) \times \Omega$

Corollary

total energy defect $= 0$ on a neighborhood of the boundary $\partial\Omega$

convergence is only weak in some part of Ω

$$\Rightarrow$$

$[\varrho, \mathbf{m}]$ $\boxed{\text{is not}}$ a weak solution of the Euler system

Numerical scheme, I

Mesh

$$\mathbb{T}^d = \bigcup_{K\in\mathcal{T}} K, K = \prod_{i=1}^{d}[0, h_i]+x_K,\ 0 < \lambda h \leq h_i \leq h,\ i = 1, \ldots, d,\ 0 < \lambda < 1$$

Function spaces

$$\mathcal{Q}_h = \left\{ v \in L^\infty(\mathbb{T}^d) \;\middle|\; v|_K = v_K \right\}$$

$$\Pi_\mathcal{T} : L^1(\mathbb{T}^d) \to \mathcal{Q}_h,\ \Pi_\mathcal{T} v = \sum_{K\in\mathcal{T}} 1_K \frac{1}{|K|}\int_K v\mathrm{d}x.$$

Numerical scheme, II

Notation

$$v^{\mathrm{out}}(x) = \lim_{\delta\to 0+} v(x+\delta\mathbf{n}), \;\; v^{\mathrm{in}}(x) = \lim_{\delta\to 0+} v(x-\delta\mathbf{n})$$

$$\overline{v}(x) = \frac{v^{\mathrm{in}}(x)+v^{\mathrm{out}}(x)}{2}, \;\; [[v(x)]] = v^{\mathrm{out}}(x) - v^{\mathrm{in}}(x)$$

Upwind

$$\mathrm{Up}[r,\mathbf{v}] = r^{\mathrm{up}}\mathbf{v}\cdot\mathbf{n} = r^{\mathrm{in}}[\mathbf{v}\cdot\mathbf{n}]^{+} + r^{\mathrm{out}}[\mathbf{v}\cdot\mathbf{n}]^{-} = \overline{r}\;\overline{\mathbf{v}}\cdot\mathbf{n} - \frac{1}{2}|\overline{\mathbf{v}}\cdot\mathbf{n}|\,[[r]]$$

$$[f]^{\pm} = \frac{f\pm|f|}{2} \quad \text{and} \quad r^{\mathrm{up}} = \begin{cases} r^{\mathrm{in}} & \text{if } \overline{\mathbf{v}}\cdot\mathbf{n}\geq 0, \\ r^{\mathrm{out}} & \text{if } \overline{\mathbf{v}}\cdot\mathbf{n} < 0. \end{cases}$$

Numerical flux

$$F_h(r,\mathbf{v}) = \mathrm{Up}[r,\mathbf{v}] - h^{\alpha}\,[[r]] = \overline{r}\;\overline{\mathbf{v}}\cdot\mathbf{n} - \frac{1}{2}\left(h^{\alpha} + |\overline{\mathbf{v}}\cdot\mathbf{n}|\right)[[r]], \quad \alpha > 0 \tag{1}$$

Numerical scheme, III

Time discretization

$$D_t v^k = \frac{v^k - v^{k-1}}{\Delta t}, \quad \text{for } k = 1, 2, \dots, N_T.$$

Numerical scheme

$$\int_\Omega D_t \varrho_h^k \varphi_h \,\mathrm{d}x - \sum_{\sigma \in \mathcal{E}} \int_\sigma F_h(\varrho_h^k, \mathbf{u}_h^k) \, [[\varphi_h]] \,\mathrm{d}\sigma = 0 \quad \text{for all } \varphi_h \in \mathcal{Q}_h,$$

$$\int_\Omega D_t(\varrho_h^k \mathbf{u}_h^k) \cdot \boldsymbol{\varphi}_h \,\mathrm{d}x - \sum_{\sigma \in \mathcal{E}} \int_\sigma \mathbf{F}_h(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) \cdot [[\boldsymbol{\varphi}_h]] \,\mathrm{d}\sigma$$

$$- \sum_{\sigma \in \mathcal{E}} \int_\sigma \overline{p(\varrho_h^k)} \mathbf{n} \cdot [[\boldsymbol{\varphi}_h]] \,\mathrm{d}\sigma = \boxed{-h^\beta \sum_{\sigma \in \mathcal{E}} \int_\sigma \left[\left[\mathbf{u}_h^k\right]\right] \cdot [[\boldsymbol{\varphi}_h]] \,\mathrm{d}\sigma} \qquad \text{(2a)}$$

$$\text{for all } \boldsymbol{\varphi}_h \in \mathcal{Q}_h(\mathbb{T}^d; R^d), \ \beta > -1$$

Properties of numerical solutions, I

Positivity of the discrete density

$$\varrho_h^k > 0$$

Discrete energy balance

$$D_t \int_\Omega \left[\frac{1}{2} \varrho_h^k |\mathbf{u}_h^k|^2 + P(\varrho_h^k) \right] \, \mathrm{d}x + \sum_{\sigma \in \mathcal{E}} \int_\sigma \left(h^\alpha \overline{\varrho_h^k} \left[\left[\mathbf{u}_h^k \right]\right]^2 + h^\beta \left[\left[\mathbf{u}_h^k \right]\right]^2 \right) \mathrm{d}\sigma$$

$$= -\frac{\Delta t}{2} \int_\Omega P''(\xi) |D_t \varrho_h^k|^2 \, \mathrm{d}x - \frac{1}{2} \sum_{\sigma \in \mathcal{E}} \int_\sigma P''(\eta) \left[\left[\varrho_h^k \right]\right]^2 \left(h^\alpha + |\overline{\mathbf{u}_h^k} \cdot \mathbf{n}| \right) \mathrm{d}\sigma$$

$$- \frac{\Delta t}{2} \int_\Omega \varrho_h^{k-1} |D_t \mathbf{u}_h^k|^2 \, \mathrm{d}x - \frac{1}{2} \sum_{\sigma \in \mathcal{E}} \int_\sigma (\varrho_h^k)^{\mathrm{up}} |\overline{\mathbf{u}_h^k} \cdot \mathbf{n}| \left[\left[\mathbf{u}_h^k \right]\right]^2 \mathrm{d}\sigma,$$

Properties of numerical solutions, II

Consistency

$$-\int_{\Omega}\varrho_h^0\varphi(0,\cdot)\ \mathrm{d}x=\int_0^T\int_{\mathbb{T}^d}\Big[\varrho_h\partial_t\varphi+\varrho_h\mathbf{u}_h\cdot\nabla_x\varphi\Big]\mathrm{d}x\mathrm{d}t+\int_0^T\int_{\mathbb{T}^d}e_1(t,h,\varphi)\mathrm{d}x\mathrm{d}t$$

for any $\varphi\in C_c^3([0,T)\times\mathbb{T}^d)$

$$\begin{aligned}&-\int_{\mathbb{T}^d}\varrho_h^0\mathbf{u}_h^0\cdot\boldsymbol{\varphi}(0,\cdot)\mathrm{d}x\\&=\int_0^T\int_{\Omega}\Big[\varrho_h\mathbf{u}_h\cdot\partial_t\boldsymbol{\varphi}+\varrho_h\mathbf{u}_h\otimes\mathbf{u}_h:\nabla_x\boldsymbol{\varphi}+p(\varrho_h)\mathrm{div}_x\boldsymbol{\varphi}\Big]\ \mathrm{d}x\mathrm{d}t\\&-h^{\beta}\int_0^T\sum_{\sigma\in\mathcal{E}}\int_{\sigma}\Big[\Big[\mathbf{u}_h^k\Big]\Big]\cdot[[\Pi_{\mathcal{T}}\boldsymbol{\varphi}]]\ \mathrm{d}\sigma\mathrm{d}t+\int_0^T\int_{\Omega}e_2(t,h,\boldsymbol{\varphi})\ \mathrm{d}x\mathrm{d}t\end{aligned}$$

for any $\boldsymbol{\varphi}\in C_c^3([0,T)\times\mathbb{T}^d;R^d)$

Conclusion

Basic properties of the numerical scheme

- Unconditional convergence to a dissipative solution (up to a subsequence)
- Strong (pointwise) convergence to the strong solution as long as the latter exists
- Pointwise convergence of Cesaro averages of Young measures