Oscillatory Solutions to Equations in Fluid Dynamics

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Wild solutions?

Charles Hermite [1822-1901]

In a letter to Stieltjes

I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives

- **Past:** What is not allowed is forbidden
- **Present:** What is not forbidden is allowed

Oscillations

Oscillatory sequence

$$g(x+a) = g(x) \text{ for all } x \in R, \int_0^a g(x)\mathrm{d}x = 0,$$

$$g_n(x) = g(nx), \;\; n = 1, 2, \ldots$$

Weak convergence (convergence in integral averages)

$$\int_R g_n(x)\varphi(x)\ \mathrm{d}x, \text{ where } \varphi \in C_c^\infty(R).$$

$$G(x) = \int_0^x g(z)\ \mathrm{d}z$$

$$\int_R g_n(x)\varphi(x)\ \mathrm{d}x = \int_R g(nx)\varphi(x)\ \mathrm{d}x = -\frac{1}{n}\int_R G(nx)\partial_x\varphi(x)\ \mathrm{d}x \to 0$$

Beware

$g_n \rightharpoonup g$ *does not* imply $H(g_n) \rightharpoonup H(g)$ if H is not linear.

Concentrations

Concentrating sequence

$$g_n(x) = ng(nx)$$

$$g \in C_c^\infty(-1,1),\ g(-x) = g(x),\ g \geq 0,\ \int_R g(x)\ \mathrm{d}x = 1.$$

- $$g_n(x) \to 0 \text{ as } n \to \infty \text{ for any } x \neq 0,\ \text{in particular } g_n \to 0 \text{ a.a. in } R;$$

- $$\|g_n\|_{L^1(R)} = \int_R g_n(x)\ \mathrm{d}x = \int_R g(x)\ \mathrm{d}x = 1 \text{ for any } n = 1, 2, \dots.$$

Convergence in the space of measures

$$\int_R g_n(x)\varphi(x)\ \mathrm{d}x = \int_{-1/n}^{1/n} g_n(x)\varphi(x)\ \mathrm{d}x$$

$$\in \left[\min_{x\in[-1/n,1/n]} \varphi(x),\ \max_{x\in[-1/n,1/n]} \varphi(x) \right] \to \varphi(0)\ \Rightarrow\ g_n \to \delta_0$$

Equations preventing oscillations

Elliptic problems

$$-\Delta_x u = f \text{ in a bounded domain } \Omega, \ u|_{\partial\Omega} = 0$$

Compactness argument

- A priori bounds:

$$\int_\Omega |\nabla_x u|^2 \ \mathrm{d}x = \int_\Omega fu \ \mathrm{d}x \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}.$$

- Poincaré inequality

$$\|u\|_{L^2(\Omega)} \lesssim \|\nabla_x u\|_{L^2(\Omega)},$$

- Rellich–Kondraschev theorem

$$W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$$

Scalar conservation laws

Burgers equation

$$\partial_t u + \partial_x f(u) = 0, \;\; u(t,0) = u(t,1), \;\; t \in (0, T)$$

Entropies

$$\partial_t S(u) + \partial_x F(u) = 0, \;\; \text{where } F'(z) = f'(z)S'(z)$$

Maximum principle

$$S(z) = \begin{cases} 0 \text{ for } z \leq L \\ > 0 \text{ for } z > L \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 S(u(t,x)) \, \mathrm{d}x = 0.$$

Entropy inequality

$$\partial_t S(u) + \partial_x F(u) \leq 0, \;\; \text{where } S \text{ is convex}, \;\; F'(z) = f'(z)S'(z)$$

Example of blow up

Burger's equation

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) = 0, \;\; u(0,x) = u_0(x)$$

Characteristics

$$u(t, x + tu_0(x)) = u_0(x)$$

Schock development in finite time

$$u_0(x_1) > u_0(x_2), \;\; x_1 < x_2$$

$$u(t, x_1 + tu_0(x_1)) = u_0(x_1) \neq u_0(x_2) = u(t, x_2 + tu_0(x_2))$$

but

$$x_1 + tu_0(x_1) = x_2 + tu_0(x_2) \;\; \Leftrightarrow t = \frac{x_2 - x_1}{u_0(x_1) - u_0(x_2)} > 0$$

Compactness for scalar conservation laws

Entropy formulation

$$\partial_t S(u) + \partial_x F(u) \leq 0, \text{ where } S \text{ is convex}, \; F'(z) = f'(z)S'(z)$$

Solution sequence

$$|u_n(t,x)| \leq c \text{ for all } (t,x)$$

Weak convergence

$$u_n \to u \text{ weakly-(*) in } L^\infty,$$

$$S(u_n) \to \overline{S(u)} \text{ weakly-(*) in } L^\infty,$$

$$F(u_n) \to \overline{F(u)} \text{ weakly-(*) in } L^\infty$$

for any convex entropy S with the corresponding flux F

Compensated compactness

Div-Curl Lemma

Let $\{\mathbf{U}_n\}_{n=1}^{\infty}$, $\{\mathbf{V}_n\}_{n=1}^{\infty}$ be two sequences of vector valued defined on a set $Q \subset R^N$ such that

$$\mathbf{U}_n \to \mathbf{U} \text{ weakly in } L^p(Q; R^N),$$
$$\mathbf{V}_n \to \mathbf{V} \text{ weakly in } L^q(Q; R^N),$$

where

$$\frac{1}{p} + \frac{1}{q} < 1.$$

In addition, let

$$\{\mathrm{div}\mathbf{U}_n\}_{n=1}^{\infty} \text{ be precompact in } W^{-1,s}(Q),$$
$$\{\mathbf{curl}\ \mathbf{V}_n\}_{n=1}^{\infty} \text{ be precompact in } W^{-1,s}(Q; R^{N\times N})$$

for some $s > 1$.
Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \to \mathbf{U} \cdot \mathbf{V} \text{ weakly in } L^r(Q),\ \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

Application to scalar conservation laws, I

Ansatz for Div-Curl Lemma

$$N = 2,\ \mathrm{div}_{t,x}[S_1, F_1] = \partial_t S_1 + \partial_x F_1,\ \mathrm{curl}_{t,x}[F_2, -S_2] = \partial_t S_2 + \partial_x F_2$$

Conclusion – Tartar's identity

$$S_1(u_n)F_2(u_n) - F_1(u_n)S_2(u_n) \to \overline{S_1(u)}\ \overline{F_2(u)} - \overline{F_1(u)}\ \overline{S_2(u)}$$

Application to scalar conservation laws, II

Ansatz for Div-Curl Lemma

$$S_1(u) = u, \;\; F_1(u) = f(u)$$

$$S_2(u) = |u - U|, \;\; F_2(u) = \mathrm{sgn}(u - U)(f(u) - f(U)), \;\; U - \text{constant}$$

$$\begin{aligned}
&\lim_{n\to\infty}\Big[u_n \mathrm{sgn}(u_n - U)(f(u_n) - f(U)) - |u_n - U| f(u_n)\Big] \\
&= \lim_{n\to\infty} u_n \lim_{n\to\infty}\Big[\mathrm{sgn}(u_n - U)(f(u_n) - f(U))\Big] \\
&- \lim_{n\to\infty} |u_n - U| \lim_{n\to\infty} f(u_n)
\end{aligned}$$

Conclusion

$$\lim_{n\to\infty}\Big[|u_n - U|\Big(\overline{f(u)} - f(U)\Big)\Big] = (u - U)\overline{\mathrm{sgn}(u - U)(f(u) - f(U))}$$

Application to scalar conservation laws, III

Lebesgue points

$$U = u(\tau, y),\ (\tau, y)\ \text{– a Lebesgue point of u}$$

$$\lim_{r\to 0} \frac{1}{|B_r(\tau, y)|} \int_{B_r(\tau,y)} |u - u(\tau, y)| \,\mathrm{d}x\mathrm{d}t = 0.$$

$$\lim_{r\to 0} \frac{1}{|B_r(\tau, y)|} \int_{B_r(\tau,y)} \left(\lim_{n\to\infty} |u_n - u| \right) \left(\overline{f(u)} - f(u) \right) \mathrm{d}x\mathrm{d}t$$

$$= \lim_{r\to 0} \frac{1}{|B_r(\tau, y)|} \int_{B_r(\tau,y)} \left[\lim_{n\to\infty} \left[|u_n - u(\tau, y)| \left(\overline{f(u)} - f(u(\tau, y)) \right) \right] \right] \mathrm{d}x\mathrm{d}t$$

$$= \lim_{r\to 0} \frac{1}{|B_r(\tau, y)|} \int_{B_r(\tau,y)} (u - u(\tau, y)) \overline{\mathrm{sgn}(u - u(\tau, y))(f(u) - f(u(\tau, y)))} \,\mathrm{d}x\mathrm{d}t = 0$$

Conclusion

For a.a. (τ, y):

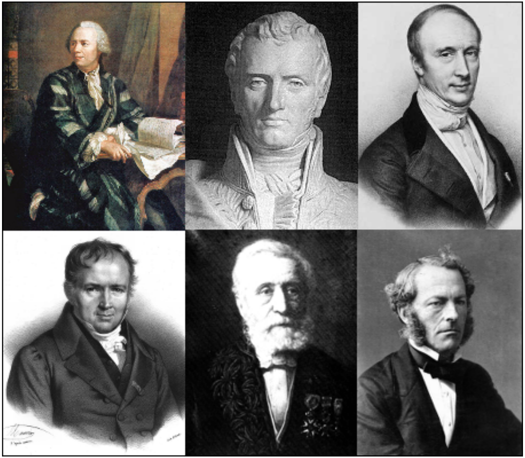
- either

$$(\text{weak}) - \lim_{n\to\infty} |u_n - u| = 0,$$

- or

$$\overline{f(u)} = f(u).$$

Perfect fluids



Who is who in fluid mechanics...

Iconic model of compressible inviscid fluid

Equation of continuity

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

Impermeable boundary

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Initial conditions

$$\varrho(0,\cdot) = \varrho_0, \ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

First and Second law – energy

Energy

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ \ \mathbf{m} = \varrho\mathbf{u}$$

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

$$p' \geq 0 \Rightarrow [\varrho, \mathbf{m}] \mapsto \begin{cases} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0 \\ P(\varrho) \text{ if } |\mathbf{m}| = 0 \\ \infty \text{ if } \varrho = 0, |\mathbf{m}| \neq 0 \end{cases} \quad \text{is convex l.s.c}$$

Energy balance (conservation)

$$\partial_t E + \mathrm{div}_x(E\mathbf{u}) + \mathrm{div}_x(p\mathbf{u}) = 0$$

Energy dissipation

$$\partial_t E + \mathrm{div}_x(E\mathbf{u}) + \mathrm{div}_x(p\mathbf{u}) \boxed{\leq} 0$$

$$\partial_t E \leq 0,\ \ E(0+) = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0)\right]\ \mathrm{d}x$$

Possible singularities

Toy model

$$\partial_t U + U\partial_x U = 0,\ \ t > 0,\ \ x \in R$$

$$U(0) = U_0$$

Explicit solution

$$U(t, x + tU_0(x)) = U_0(x)$$

Shock at finite time

$$U_0(x_1) > U_0(x_2),\ \ x_1 < x_2\ \ \Rightarrow U(t, x_1 + tU_0(x_1)) > U(t, x_2 + tU_0(x_2))$$

$$t_{shock} = \frac{x_2 - x_1}{U_0(x_1) - U_0(x_2)} \Rightarrow x_1 + tU_0(x_1) = x_2 + tU_0(x_2)$$

Weak solutions

Field equations

$$\int_0^\infty \int_\Omega [\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi] \ \mathrm{d}x\mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0, \cdot) \ \mathrm{d}x, \ \ \varphi \in C^1_c([0, \infty) \times \overline{\Omega})$$

$$\int_0^\infty \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x, \ \ \boldsymbol{\varphi} \in C^1_c([0, T) \times \overline{\Omega}; R^N), \ \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Dissipative weak solutions

$$\int_0^\infty \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x \ \partial_t \psi \ \mathrm{d}t \geq \psi(0) \int_\Omega \frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \ \mathrm{d}x$$

$$\psi \in C^1_c[0, \infty), \ \ \psi \geq 0$$

Admissible (dissipative) solutions

Energy inequality

$$\int_0^\infty \int_\Omega \left[E(\varrho, \mathbf{m}) \partial_t \varphi + E(\varrho, \mathbf{m}) \frac{\mathbf{m}}{\varrho} \cdot \nabla_x \varphi + p(\varrho) \frac{\mathbf{m}}{\varrho} \cdot \nabla_x \varphi \right] \ \mathrm{d}x \mathrm{d}t$$

$$\geq \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \varphi(0, \cdot) \ \mathrm{d}x$$

$$\varphi \in C^1_c([0, \infty) \times \overline{\Omega}), \ \varphi \geq 0$$

Well posedness

Classical solutions [Matsumura–Nishida], [Tani]

$$\varrho_0 \in W^{3,2}(\Omega),\ \varrho_0 > 0,\ \mathbf{m}_0 \in W^{3,2}(\Omega; R^N) + \text{compatibility conditions}$$

$$\Rightarrow$$

classical solution

$$\varrho \in C([0, T_{\max}); W^{3,2}(\Omega)),\ \mathbf{m} \in C([0, T_{\max}); W^{3,2}(\Omega; R^N)),\ N = 2, 3$$

$$T_{\max} < \infty \text{ for a "generic" class of initial data}$$

Weak–Strong uniqueness [Dafermos]

An *admissible* weak solution coincides with the strong solution emanating from the same initial data on the time interval $[0, T_{\max})$

Euler as symmetric hyperbolic system

Isentropic pressure

$$p(\varrho) = a\varrho^{\gamma}, \; r = \sqrt{\frac{2a\gamma}{\gamma - 1}}\varrho^{\frac{\gamma-1}{2}}$$

Symmetric hyperbolic system

$$\partial_t r + \mathbf{u} \cdot \nabla_x r + \frac{\gamma - 1}{2} r \mathrm{div}_x \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u} + \frac{\gamma - 1}{2} r \nabla_x r = 0.$$

A priori bounds

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} |\partial_x^{\alpha} \varrho|^2 + |\partial_x^{\alpha} \mathbf{u}|^2 \ \mathrm{d}x \leq c + \left(\int_{\Omega} |\partial_x^{\alpha} \varrho|^2 + |\partial_x^{\alpha} \mathbf{u}|^2 \ \mathrm{d}x \right)^M, \; M > 0$$

$$\alpha = [N/2] + 1$$

Well/ill posedness

Global existence well/ill posedness [Chiodaroli, E.F.]

$$\varrho_0 \in C^3(\overline{\Omega}),\ \varrho_0 > 0,\ \mathbf{m}_0 \in C^3(\overline{\Omega}; R^N),\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$\Rightarrow$$

infinitely many weak solutions

$$\varrho \in L^\infty_{\rm loc}([0,\infty) \times \Omega),\ \mathbf{m} \in L^\infty_{\rm loc}([0,\infty) \times \Omega; R^N)$$

$$\varrho > 0,\ {\rm div}_x \mathbf{m} \in L^\infty_{\rm loc}([0,\infty) \times \Omega),\ \mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Well/ill posedness of dissipative solutions [Chiodaroli, E.F.]

$$\varrho_0 \in C^3(\overline{\Omega}),\ \varrho_0 > 0,\ \nabla_x \Phi_0 \in C^3(\overline{\Omega}),\ \nabla_x \Phi_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$\Rightarrow$$

there exist (infinitely many) $\mathbf{v}_0 \in L^\infty(\Omega; R^N),\ {\rm div}_x \mathbf{v}_0 = 0$

and *infinitely many* dissipative weak solutions

$$\varrho \in L^\infty_{\rm loc}([0,\infty) \times \Omega),\ \mathbf{m} \in L^\infty_{\rm loc}([0,\infty) \times \Omega; R^N)$$

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{m}(0,\cdot) = \mathbf{v}_0 + \nabla_x \Phi_0$$

Strong vs. weak continuity

Strong discontinuities, A.Abbatiello, E.F.

Let $\{\tau_n\}_{n=1}^{\infty} \subset (0, T)$ be an arbitrary (dense) countable family of times. Let the initial data belong to the class

$$\varrho_0 \in C(\Omega),\ \varrho_0 > 0,\ \mathbf{u}_0 \in C(\Omega; R^N),\ \mathrm{div}_x \mathbf{u}_0 \in C(\Omega), \Omega = \mathcal{T}^N,\ N = 2, 3.$$

Then the compressible Euler system admits *infinitely many* weak solutions emanating from the initial state $[\varrho_0, \mathbf{u}_0]$ such that the mapping

$$t \mapsto [\varrho(t, \cdot), \mathbf{m} = \varrho \mathbf{u}(t, \cdot)]$$

is not strongly L^1-continuous at any of the times τ_n.

Admissible weak solutions

Global existence well/ill posedness [Chiodaroli, E.F., Luo, Xie and Xin]

$$\varrho_0 \text{ piecewise Lipschitz}, \ \varrho_0 > 0$$

$$\Rightarrow$$

$$\text{there exist (infinitely many) } \mathbf{m}_0 \in L^\infty(\Omega; R^N)$$

and *infinitely many* admissible weak solutions

$$\varrho \in L^\infty_{\rm loc}([0,\infty) \times \Omega), \ \mathbf{m} \in L^\infty_{\rm loc}([0,\infty) \times \Omega; R^N)$$

$$\varrho(0,\cdot) = \varrho_0, \ \mathbf{m}(0,\cdot) = \mathbf{m}_0$$

Energy conserving solutions [Luo, Xie and Xin]

If ϱ_0 is piecewise constant, one can find $\mathbf{m}_0$ as above such that the solution satisfy the energy equation (energy conserving solutions).

Lipschitz initial data

Ill posedness for regular data [Chiodaroli, DeLellis, Kreml]

Let $T > 0$ be given.
Then there exist (infinitely many) *Lipschitz* initial data ϱ_0, $\mathbf{m}_0$ such that the barotropic Euler system admits infinitely many admissible weak solutions on the time interval $[0, T]$.

Ill-posedness in fluid dynamics

Johann von Neumann
[1903-1957]

Motto

In mathematics you don't understand things. You just get used to them.

Rewriting Euler system

Helmholtz decomposition

$$\mathbf{m} = \mathbf{H}[\mathbf{m}] + \mathbf{H}^{\perp}[\mathbf{m}] = \mathbf{v} + \nabla_x \Phi$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{H}^{\perp}[\mathbf{m}] = \nabla_x \Phi, \ \ \Delta_x \Phi = \mathrm{div}_x \mathbf{m}, \ \ (\nabla_x \Phi - \mathbf{m}) \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Euler system - volume preserving part

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathbf{H}\left[\mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} - \frac{1}{N}\frac{|\mathbf{m}|^2}{\varrho}\mathbb{I}\right)\right] = 0$$

Euler system - acoustic part

$$\partial_t \varrho + \Delta_x \Phi = 0$$

$$\partial_t \nabla_x \Phi + \nabla_x p(\varrho) + \nabla_x \left(\frac{1}{N}\frac{|\mathbf{m}|^2}{\varrho}\right) + \mathbf{H}^{\perp}\left[\mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} - \frac{1}{N}\frac{|\mathbf{m}|^2}{\varrho}\mathbb{I}\right)\right] = 0$$

Convex integration ansatz

This is a BIG crime...

Volume preserving part

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \left[\mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} - \frac{1}{N} \frac{|\mathbf{v} + \nabla_x \Phi|^2}{\varrho} \mathbb{I} \right) \right] = 0$$

Acoustic part

$$\partial_t \varrho + \Delta_x \Phi = 0$$

$$\partial_t \nabla_x \Phi + \nabla_x p(\varrho) + \nabla_x \left(\frac{1}{N} \frac{|\mathbf{v} + \nabla_x \Phi|^2}{\varrho} \right) = 0$$

Acoustic ansatz - continuing committing crimes

Acoustic part - fixing the data

$$\partial_t \varrho + \Delta_x \Phi = 0$$

$$\varrho(0,\cdot) = \varrho_0,\ \Phi(0,\cdot) = \Phi_0,\ \nabla_x \Phi_0 = \mathbf{H}^{\perp}[\mathbf{m}_0]$$

$$\varrho = \varrho(t,\cdot),\ \Phi(t,\cdot) = -\Delta_N^{-1}[\partial_t \varrho]$$

Fixing kinetic energy

$$\partial_t \nabla_x \Phi + \nabla_x p(\varrho) + \nabla_x \left(\frac{1}{N} \frac{|\mathbf{v} + \nabla_x \Phi|^2}{\varrho} \right) = 0$$

$$\frac{1}{N} \frac{|\mathbf{v} + \nabla_x \Phi|^2}{\varrho} = -\partial_t \Phi - p(\varrho) + \boxed{\Lambda(t)}$$

Solving the volume preserving part

Abstract "Euler system"

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \left[\mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} - \frac{1}{N} \frac{|\mathbf{v} + \nabla_x \Phi|^2}{\varrho} \mathbb{I} \right) \right] = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Given kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v} + \nabla_x \Phi|^2}{\varrho} = E$$

Given "data"

$$\mathbf{v}_0, \ \mathrm{div}_x \mathbf{v}_0 = 0, \ \nabla_x \Phi, \ \rho, \ E$$

Solving a special case

Incompressible Euler system with given pressure

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \mathbb{I} \right) = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Given kinetic energy

$$\frac{1}{2} |\mathbf{v}|^2 = E$$

Weak solutions

$$\mathbf{v} \in C_{\mathrm{weak}}([0, T]; L^2(\Omega; R^N)), \ \mathbf{v} \in L^\infty((0, T) \times \Omega; R^N)$$

Weak formulation of the basic problem

Regularity class

$$\mathbf{v} \in C_{\mathrm{weak}}([0,T]; L^2(\Omega; R^N)), \ \mathbf{v} \in L^\infty((0,T) \times \Omega; R^N)$$

Incompressibility condition

$$\int_0^T \int_\Omega \mathbf{v} \cdot \nabla_x \varphi \ \mathrm{d}x = 0, \ \text{for any } \varphi \in \boxed{C^1([0,T] \times \overline{\Omega})}$$

Field equations

$$\int_0^T \int_\Omega \left[\mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + \mathbf{v} \otimes \mathbf{v} : \nabla_x \boldsymbol{\varphi} - \frac{1}{N} |\mathbf{v}|^2 \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t = - \int_\Omega \mathbf{v}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x$$

for any $\boldsymbol{\varphi} \in \boxed{C^1([0,T) \times \overline{\Omega}; R^N)}$

Energy

$$\frac{1}{2} |\mathbf{v}|^2 = E \ \boxed{a.a.} \ \text{in } (0,T) \times \Omega$$

Pasting together spatial domains

Domain decomposition

$$\overline{\Omega} = \cup_{i \in L} \overline{\Omega}_i, \ \Omega_i \cap \Omega_j = \emptyset, \ i \neq j$$

Incompressible Euler system with piece–wise constant density

$$\mathrm{div}_x \mathbf{v} = 0 \text{ in } \Omega$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{\mathbf{v} \otimes \mathbf{v}}{\varrho_i} - \frac{1}{N} \frac{|\mathbf{v}|^2}{\varrho_i} \mathbb{I} \right) = 0 \text{ in } \Omega_i, \ \varrho_i > 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v}|^2}{\varrho_i} = E_i \text{ a.a. in } \Omega_i$$

Results for piecewise constant density

Initial data

$$\overline{\Omega} = \cup_{i \in L} \overline{\Omega}_i,\ \Omega_i \cap \Omega_j = \emptyset,\ i \neq j,\ \varrho = \varrho_i \in \Omega_i$$

Incompressible Euler system with piece–wise constant density

$$\partial_t \varrho + \mathrm{div}_x \mathbf{v} = 0 \text{ in } \Omega$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{\mathbf{v} \otimes \mathbf{v}}{\varrho} + \left(p(\varrho) - \frac{N}{2} \Lambda \right) \mathbb{I} \right) = 0 \text{ in } \Omega,$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v}|^2}{\varrho_i} = E_i = \frac{N}{2} \Lambda - \frac{N}{2} p(\varrho_i) \text{ a.a. in } \Omega_i$$

Solutions of the Euler system with given pressure

However beautiful the strategy, you should occasionally look at the results...
Sir Winston Churchill
[1874-1965]

Reformulation

Original problem

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \mathbb{I} \right) = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0,\ \frac{1}{2} |\mathbf{v}|^2 = E$$

Reformulation

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0,\ \mathbb{U}(t, x) \in R^{N \times N}_{0,\mathrm{sym}}$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

$$\mathbb{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \mathbb{I},\ \frac{1}{2} |\mathbf{v}|^2 = E$$

Relaxation

$$\frac{1}{2} |\mathbf{v}|^2 \leq \frac{N}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right] < E$$

Subsolutions

Regularity class

$$\mathbf{v} \in C_c^\infty([0,T] \times \Omega; R^N), \ \mathbb{U} \in C_c^\infty([0,T] \times \Omega; R^{N\times N}_{0,\mathrm{sym}})$$

System of equation

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0$$

Convex constraint

$$\frac{N}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right] < E$$

$$[\mathbf{v}, \mathbb{U}] \mapsto \frac{N}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right] \ \text{convex}$$

Topology on the space of subsolutions

Basic space

$$X_0 = \left\{ \mathbf{v} \;\middle|\; \mathbf{v},\ \mathbb{U} \text{ is a subsolution} \right\}$$

Boundedness

$$\frac{1}{2}|\mathbf{v}|^2 \leq \frac{N}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v} - \mathbb{U}\right] < E \Rightarrow X_0 \subset \text{ bounded set in } L^\infty$$

Metric on X_0

$$d[\mathbf{v},\mathbf{w}] = \sup_{t\in[0,T]} \sum_n \frac{1}{2^n} \frac{\left|\int_\Omega (\mathbf{v}(t,\cdot) - \mathbf{w}(t,\cdot)) \cdot \varphi_n \, \mathrm{d}x\right|}{1 + \left|\int_\Omega (\mathbf{v}(t,\cdot) - \mathbf{w}(t,\cdot)) \cdot \varphi_n \, \mathrm{d}x\right|}$$

Closure

$$X = \mathrm{closure}_d[X_0] \text{ – compact metric space}$$

Closure of the subsolution space

Regularity class

$$\mathbf{v} \in X \Rightarrow \mathbf{v} \in C_{\rm weak}([0,T]; L^2(\Omega; R^N)) \cap L^\infty((0,T)\times\Omega), \ \ \|\mathbf{v}\|^2_{L^\infty} \lesssim E$$

System of equation

$$\int_0^T \int_\Omega \mathbf{v} \cdot \nabla_x \varphi \ {\rm d}x{\rm d}t = 0 \text{ for all } \varphi \in C^\infty_c([0,T] \times \overline{\Omega})$$

$$\int_0^T \int_\Omega \left[\mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + \mathbb{U} : \nabla_x \boldsymbol{\varphi} \right] \ {\rm d}x \ {\rm d}t = - \int_\Omega \mathbf{v}_0 \cdot \boldsymbol{\varphi}(0,\cdot) \ {\rm d}x$$

for some $\mathbb{U} \in L^\infty((0,T)\times\Omega; R^{N\times N}_{0,{\rm sym}})$, and for all $\boldsymbol{\varphi} \in C^\infty_c([0,T)\times\overline{\Omega}; R^N)$

Convex constraint

$$\frac{1}{2}|\mathbf{v}|^2 \leq \frac{N}{2} \lambda_{\rm max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right] \leq E \text{ a.a. in } (0,T)\times\Omega$$

Distance to "extremal" points

"Distance"

$$I[\mathbf{v}] = \int_0^T \int_\Omega \left(E - \frac{1}{2} |\mathbf{v}|^2 \right) \, \mathrm{d}x \, \mathrm{d}t, \ \mathbf{v} \in X$$

Properties

- I is a concave functional on X; whence upper semi–continuous on X
- $$I \geq 0 \text{ on } X$$
  $$I[\mathbf{v}] = 0 \Rightarrow \mathbb{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \mathbb{I}, \ \frac{1}{2} |\mathbf{v}|^2 = E \text{ a.a. in } (0, T) \times \Omega$$
  $$\Leftrightarrow$$
  $\mathbf{v}$ solves the Euler system with given energy and pressure

Baire category argument

Point of continuity of I on X form a residual set, in particular, they are dense in X

Existence of infinitely many solutions

Existence of at least one subsolution

$$E > 0 \text{ large enough} \Rightarrow X_0 \text{ is non–empty}$$

Points of continuity of I

Claim:

$$\mathbf{v} \in X - \text{point of continuity of } I \Rightarrow I[\mathbf{v}] = 0$$

Oscillatory Lemma

Hypotheses

$$\mathbf{v} \in X_0 \text{ with the associated flux } \mathbb{U}$$

Conclusion

There exists a sequence $\mathbf{w}_n$, $\mathbb{V}_n$, $n = 1, 2, \dots$ satisfying:

- $$\mathbf{w}_n \in C_c^\infty((0,T) \times \Omega; R^N),\ \mathbb{V}_n \in C_c^\infty((0,T) \times \Omega; R^{N \times N}_{0,\mathrm{sym}})$$
- $$(\mathbf{v} + \mathbf{w}_n) \in X_0 \text{ with the flux } \mathbb{U} + \mathbb{V}_n$$
- $$\mathbf{w}_n \to 0 \text{ in } C_{\mathrm{weak}}([0,T]; L^2(\Omega; R^N))$$
- $$\liminf_{n \to \infty} \int_0^T \int_\Omega |\mathbf{w_n}|^2 \ \mathrm{d}x\mathrm{d}t \geq c(E,N) \int_0^T \int_\Omega \left(E - \frac{1}{2} |\mathbf{v}|^2 \right)^2 \ \mathrm{d}x\mathrm{d}t$$

Infinitely many solutions via Oscillatory Lemma, I

Claim: $\mathbf{v} \in X$ – point of continuity of $I \Rightarrow I[\mathbf{v}] = 0$

- $$I[\mathbf{v}] = \delta > 0,\ \mathbf{v} \in X \Rightarrow \mathbf{v}_m \in X_0,\ \mathbf{v}_m \to \mathbf{v} \in X,\ I[\mathbf{v}_m] \to \delta \text{ as } m \to \infty$$

- $$\textbf{Oscillatory lemma} \Rightarrow \mathbf{w}_{m,n} \in X_0, \mathbf{v}_m + \mathbf{w}_{m,n} \to \mathbf{v}_m \ \text{ in as } n \to \infty$$

- $$\liminf_{n\to\infty} \int_0^T \int_\Omega |\mathbf{v}_m + \mathbf{w}_{m,n}|^2 \ \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_\Omega |\mathbf{v}_m|^2 \ \mathrm{d}x + \liminf_{n\to\infty} \int_0^T \int_\Omega |\mathbf{w}_{m,n}|^2 \ \mathrm{d}x\mathrm{d}t$$
$$\geq \int_0^T \int_\Omega |\mathbf{v}_m|^2 \ \mathrm{d}x \ \mathrm{d}t + c(N,E) \int_0^T \int_\Omega \left(E - \frac{1}{2}|\mathbf{v}_m|^2 \right)^2 \ \mathrm{d}x\mathrm{d}t$$
$$\geq \int_0^T \int_\Omega |\mathbf{v}_m|^2 \ \mathrm{d}x + c(N,E,T,\Omega) I^2([\mathbf{v}_m])$$

Infinitely many solutions via Oscillatory Lemma, II

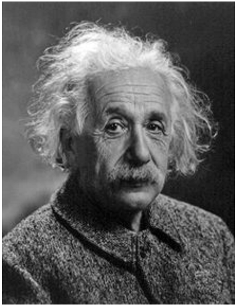
Convergence of functional

$$\lim_{m\to\infty}\lim_{n\to\infty} I[\mathbf{v}_m+\mathbf{w}_{m,n}] = \int_0^T\int_\Omega E - \frac{1}{2}|\mathbf{v}_n+\mathbf{w}_{m,n}|^2 \,\mathrm{d}x\mathrm{d}t$$

$$\leq \lim_{m\to\infty}\left(\int_0^T\int_\Omega E\,\mathrm{d}x - \liminf_{n\to\infty}\int_0^T\int_\Omega \frac{1}{2}|\mathbf{v}_n+\mathbf{w}_{m,n}|^2 \,\mathrm{d}x\mathrm{d}t\right)$$

$$\leq \lim_{m\to\infty}\left(\int_0^T\int_\Omega E - \frac{1}{2}|\mathbf{v}_m|^2\,\mathrm{d}x\mathrm{d}t - cI^2[\mathbf{v}_m]\right)$$

Proof of Oscillatory Lemma



Albert Einstein [1879-1955]

Everything should be made as simple as possible, but not simpler...

- **Past:** What is not allowed is forbidden
- **Present:** What is not forbidden is allowed

Oscillatory Lemma revisited

Hypotheses

- $$\mathbf{v} \in R^N,\ \mathbb{U} \in R^{N\times N}_{0,\mathrm{sym}},\ \frac{N}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v} - \mathbb{U}\right] < E,\ Q = (-1,1)\times(-1,1)^N$$

Conclusion

- $$\mathbf{w}_n \in C^\infty_c(Q; R^N),\ \mathbb{V}_n \in C^\infty_c(Q; R^{N\times N}_{0,\mathrm{sym}})$$
$$\mathrm{div}_x \mathbf{w}_n = 0,\ \partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{V}_n = 0$$
$$\frac{N}{2}\lambda_{\max}\left[(\mathbf{v}+\mathbf{w}_n)\otimes(\mathbf{v}+\mathbf{w}_n) - (\mathbb{U}+\mathbb{V}_n)\right] < E$$

- $$\mathbf{w}_n \to 0 \text{ in } L^2(Q; R^N))$$
$$\liminf_{n\to\infty} \int_Q |\mathbf{w}_n|^2\ \mathrm{d}x\ \mathrm{d}t \geq c(N,E)\int_Q \left(E - \frac{1}{2}|\mathbf{v}|^2\right)^2 \mathrm{d}x\ \mathrm{d}t$$

Oscillatory Lemma - Geometry, I

Convex set

$$\mathfrak{C} = \left\{ \mathbf{v} \in R^N,\ \mathbb{U} \in R^{N \times N}_{0,\mathrm{sym}} \ \middle|\ \frac{N}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right] < E \right\} \subset R^n$$

$$n = \frac{N(N+3)}{2} - 1$$

Properties

- $$\mathfrak{C} \subset R^n \text{ is a convex set}$$
- $$\frac{1}{2} |\mathbf{v}|^2 \leq \frac{N}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right]$$
- $$\frac{1}{2} |\mathbb{U}|^2 \leq \frac{N-1}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right]$$
- $$\mathrm{ext} \overline{\mathfrak{C}} \subset \left\{ [\mathbf{v}, \mathbb{U}] \ \middle|\ \frac{1}{2} |\mathbf{v}|^2 = E,\ \mathbb{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \mathbb{I} \right\}$$

Oscillatory Lemma - Geometry, II

Extremal points

$$\frac{1}{2}|\mathbf{v}|^2 \boxed{\leq} \frac{N}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v} - \mathbb{U}\right] \boxed{=} E$$

Extremal points

$$\frac{1}{2}|\mathbf{v}|^2 < \frac{N}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v} - \mathbb{U}\right] \ \Rightarrow \ \lambda_{\max} > \lambda_{\min}$$

$[\mathbf{e}_1, \dots, \mathbf{e}_N]$ the normalized eigenvectors of $\left[\mathbf{v}\otimes\mathbf{v} - \mathbb{U}\right]$, $\lambda_N = \lambda_{\min}$

$$(\mathbf{v} + \varepsilon\mathbf{e}_N) \otimes (\mathbf{v} + \varepsilon\mathbf{e}_N) - \left[\mathbb{U} + \varepsilon\sum_{i=1}^{N-1} v_i\left((\mathbf{e}_i \otimes \mathbf{e}_N) + (\mathbf{e}_N \otimes \mathbf{e}_i)\right)\right]$$

$$= \mathbf{v}\otimes\mathbf{v} - \mathbb{U} + (2\varepsilon v_N + \varepsilon^2)\mathbf{e}_n \otimes \mathbf{e}_N$$

Oscillatory Lemma - Geometry, III

Segment Lemma

For any $[\mathbf{v}, \mathbb{U}] \in \mathfrak{C}$ there exist **a**, **b** enjoying the following properties:

- $$\frac{1}{2}|\mathbf{a}|^2 = \frac{1}{2}|\mathbf{b}|^2 = E,\ |\mathbf{a} \pm \mathbf{b}| > 0$$
- there is $L > 0$ such that
$$[\mathbf{v} + \lambda(\mathbf{a} - \mathbf{b}), \mathbb{U} + \lambda\left(\mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b}\right)] \in \mathfrak{C},$$
$$\mathrm{dist}\left[\ [\mathbf{v} + \lambda(\mathbf{a} - \mathbf{b}), \mathbb{U} + \lambda\left(\mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b}\right)]\, ; \partial\mathfrak{C}\ \right]$$
$$\geq \frac{1}{2}\mathrm{dist}\left[\ [\mathbf{v}, \mathbb{U}]\, ; \partial\mathfrak{C}\ \right]$$
for all $-L \leq \lambda \leq L$
- $$L|\mathbf{a} - \mathbf{b}| \geq C(N)\frac{1}{\sqrt{E}}\left(E - \frac{1}{2}|\mathbf{v}|^2\right).$$

Oscillatory Lemma - Geometry, IV

Proof of Segment Lemma

1.
$$[\mathbf{v},\mathbb{U}]\in\mathfrak{C},\ \ \mathfrak{C}\text{ convex}\Rightarrow[\mathbf{v},\mathbb{U}]=\sum_{\text{fin}}\alpha_i\left[\mathbf{a}_i,\mathbf{a}_i\otimes\mathbf{a}_i-\frac{1}{N}E\mathbb{I}\right]$$

2. Caratheodory Theorem:
$$[\mathbf{v},\mathbb{U}]=\sum_{i=1}^{n}\alpha_i\left[\mathbf{a}_i,\mathbf{a}_i\otimes\mathbf{a}_i-\frac{1}{N}E\mathbb{I}\right],\ \alpha_1\geq\alpha_2\geq\cdots\geq\alpha_n$$
$$[\mathbf{v}+\lambda(\mathbf{a}_j-\mathbf{a}_1),\mathbb{U}+\lambda\left(\mathbf{a}_j\otimes\mathbf{a}_j-\mathbf{a}_1\otimes\mathbf{a}_1\right)],\ \lambda\in[-\alpha_j,\alpha_j]$$

3.
$$\alpha_j|\mathbf{a}_j-\mathbf{a}_1|\geq\alpha_k|\mathbf{a}_k-\mathbf{a}_1|\text{ for all }k>1,\ \mathbf{a}=\mathbf{a}_1,\ \mathbf{b}=\mathbf{a}_j,\ L=\frac{1}{2}\alpha_j$$

4.
$$|\mathbf{a}-\mathbf{v}|=\left|\sum_{k=1}^{n}\alpha_k(\mathbf{a}_1-\mathbf{a}_k)\right|\leq n\alpha_j|\mathbf{a}_j-\mathbf{a}_1|=2nL|\mathbf{a}-\mathbf{b}|$$
$$2E-|\mathbf{v}|^2\leq2\sqrt{2E}(\sqrt{2E}-|\mathbf{v}|)=2\sqrt{2E}(|\mathbf{a}|-|\mathbf{v}|)\leq4\sqrt{2E}nL|\mathbf{a}-\mathbf{b}|$$

Oscillatory Lemma - Fourier analysis, I

Fourier transform

$$\widehat{\mathbf{w}}(\xi_0, \dots \xi_N) = \mathcal{F}_{(t,x)\to(\xi_1,\xi_1,\dots,\xi_N)}\mathbf{w}(t,x)$$

$$\widehat{\mathbb{V}}(\xi_0, \dots \xi_N) = \mathcal{F}_{(t,x)\to(\xi_1,\xi_1,\dots,\xi_N)}\mathbb{V}(t,x)$$

Field equations

$$\mathrm{div}_x \mathbf{w} = 0 \Leftrightarrow \sum_{i=1}^{N} \xi_i \widehat{w}_i = 0$$

$$\partial_t \mathbf{w} + \mathrm{div}_x \mathbb{V} = 0 \Leftrightarrow \xi_0 \widehat{w}_i + \sum_{i=j}^{N} \xi_j \widehat{V}_{i,j} = 0, \ i = 1, \dots, N$$

Vector formulation

$$\mathrm{DIV}_{t,x} \left[\begin{array}{cc} 0 & \mathbf{w} \\ \mathbf{w} & \mathbb{V} \end{array} \right] = 0 \Leftrightarrow \boldsymbol{\xi} \cdot \left[\begin{array}{cc} 0 & \widehat{\mathbf{w}} \\ \widehat{\mathbf{w}} & \widehat{\mathbb{V}} \end{array} \right] = 0$$

Oscillatory Lemma - Fourier analysis, II

Operator ansatz

$$\xi = [\xi_0, \xi_1, \dots, \xi_N] \mapsto \mathbb{A}_{\mathbf{a},\mathbf{b}}(\xi) \in R^{(N+1)\times(N+1)}_{0,\mathrm{sym}},$$

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\xi) = \frac{1}{2}\left((\mathbb{R}\cdot\xi)\otimes(\mathbb{Q}(\xi)\cdot\xi) + (\mathbb{Q}(\xi)\cdot\xi)\otimes(\mathbb{R}\cdot\xi)\right)$$

where

$$\mathbb{Q} = \xi\otimes e_0 - e_0\otimes\xi,\ \ \mathbb{R} = ([0,\mathbf{a}]\otimes[0,\mathbf{b}]) - ([0,\mathbf{b}]\otimes[0,\mathbf{a}])\,,$$

$$e_0 = [1,0,\dots,0],\ \ \mathbf{a},\mathbf{b}\in R^N,\ \ \frac{1}{2}|\mathbf{a}|^2 = \frac{1}{2}|\mathbf{b}|^2 = E > 0,\ \ |\mathbf{a}\pm\mathbf{b}| > 0.$$

Third order differential operator

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial) = \mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial_t, \partial_{x_1}, \dots, \partial x_N)$$

Oscillatory Lemma - Fourier analysis, III

Properties of the differential operator

- $$\begin{bmatrix} 0 & \mathbf{w} \\ \mathbf{w} & \mathbb{V} \end{bmatrix} = \mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial)[\varphi],\ \varphi \in C_c^\infty(R^{N+1})$$
  $$\Rightarrow \mathrm{div}_x \mathbf{w} = 0,\ \partial_t \mathbf{w} + \mathrm{div}_x \mathbb{V} = 0$$
- $$\psi \in C_c^\infty(R)$$
  $$\eta_{\mathbf{a},\mathbf{b}} = -\frac{1}{(|\mathbf{a}||\mathbf{b}| + \mathbf{a}\cdot\mathbf{b})^{2/3}}\Big[[0,\mathbf{a}] + [0,\mathbf{b}] - (|\mathbf{a}||\mathbf{b}| + \mathbf{a}\cdot\mathbf{b})\, e_0\Big]$$
  $$\Rightarrow$$
  $$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial)[\psi([t,\mathbf{x}]\cdot\eta_{\mathbf{a},\mathbf{b}})] = \psi'''([t,\mathbf{x}]\cdot\eta_{\mathbf{a},\mathbf{b}}) \begin{bmatrix} 0 & \mathbf{a}-\mathbf{b} \\ \mathbf{a}-\mathbf{b} & \mathbf{a}\otimes\mathbf{a} - \mathbf{b}\otimes\mathbf{b} \end{bmatrix}$$

Construction of the differential operator $N = 2$, I

Abstract ansatz

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\xi) = \begin{bmatrix} 0 & A^{0,1}(\xi) & A^{0,2}(\xi) \\ A^{1,0}(\xi) & A^{1,1}(\xi) & A^{1,2}(\xi) \\ A^{2,0}(\xi) & A^{2,1}(\xi) & A^{2,2}(\xi) \end{bmatrix}$$

Symmetry properties

$$A^{1,1}(\xi) = -A^{2,2}(\xi),\ A^{0,1}(\xi) = A^{1,0}(\xi),\ A^{0,2}(\xi) = A^{2,0}(\xi),\ A^{1,2}(\xi) = A^{2,1}(\xi)$$

Reduced form

$$B = \frac{A^{0,1}}{A^{1,1}},\ C = \frac{A^{0,2}}{A^{1,1}},\ D = \frac{A^{1,2}}{A^{1,1}}$$

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\xi) = \begin{bmatrix} 0 & B(\xi) & C(\xi) \\ B(\xi) & 1 & D(\xi) \\ C(\xi) & D(\xi) & -1 \end{bmatrix} A^{1,1}(\xi)$$

Construction of the differential operator $N = 2$, II

Differential constraints

$$\xi_1 B + \xi_2 C = 0,\ \xi_0 B + \xi_1 + \xi_2 D = 0,\ \xi_0 C + \xi_1 D - \xi_2 = 0$$

unique solution

$$B(\xi) = \frac{1}{2}\frac{\xi_2^2 - \xi_1^2}{\xi_0\xi_1} - \frac{\xi_2^2}{\xi_0\xi_1},\ C(\xi) = \frac{\xi_2}{\xi_0} - \frac{1}{2}\frac{\xi_2^2 - \xi_1^2}{\xi_0\xi_2},\ D(\xi) = \frac{1}{2}\frac{\xi_2^2 - \xi_1^2}{\xi_1\xi_2}.$$

Final form

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\xi) = A_{\mathbf{a},\mathbf{b}} \begin{bmatrix} 0 & -\frac{\xi_2}{2}(\xi_1^2 + \xi_2^2) & \frac{\xi_1}{2}(\xi_1^2 + \xi_2^2) \\ -\frac{\xi_2}{2}(\xi_1^2 + \xi_2^2) & \xi_0\xi_1\xi_2 & \frac{\xi_0}{2}(\xi_2^2 - \xi_1^2) \\ \frac{\xi_1}{2}(\xi_1^2 + \xi_2^2) & \frac{\xi_0}{2}(\xi_2^2 - \xi_1^2) & -\xi_0\xi_1\xi_2 \end{bmatrix}.$$

Construction of the differential operator $N = 2$, III

Fixing the coefficients

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial)\psi\left([t,x]\cdot\eta\right)$$

$$= A_{\mathbf{a},\mathbf{b}}\psi'''\left([t,x]\cdot\eta\right)\begin{bmatrix} 0 & -\frac{\eta_2}{2}(\eta_1^2+\eta_2^2) & \frac{\eta_1}{2}(\eta_1^2+\eta_2^2) \\ -\frac{\eta_2}{2}(\eta_1^2+\eta_2^2) & \eta_0\eta_1\eta_2 & \frac{\eta_0}{2}(\eta_2^2-\eta_1^2) \\ \frac{\eta_1}{2}(\eta_1^2+\eta_2^2) & \frac{\eta_0}{2}(\eta_2^2-\eta_1^2) & -\eta_0\eta_1\eta_2 \end{bmatrix}$$

$$\frac{A}{2}(\eta_1^2+\eta_2^2)\left[-\eta_2,\eta_1\right] = \mathbf{a}-\mathbf{b},$$

$$A\eta_0\begin{bmatrix} \eta_1\eta_2 & \frac{1}{2}(\eta_2^2-\eta_1^2) \\ \frac{1}{2}(\eta_2^2-\eta_1^2) & -\eta_1\eta_2 \end{bmatrix} = \mathbf{a}\otimes\mathbf{a}-\mathbf{b}\otimes\mathbf{b},\ \frac{1}{2}|\mathbf{a}|^2 = \frac{1}{2}|\mathbf{b}|^2 = E.$$

Ansatz

$$[\eta_1,\eta_2] = \Lambda(\mathbf{a}+\mathbf{b})$$

find Λ, A, η_0

Proof of Oscillatory Lemma

Ansatz

1. for given $[\mathbf{v}, \mathbb{U}] \in \mathfrak{C}$ identify the vectors **a**, **b** via Segment Lemma
2. consider the operator $\mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial)$ and the direction $\eta_{\mathbf{a},\mathbf{b}}$
3. $$\begin{bmatrix} 0 & \mathbf{w}_n \\ \mathbf{v}_n & \mathbb{V}_n \end{bmatrix} = \mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial) \left[\varphi \frac{L}{n^3} \cos\left(n[t,x] \cdot \eta_{\mathbf{a},\mathbf{b}}\right) \right], \ \varphi \in C_c^\infty(Q)$$
$$0 \le \varphi \le 1, \ \varphi(t,x) = 1$$
$$\text{whenever } -\frac{1}{2} \le t \le \frac{1}{2}, \ -\frac{1}{2} \le x_j \le \frac{1}{2}, \ j = 1, \dots, N.$$
4. $$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial) \left[\varphi \frac{L}{n^3} \cos\left(n[t,x] \cdot \eta_{\mathbf{a},\mathbf{b}}\right) \right]$$
$$= \varphi \sin\left(n[t,x] \cdot \eta_{\mathbf{a},\mathbf{b}}\right) L \begin{bmatrix} 0 & (\mathbf{a} - \mathbf{b}) \\ (\mathbf{a} - \mathbf{b}) & \mathbf{a} \otimes \mathbf{a} - \mathbf{b} \otimes \mathbf{b} \end{bmatrix} + \frac{1}{n} R_n, \ |R_n| \le C$$

Oscillatory Lemma - piecewise constant functions

Hypotheses

- $$Q = \cup \overline{Q}_i,\ \ Q_i \cap Q_j = \emptyset \text{ for } i \neq j,\ \ Q_i = (-a_i, a_i) \times (-a_i, a_i)^N$$
  $$\mathbf{v} = \sum \mathbf{v}^i 1_{Q_i},\ \ \mathbb{U} = \sum \mathbb{U}_i 1_{\Omega_i}$$
  $$\mathbf{v}_i \in R^N,\ \ \mathbb{U}_i \in R^{N\times N}_{0,\mathrm{sym}},\ \ \frac{N}{2}\lambda_{\max}\left[\mathbf{v}_i \otimes \mathbf{v}_i - \mathbb{U}_i\right] < E_i$$

Conclusion

- $$\mathbf{w}_n \in C^\infty_c(Q; R^N),\ \ \mathbb{V}_n \in C^\infty_c(Q; R^{N\times N}_{0,\mathrm{sym}})$$
  $$\mathrm{div}_x \mathbf{w}_n = 0,\ \ \partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{V}_n = 0$$
  $$\frac{N}{2}\lambda_{\max}\left[(\mathbf{v} + \mathbf{w}_n) \otimes (\mathbf{v} + \mathbf{w}_n) - (\mathbb{U} + \mathbb{V}_n)\right] < E$$
- $$\mathbf{w}_n \to 0 \text{ in } L^2(Q; R^N))$$
  $$\liminf_{n\to\infty} \int_Q |\mathbf{w}_n|^2\ \mathrm{d}x\ \mathrm{d}t \geq c(N, E) \int_Q \left(E - \frac{1}{2}|\mathbf{v}|^2 \right)^2 \mathrm{d}x\ \mathrm{d}t$$

Oscillatory Lemma - continuous functions

Decomposition

$$Q = \cup \overline{Q}_i,\ \ Q_i \cap Q_j = \emptyset \text{ for } i \neq j,\ \ Q_i = (-a_i, a_i) \times (-a_i, a_i)^N$$

$$\frac{3}{2}\lambda_{\max}\left[\mathbf{v}_i \otimes \mathbf{v}_i - \mathbb{U}_i\right] < E_i - \delta \text{ in } Q^i,\ \ i = 1, \dots, m$$

for arbitrary constant quantities

$$\mathbf{v}_i = \mathbf{v}(t_{i,v}, x_{i,v}),\ \ \mathbb{U}_i = \mathbb{U}(t_{i,u}, r_{i,u}),\ \ E_i = E(t_{i,e}, x_{i,e}),\ \ (t_{i,\cdot}, x_{i,\cdot}) \in Q^i.$$

$$\Big| \frac{3}{2}\lambda_{\max}\left[(\mathbf{v}_i + \mathbf{w}) \otimes (\mathbf{v}_i + \mathbf{w}) - (\mathbb{U}_i + \mathbb{V})\right]$$
$$- \frac{3}{2}\lambda_{\max}\left[(\mathbf{v} + \mathbf{w}) \otimes (\mathbf{v} + \mathbf{w}) - (\mathbb{U} + \mathbb{V})\right] \Big| < \frac{\delta}{2}$$

Conclusion

$$\liminf_{n \to \infty} \int_Q |\mathbf{w}_n|^2 \ \mathrm{d}x \ \mathrm{d}t \geq c(N, E) \int_Q \left(E - \delta - \frac{1}{2}|\mathbf{v}|^2 \right)^2 \mathrm{d}x \ \mathrm{d}t$$

Abstract formulation

Variable coefficients "Euler system"

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{h}[\mathbf{v}]) \otimes (\mathbf{v} + \mathbf{h}[\mathbf{v}])}{r[\mathbf{v}]} - \frac{1}{N} \frac{|\mathbf{v} + \mathbf{h}[\mathbf{v}]|^2}{r[\mathbf{v}]} + \mathbb{M}[\mathbf{v}] \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0,$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v} + \mathbf{h}[\mathbf{v}]|^2}{r[\mathbf{v}]} = E[\mathbf{v}]$$

Data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Abstract operators

Boundedness

b maps bounded sets in $L^\infty((0,T)\times\Omega; R^N)$ on bounded sets in $C_b(Q, R^M)$

Continuity

$$b[\mathbf{v}_n] \to b[\mathbf{v}] \text{ in } C_b(Q; R^M) \text{ (uniformly for } (t,x)\in Q \text{)}$$

whenever

$$\mathbf{v}_n \to \mathbf{v} \text{ in } C_{\mathrm{weak}}([0,T]; L^2(\Omega; R^N))$$

Causality

$$\mathbf{v}(t,\cdot) = \mathbf{w}(t,\cdot) \text{ for } 0 \le t \le \tau \le T \text{ implies } b[\mathbf{v}] = b[\mathbf{w}] \text{ in } [(0,\tau]\times\Omega]$$

Results

Result (A)

The set of subsolutions is non-empty $\Rightarrow$ there exists infinitely many weak solutions of the problem with the same initial data

Initial energy jump

$$\frac{1}{2}\frac{|\mathbf{v_0}+\mathbf{h}[\mathbf{v_0}]|^2}{r[\mathbf{v_0}]} \boxed{<} \liminf_{t\to 0+} \frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}[\mathbf{v}]|^2}{r[\mathbf{v}]}$$

Result (B)

The set of subsolutions is non-empty $\Rightarrow$ there exists a dense set of times $\{t_i\}$ such that the values $\mathbf{v}(t_i)$ give rise to non-empty subsolution set with

$$\frac{1}{2}\frac{|\mathbf{v}(t_i)+\mathbf{h}[\mathbf{v}](t_i)|^2}{r[\mathbf{v}](t_i)} \boxed{=} \lim_{t\to t_i+} \frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}[\mathbf{v}]|^2}{r[\mathbf{v}]}$$

Savage-Hutter model for avalanches

Unknowns

flow height . . . . $h = h(t, x)$
depth-averaged velocity . . . . $\mathbf{u} = \mathbf{u}(t, x)$

$$\partial_t h + \mathrm{div}_x(h\mathbf{u}) = 0$$

$$\partial_t(h\mathbf{u}) + \mathrm{div}_x(h\mathbf{u} \otimes \mathbf{u}) + \nabla_x(ah^2) = h\left(-\gamma\frac{\mathbf{u}}{|\mathbf{u}|} + \mathbf{f}\right)$$

Periodic boundary conditions

$$\Omega = \left([0, 1]|_{\{0,1\}}\right)^2$$

Application to Savage-Hutter model

Theorem

(joint work with P. Gwiazda and A. Świerczewska-Gwiazda)

(i) Let the initial data

$$h_0 \in C^2(\Omega),\ \mathbf{u}_0 \in C^2(\Omega; R^2), h_0 > 0 \text{ in } \Omega$$

be given, and let $\mathbf{f}$ and a be smooth.
Then the Savage-Hutter system admits infinitely many weak solutions in $(0, T) \times \Omega$.

(ii) Let $T > 0$ and

$$h_0 \in C^2(\Omega),\ h_0 > 0$$

be given.
Then there exists

$$\mathbf{u}_0 \in L^\infty(\Omega; R^2)$$

such that the Savage-Hutter system admits infinitely many weak solutions in $(0, T) \times \Omega$ satisfying the energy inequality.

Transformation - Step I

Helmholtz decomposition

$$h\mathbf{u} = \mathbf{v} + \mathbf{V} + \nabla_x \Psi$$

where

$$\mathrm{div}_x \mathbf{v} = 0, \ \int_\Omega \Psi \ \mathrm{d}x = 0, \ \int_\Omega \mathbf{v} \ \mathrm{d}x = 0, \ \mathbf{V} \in R^2$$

Fixing h and the potential Ψ

$$\partial_t h + \Delta \Psi = 0$$

$$h(0, \cdot) = h_0, \ -\partial_t h(0, \cdot) = \Delta \Psi_0$$

Problem I

Equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V} + \nabla_x \Psi) \otimes (\mathbf{v} + \mathbf{V} + \nabla_x \Psi)}{h} + \left(ah^2 + \partial_t \Psi \right) \mathbb{I} \right)$$

$$+ \partial_t \mathbf{V}$$

$$= h \left(-\gamma \frac{\mathbf{v} + \mathbf{V} + \nabla_x \Psi}{|\mathbf{v} + \mathbf{V} + \nabla_x \Psi|} + \mathbf{f} \right),$$

Constraints and initial conditions

$$\mathrm{div}_x \mathbf{v} = 0, \ \int_\Omega \mathbf{v}(t, \cdot) \ \mathrm{d}x = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{V}(0) = \mathbf{V}_0$$

Transformation - Step II

Prescribing the kinetic energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}+\nabla_x\Psi|^2}{h} = E \equiv \Lambda(t) - ah^2 - \partial_t\Psi$$

Problem II

$$\partial_t\mathbf{v} + \partial_t\mathbf{V}$$

$$+\mathrm{div}_x\left(\frac{(\mathbf{v}+\mathbf{V}+\nabla_x\Psi)\otimes(\mathbf{v}+\mathbf{V}+\nabla_x\Psi)}{h} - \frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}+\nabla_x\Psi|^2}{h}\mathbb{I}\right)$$

$$= -\gamma\left(\frac{h}{2E}\right)^{1/2}(\mathbf{v}+\mathbf{V}+\nabla_x\Psi) + h\mathbf{f}$$

Transformation - Step III

Determining function V

$$\partial_t \mathbf{V} - \left[\frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} \, \mathrm{d}x \right] \mathbf{V}$$

$$= + \frac{1}{|\Omega|} \int_\Omega \left[\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \nabla_x \Psi) + h\mathbf{f} \right] \, \mathrm{d}x, \;\; \mathbf{V}(0) = \mathbf{V}_0$$

Problem III

Equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi) \odot (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi)}{h} \right)$$

$$= -\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi)$$

$$+ \frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi) \ \mathrm{d}x + h\mathbf{f} - \frac{1}{|\Omega|} \int_\Omega h\mathbf{f} \ \mathrm{d}x$$

$$\mathbf{v} \odot \mathbf{w} = \mathbf{v} \otimes \mathbf{w} - \frac{1}{2} \mathbf{v} \cdot \mathbf{w} \mathbb{I}$$

Transformation - Step IV

Solving elliptic problem

$$\mathrm{div}_x \mathbb{M} \equiv \mathrm{div}_x \left(\nabla_x \mathbf{m} + \nabla_x^t \mathbf{m} - \mathrm{div}_x \mathbf{m} \mathbb{I} \right)$$

$$= -\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi)$$

$$+ \frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi) \ \mathrm{d}x + h\mathbf{f} - \frac{1}{|\Omega|} \int_\Omega h\mathbf{f} \ \mathrm{d}x,$$

$$\int_\Omega \mathbb{M}(t, \cdot) \ \mathrm{d}x = 0 \text{ for any } t \in [0, T].$$

Abstract formulation

Variable coefficients "Euler system"

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \odot (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}] \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0,$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

Data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Euler-Fourier system

(joint work with E. Chiodaroli and O.Kreml)

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\Big[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\Big] - \Delta\vartheta = -\varrho\vartheta \mathrm{div}_x \mathbf{u}$$

Transformation

Ansatz

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi, \ \mathrm{div}_x \mathbf{v} = 0$$

Equations

$$\partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x (\partial_t \Psi + \varrho \vartheta) = 0$$

$$\frac{3}{2} \Big(\partial_t (\varrho \vartheta) + \mathrm{div}_x \Big(\vartheta (\mathbf{v} + \nabla_x \Psi) \Big) \Big) - \Delta \vartheta = - \varrho \vartheta \mathrm{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

Energy

$$E = \Lambda(t) - \frac{3}{2} \varrho \vartheta [\mathbf{v}] - \frac{3}{2} \partial_t \Psi$$

Solution

Construction of solutions

1. Fix ϱ and compute the acoustic potential Ψ

$$-\Delta \Psi = \partial_t \varrho$$

2. Compute $\vartheta = \vartheta[\mathbf{v}]$ for $\mathbf{v} \in L^\infty$

$$\frac{3}{2}\left(\partial_t(\varrho\vartheta) + \mathrm{div}_x\Big(\vartheta(\mathbf{v} + \nabla_x \Psi)\Big)\right) - \Delta\vartheta = -\varrho\vartheta \mathrm{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho}\right)$$

3. Observe that $0 < \vartheta < \overline{\vartheta}$, $\overline{\vartheta}$ independent of $\mathbf{v}$
4. Take

$$E = \chi(t) - \frac{3}{2}\varrho\vartheta[\mathbf{v}]$$

and use the non-local variant of Oscillatory Lemma

Euler-Korteweg-Poisson system

(joint work with D.Donatelli and P.Marcati)

Mass conservation - equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equations - Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho)$$

$$= \boxed{\varrho \nabla_x \left(K(\varrho)\Delta_x \varrho + \frac{1}{2} K'(\varrho) |\nabla_x \varrho|^2 \right)} - \varrho \mathbf{u} + \varrho \nabla_x V$$

Poisson equation

$$\Delta_x V = \varrho - \overline{\varrho}$$

Euler-Cahn-Hilliard system

Model by Lowengrub and Truskinovsky

Mass conservation

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_0(\varrho, c) = \operatorname{div}_x \left(\varrho \nabla_x c \otimes \nabla_x c - \frac{\varrho}{2} |\nabla_x c|^2 \mathbb{I} \right)$$

Cahn-Hilliard equation

$$\partial_t(\varrho c) + \operatorname{div}_x(\varrho c \mathbf{u}) = \Delta \left(\mu_0(\varrho, c) - \frac{1}{\varrho} \operatorname{div}_x \left(\varrho \nabla_x c \right) \right)$$

Models of collective behavior

(joint work with J.A. Carrillo, P.Gwiazda, A.Swierczewska–Gwiazda)

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

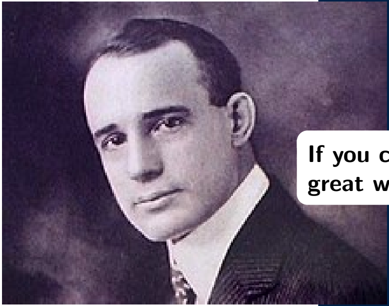
Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u})$$

$$= -\nabla_x p(\varrho) + \left(1 - H\left(|\mathbf{u}|^2\right)\right) \varrho \mathbf{u}$$

$$-\varrho \nabla_x K * \varrho + \varrho \psi * \left[\varrho\Big(\mathbf{u} - \mathbf{u}(\cdot)\Big)\right]$$

Inviscid limits



If you cannot do great things, do small things in a great way

Napoleon Hill [1883-1970]

Oliver Napoleon Hill was an American self-help author

Navier–Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Constitutive equations

$$p(\varrho) = a\varrho^{\gamma},\ \ a > 0,\ \ \gamma > 1$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Far field conditions

$$\varrho \to \varrho_\infty > 0,\ \ \mathbf{u} \to 0 \text{ as } |x| \to \infty$$

Finite energy solutions

Energy inequality

$$\int_{R^N} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty) \right] (\tau, \cdot) \, \mathrm{d}x$$

$$+ \int_0^\tau \int_{R^N} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \mathrm{d}t$$

$$\leq \int_{R^N} \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) - P'(\varrho_\infty)(\varrho_0 - \varrho_\infty) - P(\varrho_\infty) \right] \, \mathrm{d}x$$

Bounded energy solutions

$$\int_{R^N} \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty) \right] (\tau, \cdot) \, \mathrm{d}x$$

$$+ \int_0^\tau \int_{R^N} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \mathrm{d}t \leq E_0$$

Uniform bounds

Pressure potential

$$P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty) \gtrsim \begin{cases} |\varrho - \varrho_\infty|^2 \text{ if } \frac{\varrho_\infty}{2} < \varrho < 2\varrho_\infty \\ \\ 1 + P(\varrho) \text{ otherwise} \end{cases}$$

Viscous stress

$$\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} = \frac{\mu}{2} \left| \nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{N} \mathrm{div}_x \mathbf{u} \right|^2 + \eta |\mathrm{div}_x \mathbf{u}|^2$$

Vanishing viscosity limit

Viscosity coefficients

$$\mu = \mu_n \searrow 0, \ \eta = \eta_n \searrow 0$$

Approximate solutions

$$(\varrho_n - \varrho_\infty) \text{ bounded in } L^\infty(0,T;(L^\gamma + L^2)(R^N))$$

$$\mathbf{m}_n \equiv \varrho_n \mathbf{u}_n \text{ bounded in } L^\infty(0,T;(L^{\frac{2\gamma}{\gamma+1}} + L^2)(R^N))$$

Convergence in the sense of distributions

$$\varrho_n \to \varrho \text{ in } \mathcal{D}'((0,T)\times R^N), \ \varrho - \varrho_\infty \in L^\infty(0,T;(L^\gamma + L^2)(R^N))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ in } \mathcal{D}'((0,T)\times R^N;R^N), \ \mathbf{m} \in L^\infty(0,T;(L^{\frac{2\gamma}{\gamma+1}} + L^2)(R^N))$$

Strong convergence to weak solutions

Theorem - EF, M.Hofmanová

Suppose that ϱ, $\mathbf{m}$ is a weak solution of the Euler system in $\mathcal{D}'((0,T) \times R^N)$.
Then

$$\left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) - P'(\varrho_\infty)(\varrho_n - \varrho_\infty) - P(\varrho_\infty) \right]$$

$$\to \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty) \right]$$

in $L^1((0,T) \times R^N)$, in particular

$$(\varrho_n - \varrho_\infty) \to (\varrho - \varrho_\infty) \text{ in } (L^\gamma + L^2)((0,T) \times R^N)$$

$$\mathbf{m}_n \to \mathbf{m} \text{ in } (L^{\frac{2\gamma}{\gamma+1}} + L^2)((0,T) \times R^N, R^N)$$

Energy defect

Bounded energy

$$e(\varrho_n, \mathbf{m}_n) \equiv \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) - P'(\varrho_\infty)(\varrho_n - \varrho_\infty) - P(\varrho_\infty) \right]$$

bounded in $L^\infty(0, T; L^1(R^N))$.

Duality

$$L^1(R^N) \hookrightarrow \mathcal{M}(R^N) - \text{ the space of } \boxed{\textit{finite}} \text{ Borel measures}$$

$$\mathcal{M}(R^N) = \left[C_0(R^N) \right]^*$$

$$L^\infty(0, T; L^1(R^N)) \hookrightarrow [L^1(0, T; C_0(R^N))]^*$$

Weak convergence

$$e(\varrho_n, \mathbf{m}_n) \to \overline{e(\varrho, \mathbf{m})}$$

$$P(\varrho_n) - P'(\varrho_\infty)(\varrho_n - \varrho_\infty) - P(\varrho_\infty) \to \overline{P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty)}$$

weakly-(*) in $L^\infty(0, T; \mathcal{M}(R^N))$

Pressure defect, I

Isentropic pressure, weak convergence

$$p(\varrho) = (\gamma - 1)P(\varrho)$$

$$p(\varrho_n) - p'(\varrho_\infty)(\varrho_n - \varrho_\infty) - p(\varrho_\infty)$$
$$\to (\gamma - 1)\overline{P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty)}$$

weakly-(*) in $L^\infty(0, T; \mathcal{M}(R^N))$

Compatibility

$$\lim_{n\to\infty} \int_0^T \int_{R^N} [p(\varrho_n) - p(\varrho)] \, \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t$$
$$= \lim_{n\to\infty} \int_0^T \int_{R^N} \left[p(\varrho_n) - p'(\varrho_\infty)(\varrho_n - \varrho_\infty) - p(\varrho_\infty)\right] \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t$$
$$- \int_0^T \int_{R^N} \left[p(\varrho) - p'(\varrho_\infty)(\varrho - \varrho_\infty) - p(\varrho_\infty)\right] \mathrm{div}_x \boldsymbol{\varphi} \, \mathrm{d}x\mathrm{d}t$$

$\boldsymbol{\varphi} \in C_c^\infty((0, T) \times R^N; R^N)$

Pressure defect, II

Internal energy defect

$$\overline{P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty)} - \Big[P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty) \Big] \geq 0$$

Pressure defect

$$\mathfrak{R}_p = (\gamma - 1)\overline{P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty)}$$

$$-(\gamma - 1)\Big[P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty) \Big] \geq 0$$

$$\mathfrak{R}_p \in L^\infty(0, T; \mathcal{M}^+(R^N))$$

Compatibility

$$\lim_{n \to \infty} \int_0^T \int_{R^N} [p(\varrho_n) - p(\varrho)] \mathrm{div}_x \varphi \ \mathrm{d}x\mathrm{d}t = \int_0^T \int_{R^N} \mathrm{div}_x \varphi \ \mathrm{d}\mathfrak{R}_p$$

Turbulent stress

Convective term

$$\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} \in L^\infty(0, T; L^1(R^N; R^{N\times N}_{\rm sym}))$$

Weak convergence

$$\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} \to \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}}$$

weakly-(*) in $L^\infty(0, T; \mathcal{M}(R^N; R^{N\times N}_{\rm sym}))$

Turbulent viscous stress (defect)

$$\mathfrak{R}_v = \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} - \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \in L^\infty(0, T; \mathcal{M}^+(R^N; R^{N\times N}_{\rm sym}))$$

Positivity of the turbulent stress

Positivity of a matrix valued measure

$$\mathbb{D} \in \mathcal{M}^+(R^N; R^{N\times N}_{\rm sym}) \Leftrightarrow \int_{R^N} \varphi(\xi \otimes \xi) : {\rm d}\mathbb{D} \geq 0$$

for any $\varphi \in C_c^\infty(R^N)$, $\varphi \geq 0$, $\xi \in R^N$

Positivity of the turbulent viscous stress

$$\int_0^T \int_{R^3} \varphi(\xi \otimes \xi) : {\rm d}\mathfrak{R}_v$$

$$= \lim_{n\to\infty} \int_0^T \int_{R^N} \left[\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} - \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right] : (\xi \otimes \xi)\varphi \ {\rm d}x{\rm d}t$$

$$= \lim_{n\to\infty} \int_0^T \int_{R^N} \left[\frac{|\mathbf{m}_n \cdot \xi|^2}{\varrho_n} - \frac{|\mathbf{m} \cdot \xi|^2}{\varrho} \right] \varphi \ {\rm d}x{\rm d}t \geq 0$$

Limit process

Consistency

$$\int_0^T \int_{R^N} \left[\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi \right] \, \mathrm{d}x\mathrm{d}t \to 0$$

for any $\varphi \in C_c^\infty((0,T) \times R^N)$

$$\int_0^T \int_{R^N} \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \boldsymbol{\varphi} + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \, \mathrm{d}x\mathrm{d}t \to 0$$

for any $\boldsymbol{\varphi} \in C_c^\infty((0,T) \times R^N; R^N)$

Reorganizing

$$\begin{aligned}
&\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \boldsymbol{\varphi} + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \\
&\quad = \left(\frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} - \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) : \nabla_x \boldsymbol{\varphi} + (p(\varrho_n) - p(\varrho)) \mathrm{div}_x \boldsymbol{\varphi} \\
&\quad + \mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi}
\end{aligned}$$

Limit system (dissipative solutions)

Equations

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \left(\mathfrak{R}_v + \mathfrak{R}_p \mathbb{I} \right)$$

$$\text{in } \mathcal{D}'((0,T) \times R^N)$$

Turbulent defect measures

$$\mathfrak{R}_v \in L^\infty(0,T; \mathcal{M}^+(R^N; R^{N\times N}_{\mathrm{sym}}))$$

$$\mathfrak{R}_p \in L^\infty(0,T; \mathcal{M}^+(R^N))$$

Energy

$$\overline{e(\varrho, \mathbf{m})} = e(\varrho, \mathbf{m}) + \frac{1}{2}\mathrm{trace}[\mathfrak{R}_v] + \frac{1}{\gamma - 1}\mathfrak{R}_p$$

Solvability of the problem

Basic equation

$$\mathbb{D} = \mathfrak{R}_v(t) + \mathfrak{R}_p(t)\mathbb{I} \in \mathcal{M}^+(R^N; R^{N\times N}_{\rm sym}) \text{ for a.a. } t \in (0,T)$$

$$\mathrm{div}_x \mathbb{D} = 0 \text{ in } \mathcal{D}'(R^N)$$

Weak formulation

$$\int_{R^N} \nabla_x \varphi : \mathrm{d}\mathbb{D} = 0 \text{ for any } \varphi \in C_c^\infty(R^N)$$

Larger class of test functions

Weak formulation

$$\int_{R^N} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} = 0 \text{ for any } \boldsymbol{\varphi} \in C_c^\infty(R^N)$$

for any $\boldsymbol{\varphi} \in C_c^\infty(R^N)$

Cut-off

$$0 \leq \psi_R \leq 1,\ \psi_R \in C_c^\infty(R^N)$$

$$\psi_R(Y) = 1 \text{ if } |Y| < r,\ \psi_R(Y) = 0 \text{ if } |Y| > 2r,\ |\nabla_x \psi_R| \leq \frac{2}{R}$$

Globally Lipschitz test functions

$$0 = \int_{R^N} \nabla_x(\psi_R \boldsymbol{\varphi}) : \mathrm{d}\mathbb{D} = \int_{R^N} \psi_R \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} + \int_{R^N} (\nabla_x \psi_R \otimes \boldsymbol{\varphi}) : \mathrm{d}\mathbb{D}$$

$$= \int_{|x|<R} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mathbb{D} + \int_{|x|\geq R} [\psi_R \nabla_x \boldsymbol{\varphi} + (\nabla_x \psi_R \otimes \boldsymbol{\varphi})] : \mathrm{d}\mathbb{D}$$

Conclusion

Extending the class of text functions

$$\int_{R^N} \nabla_x \varphi : \mathrm{d}\mathbb{D} = 0$$

for any $\varphi \in C^\infty(R^N)$, $|\nabla_x \varphi| \leq c$

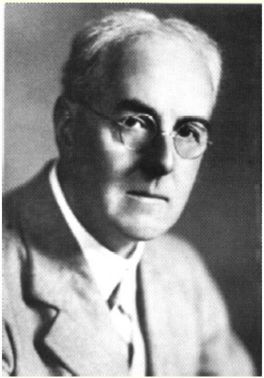
Special test function

$$\varphi, \ \varphi_i = \sum_{j=1}^{N} \xi_i \xi_j x_j$$

Conclusion

$$\int_{R^N} (\xi \otimes \xi) : \mathrm{d}\mathbb{D} = 0 \Rightarrow (\xi \otimes \xi) : \mathbb{D} = 0 \Rightarrow \mathbb{D} = 0$$

Solving ill–posed problems



L.F. Richardson [1881–1953]

Another advantage of a mathematical statement is that it is so definite that it might be definitely wrong. Some verbal statements have not this merit.

Isentropic Euler system revisited

Phase variables

mass density .... $\varrho = \varrho(t,x)$
momentum .... $\mathbf{m} = \mathbf{m}(t,x) \in R^N$
(total) energy .... $E = E(t) \in R$

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Balance of momentum

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + a \nabla_x \varrho^\gamma = 0$$

Energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq 0, \ \ E = \int_{\mathbb{T}^N} \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \frac{a}{\gamma - 1} \varrho^\gamma \right] \ \mathrm{d}x$$

Dissipative solutions

Phase variables

mass density . . . . $\varrho = \varrho(t,x)$
momentum . . . . $\mathbf{m} = \mathbf{m}(t,x) \in R^N$
"turbulent" defect measures . . . . $\mathfrak{R}_e \in L^\infty(0,T;\mathcal{M}^+(\mathbb{T}^N))$,
$\mathfrak{R}_v \in L^\infty(0,T;\mathcal{M}^+(\mathbb{T}^N;R^{N\times N}_{\rm sym}))$

Euler system

$$\partial_t \varrho + {\rm div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + {\rm div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = - {\rm div}_x \mathfrak{R}_v - (\gamma - 1) \nabla_x \mathfrak{R}_e$$

$$\frac{{\rm d}}{{\rm d}t} E(t) \leq 0$$

$$E = \int_{\mathbb{T}^N} \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \frac{a}{\gamma - 1} \varrho^\gamma \right) \ {\rm d}x + \int_{\mathbb{T}^N} \left(\frac{1}{2} {\rm tr}[\mathfrak{R}_v] + \mathfrak{R}_e \right)$$

Dissipative solutions - existence

Existence

Dissipative solutions can be constructed as limits of energy dissipating numerical schemes (Lax–Friedrichs and similar). They appear as zero viscosity limit for the Navier–Stokes system

Dissipative–strong uniqueness

A dissipative solution coincides with a strong solution starting from the same initial data on the life–span of the latter (see also similar results on measure–valued solutions - Wiedemann et al.)

Uniqueness - semigroup selection

For each initial data, one can select a global in time dissipative solution so that the resulting system forms a semigroup. The selected solutions maximize the energy dissipation

Relative energy

Relative energy

$$\mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right) \equiv \frac{1}{2}\varrho\left|\frac{\mathbf{m}}{\varrho} - \mathbf{U}\right|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r)$$
$$+ \frac{1}{2}\mathrm{tr}[\mathfrak{R}_v] + \mathfrak{R}_e, \;\; P(\varrho) = \frac{a}{\gamma - 1}\varrho^{\gamma}$$

Relative energy decomposition

$$\int_{\mathbb{T}^N} \mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right) \, \mathrm{d}x$$
$$= \int_{\mathbb{T}^N} \left[\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right] \, \mathrm{d}x + \int_{\mathbb{T}^N} \left[\frac{1}{2}\mathrm{tr}[\mathfrak{R}_v] + \mathfrak{R}_e\right]$$
$$- \int_{\mathbb{T}^N} \mathbf{m} \cdot \mathbf{U} \, \mathrm{d}x + \int_{\Omega} \varrho \left[\frac{1}{2}|\mathbf{U}|^2 - P'(r)\right] \, \mathrm{d}x$$
$$+ \int_{\mathbb{T}^N} \left[P'(r)r - P(r)\right] \, \mathrm{d}x$$

Relative energy inequality

$$\int_{\mathbb{T}^N} \mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right)(\tau,\cdot)\,\mathrm{d}x - \int_{\mathbb{T}^N} \mathcal{E}\left(\varrho, \mathbf{m} \,\middle|\, r, \mathbf{U}\right)(0,\cdot)\,\mathrm{d}x$$

$$\leq - \int_0^\tau \int_{\mathbb{T}^N} \nabla_x \mathbf{U} : \varrho \left(\mathbf{U} - \frac{\mathbf{m}}{\varrho}\right) \otimes \left(\mathbf{U} - \frac{\mathbf{m}}{\varrho}\right)\,\mathrm{d}x\mathrm{d}t$$

$$- \int_0^\tau \int_{\mathbb{T}^N} \Big(p(\varrho) - p'(r)(\varrho - r) - p(r)\Big)\mathrm{div}_x \mathbf{U}\,\mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_{\mathbb{T}^N} \Big[\partial_t (r\mathbf{U}) + \mathrm{div}_x\left(r\mathbf{U} \otimes \mathbf{U}\right) + \nabla_x p(r)\Big] \cdot \frac{1}{r}\Big(\varrho \mathbf{U} - \mathbf{m}\Big)\,\mathrm{d}x\mathrm{d}t$$

$$+ \int_0^\tau \int_{\mathbb{T}^N} \Big[\partial_t r + \mathrm{div}_x (r\mathbf{U})\Big] \left[\left(1 - \frac{\varrho}{r}\right) p'(r) + \frac{1}{r}\mathbf{U} \cdot (\mathbf{m} - \varrho \mathbf{U})\right]\,\mathrm{d}x\mathrm{d}t$$

$$- \int_0^\tau \left(\int_{\mathbb{T}^N} \nabla_x \mathbf{U} : \mathrm{d}\mathfrak{R}^v(t)\right)\mathrm{d}t - \int_0^\tau \left(\int_{\mathbb{T}^N} \mathrm{div}_x \mathbf{U}\,\mathrm{d}\mathfrak{R}^p(t)\right)\mathrm{d}t$$

Dispersive velocity weak solutions

Besov spaces

$$v \in B^{\alpha,\infty}_p(Q) \ \Leftrightarrow \ \|v\|_{L^p(Q)} + \sup_{\xi} \frac{\|v(\cdot + \xi) - v(\cdot)\|_{L^p(Q \cap (Q-\xi))}}{|\xi|^\alpha} < \infty.$$

Class $\mathcal{D}$

$$\varrho \in C([0,T]; L^1(\mathbb{T}^N)), \ \mathbf{u} \in C([0,T]; L^1(\mathbb{T}^N; R^d))$$

$$0 < \underline{\varrho} \leq \varrho \leq \overline{\varrho}, \ |\mathbf{u}| \leq \overline{\mathbf{u}} \text{ a.a. in } (0,T) \times \Omega$$

$$\varrho \in B^{\alpha,\infty}_p([\delta, T] \times \mathbb{T}^N), \ \mathbf{u} \in B^{\alpha,\infty}_p([\delta, T] \times \mathbb{T}^N; R^d)$$

$$\text{for any } 0 < \delta < T, \ \alpha > \frac{1}{2}, \ p \geq \frac{4\gamma}{\gamma - 1}$$

$$\int_{\mathbb{T}^N} \left[-\xi \cdot \mathbf{u}(\tau, \cdot)(\xi \cdot \nabla_x)\varphi + D(\tau)|\xi|^2 \varphi \right] \ \mathrm{d}x \geq 0 \text{ for a.a. } \tau \in (0,T)$$

$$\text{for any } \xi \in R^d \text{ and any } \varphi \in C^1(\mathbb{T}^N), \ \varphi \geq 0, \ \text{where } D \in L^1(0,T)$$

Weak (dissipative) – weak uniqueness

Theorem

Let ϱ, $\mathbf{m} = \varrho\mathbf{u}$ be a weak solution of the Euler system belonging to class $\mathcal{D}$, and let $\widetilde{\varrho}$, $\widetilde{\mathbf{m}}$ be a dissipative solution of the same problem starting from the same initial data.
Then

$$\varrho = \widetilde{\varrho}, \ \mathbf{m} = \widetilde{\mathbf{m}}.$$

Commutator estimates

Basic properties of Besov functions

$$[v]_\varepsilon = v * \vartheta_\varepsilon$$

$$\|[v]_\varepsilon - v\|_{L^p(Q)} \leq \varepsilon^\alpha \|v\|_{B_p^{\alpha,\infty}(Q)}, \ \|\nabla_x [v]_\varepsilon\|_{L^p(Q)} \leq \varepsilon^{\alpha-1} \|v\|_{B_p^{\alpha,\infty}(Q)}$$

Lemma

Let Q be a bounded domain in R^M. Suppose that $\mathbb{V} : \tilde{Q} \to R^k$ belongs to the Besov space $B_p^{\alpha,\infty}(Q, R^k)$, $p \geq 2$, where $\tilde{Q} \subset R^M$ is another domain containing $\overline{Q}$ in its interior. Let η^ε be a standard family of regularizing kernels, $\mathrm{supp}[\eta^\varepsilon] \subset \{|y| < \varepsilon\}$. Let $G : K \to R$ be a twice continuously differentiable function defined on an open set $K \subset R^k$ containing the closure of the range of $\mathbb{V}$. Finally, set $[v]_\varepsilon \equiv \eta^\varepsilon * v$.
Then

$$\|\nabla_y G([\mathbb{V}]_\varepsilon) - \nabla_y [G(\mathbb{V})]_\varepsilon\|_{L^{\frac{p}{2}}(Q;R^M)}$$

$$\leq \varepsilon^{2\alpha-1} c(\|G\|_{C^2(K)}) \left(1 + \|\mathbb{V}\|^2_{B_p^{\alpha,\infty}(Q;R^k)}\right)$$

for $\nabla_y = (\partial_{y_1}, \ldots, \partial_{y_M})$.

Semiflow selection

Semiflow

$$U[t, \varrho_0, \mathbf{m}_0, E_0] \mapsto [\varrho(t), \mathbf{m}(t), E(t-)], \ \ t > 0$$

Semigroup property

$$U[t_1 + t_2, \varrho_0, \mathbf{m}_0, E_0] = U\left[t_2, U[t_1, \varrho_0, \mathbf{m}_0, E_0]\right] \ \text{for any } 0 \leq t_1 \leq t_2$$

Dissipative solution

$$\varrho \in C_{\rm weak,loc}([0,\infty); L^\gamma(\mathbb{T}^N))$$

$$\mathbf{m} \in C_{\rm weak,loc}([0,\infty); L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^N; R^N))$$

$$E \in BV_{\rm loc}([0,\infty); R), \ \ \text{(non–increasing)}$$

Initial data

$$\varrho(0,\cdot) = \varrho_0, \ \ \mathbf{m}(0,\cdot) = \mathbf{m}_0, \ \ E(0+) \leq E_0$$

Semiflow

Basic properties

- **Stability of regular solutions.** Let $\widehat{\varrho}$, $\widehat{\mathbf{m}}$ be a solution in the class $\mathcal{D}$,

$$E_0 = \int_Q \left[\frac{|\mathbf{m}_0|^2}{\varrho_0} + \frac{a}{\gamma - 1} \varrho_0^\gamma \right] \mathrm{d}x,$$

  defined on a maximal time interval $[0, T_{\max})$. Then

$$U[t, \varrho_0, \mathbf{m}_0, E_0] = [\widehat{\varrho}, \widehat{\mathbf{m}}, E_0](t) \text{ for all } t \in [0, T_{\max}).$$

- **Maximal dissipation.** Let $\widehat{\varrho}$, $\widehat{\mathbf{m}}$ be a dissipative associated energy $\widehat{E}$ such that

$$\widehat{E}(t) \leq E(t) \text{ for all } t \geq 0,$$

  where E is the energy of the semiflow $U[t, \varrho_0, \mathbf{m}_0, E_0]$. Then

$$E(t) = \widehat{E}(t) \text{ for all } t \geq 0.$$

- **Stability of stationary states.** Let $\overline{\varrho} > 0$, $\mathbf{m} \equiv 0$ a stationary solution. Then

$$\varrho(T, \cdot) = \overline{\varrho},\ \mathbf{m}(T, \cdot) = 0 \ \Rightarrow \varrho(t, \cdot) = \overline{\varrho},\ \mathbf{m}(t, \cdot) = 0 \text{ for all } t \geq T$$

Abstract setting

Phase space

$$X = W^{-\ell,2}(\mathbb{T}^N) \times W^{-\ell,2}(\mathbb{T}^N; R^N) \times R$$

Data space

$$D = \left\{ [\varrho_0, \mathbf{m}_0, E_0] \in X \ \middle| \ \varrho_0 \geq 0, \ \int_{\mathbb{T}^N} \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + \frac{a}{\gamma - 1} \varrho_0^\gamma \right] \ \mathrm{d}x \leq E_0 \right\}.$$

Trajectory space

$$\Omega = C_{\rm loc}([0,\infty); W^{-\ell,2}(\mathbb{T}^N)) \times C_{\rm loc}([0,\infty); W^{-\ell,2}(\mathbb{T}^N; R^N)) \times L^1_{\rm loc}(0,\infty)$$

Method by Krylov adapted by Cardona and Kapitanski

Multi–valued solution mapping

$$\mathcal{U} : [\varrho_0, \mathbf{m}_0, E_0] \mapsto [\varrho, \mathbf{m}, E] \in 2^{\Omega}$$

Time shift

$$S_T \circ \xi, \ \ S_T \circ \xi(t) = \xi(T + t), \ \ t \geq 0.$$

Continuation

$$\xi_1 \cup_T \xi_2(\tau) = \left\{ \begin{array}{l} \xi_1(\tau) \text{ for } 0 \leq \tau \leq T, \\ \\ \xi_2(\tau - T) \text{ for } \tau > T. \end{array} \right.$$

Basic ansatz

- **(A1) Compactness:** For any $[\varrho_0, \mathbf{m}_0, E_0] \in D$, the set $\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0]$ is a non–empty compact subset of Ω
- **(A2)** The mapping

$$D \ni [\varrho_0, \mathbf{m}_0, E_0] \mapsto \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] \in 2^{\Omega}$$

  is **Borel measurable**, where the range of $\mathcal{U}$ is endowed with the Hausdorff metric on the subspace of compact sets in 2^{Ω}
- **(A3) Shift invariance:** For any

$$[\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0],$$

  we have

$$S_T \circ [\varrho, \mathbf{m}, E] \in \mathcal{U}[\varrho(T), \mathbf{m}(t), E(T-)] \text{ for any } T > 0.$$

- **(A4) Continuation:** If $T > 0$, and

$$[\varrho^1, \mathbf{m}^1, E^1] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0],\ [\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho^1(T), \mathbf{m}^1(T), E^1(T-)],$$

  then

$$[\varrho^1, \mathbf{m}^1, E^1] \cup_T [\varrho^2, \mathbf{m}^2, E^2] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0].$$

Induction argument

System of functionals

$$I_{\lambda,F}[\varrho,\mathbf{m},E] = \int_0^\infty \exp(-\lambda t) F(\varrho,\mathbf{m},E)\ \mathrm{d}t,\ \ \lambda > 0$$

where

$$F : X = W^{-\ell,2}(Q) \times W^{-\ell,2}(Q;R^N) \times R \to R$$

is a bounded and continuous functional

Semiflow reduction

$$I_{\lambda,F} \circ \mathcal{U}[\varrho_0,\mathbf{m}_0,E_0]$$
$$= \Big\{ [\varrho,\mathbf{m},E] \in \mathcal{U}[\varrho_0,\mathbf{m}_0,E_0] \ \Big|$$
$$I_{\lambda,F}[\varrho,\mathbf{m},E] \leq I_{\lambda,F}[\tilde{\varrho},\tilde{\mathbf{m}},\widetilde{E}] \text{ for all } [\tilde{\varrho},\tilde{\mathbf{m}},\widetilde{E}] \in \mathcal{U}[\varrho_0,\mathbf{m}_0,E_0] \Big\}$$

Induction argument

$$\mathcal{U} \text{ satisfies (A1) - (A4)} \ \Rightarrow I_{\lambda,F} \circ \mathcal{U} \text{ satisfies (A1) - (A4)}$$

Testing functionals

Abstract ansatz

A countable $\{\mathbf{w}_m\}_{m=1}^{\infty}$ in $L^2(\mathbb{T}^N; R^N)$, a countable set $\{\lambda_k\}_{k=1}^{\infty}$ which is dense in $(0,\infty)$, $\beta : R \to R$ bounded strictly increasing function

Functionals

$$I_{k,0,0}[\varrho, \mathbf{m}, E] = \int_0^\infty \exp(-\lambda_k t)\beta(E(t))\mathrm{d}t,$$

$$I_{k,0,m}[\varrho, \mathbf{m}, E] = \int_0^\infty \exp(-\lambda_k t)\beta\left(\int_{\mathbb{T}^N} \mathbf{m}\cdot\mathbf{w}_m \ \mathrm{d}x\right)\mathrm{d}t,$$

Lerch theorem

$$\int_0^\infty \exp(-\lambda_k t)a(t)\mathrm{d}t = \int_0^\infty \exp(-\lambda_k t)b(t)\mathrm{d}t \text{ for all } \lambda_k$$

$$\Rightarrow$$

$$a(t) = b(t) \text{ for a.a. } t > 0$$