Oscillatory solutions to problems in fluid dynamics

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Abstract

We consider the phenomenon of oscillations in the solution families to partial differential equations. To begin, we briefly discuss mechanisms preventing oscillations/concentrations and make a short excursion in the theory of compensated compactness. Pursuing the philosophy "everything what is not forbidden is allowed" we show that certain problems in fluid dynamics admit oscillatory solutions. This fact gives rise to two rather unexpected and in a way contradictory results: **(i)**many problems describing inviscid fluid motion in several space dimensions admit global-in-time (weak solution); **(ii)** the solutions are not determined uniquely by their initial data. We examine the basic analytical tool behind these rather ground breaking results - the method of convex integration applied to problems in fluid mechanics

Keywords: Oscillations, concentrations, weak solution, Euler system, Navier Stokes system, convex integration

Contents

1 Oscillations, concentrations and how to handle them

Oscillations and concentrations are phenomena that may break down stability of certain non–linear problems, notably system of conservation laws arising in fluid dynamics. Even if constrained by field equations and in general non–linear constitutive relations, solutions of these problems may develop uncontrollable oscillations and/or concentrations. These may lead to rather unpleasant violation of stability at the level of numerical schemes but, on the other hand, to unexpected results concerning existence and well/ill posedness of the associated initial value problems. The mathematical approach discussed below can be seen as a counterpart of the theory of *compensated compactness* in the spirit of DiPerna [6], [7] and Tartar [14].

Some of the arguments presented in this part are illustrative and may be not completely rigorous, in the sense that certain hypotheses are omitted to keep the presentation clear and concise.

1.1 Oscillations

Consider an a-periodic continuous function $g : R \to R$,

$$g(x+a) = g(x) \text{ for all } x \in R, \ \int_0^a g(x)\mathrm{d}x = 0,$$

and a sequence

$$g_n(x) = g(nx), \ n = 1, 2, \ldots$$

Our goal is to describe the limit $\lim_{n\to\infty} g_n$. Apparently, the sequence $g(nx)$ does not converge pointwise, not even a.a. pointwise and not even for a subsequence. To capture its asymptotic behavior, we have to consider its averaged values,

$$\int_R g_n(x)\varphi(x) \ \mathrm{d}x, \ \text{where } \varphi \in C_c^\infty(R).$$

Introducing the primite function G,

$$G(x) = \int_0^x g(z) \ \mathrm{d}z$$

we easily observe that G is also continuous and periodic. Consequently, by means of the by–parts–integration formula,

$$\int_R g_n(x)\varphi(x) \ \mathrm{d}x = \int_R g(nx)\varphi(x) \ \mathrm{d}x = -\frac{1}{n}\int_R G(nx)\partial_x\varphi(x) \ \mathrm{d}x \to 0$$

as $n \to \infty$ for any smooth function φ. We say that the sequence $\{g_n\}_{n=1}^\infty$ *converges weakly* to 0, $g_n \rightharpoonup 0$. As a straightforward consequence we deduce that a sequence

$$h_n(x) = h(nx), \ \text{where } h \text{ is } a-\text{periodic},$$

$$\text{converges weakly to the integral average} \int_0^a h(x) \ \mathrm{d}x, \ h_n \rightharpoonup \int_0^a h(x) \ \mathrm{d}x.$$

Thus the weak convergence does not, in general, commute with non–linear compositions, specifically

$$g_n \rightharpoonup g \text{ does not imply } H(g_n) \rightharpoonup H(g) \text{ if } H \text{ is not linear.}$$

Convex compositions are *weakly lower semi–continuous*,

$$g_n \rightharpoonup g \text{ implies } \int_R H(g)\varphi \, \mathrm{d}x \leq \liminf_{n\to\infty} \int_R H(g_n)\varphi \ \mathrm{d}x \text{ for any } \varphi \geq 0$$

whenever H is convex. This can be easily seen from the Taylor decomposition,

$$H(g_n) = H(g) + H'(g)(g_n - g) + \frac{1}{2}H''(\xi)(g_n - g)^2. \tag{1.1}$$

Formula (1.1) also reveals the difference between the weak limit of $H(g_n)$ and $H(g)$ due to the quadratic term. Indeed one can deduce that g_n converges strongly to g only if there exists a *strictly convex* function H such that $H(g_n) \rightharpoonup H(g)$.

1.2 Concentrations

Consider a sequence

$$g_n(x) = ng(nx), \text{ where } g \in C_c^\infty(-1,1), \ g(-x) = g(x), \ g \geq 0, \ \int_R g(x) \ \mathrm{d}x = 1.$$

It is easy to check that

- $$g_n(x) \to 0 \text{ as } n \to \infty \text{ for any } x \neq 0, \text{ in particular } g_n \to 0 \text{ a.a. in } R;$$

- $$\|g_n\|_{L^1(R)} = \int_R g_n(x) \ \mathrm{d}x = \int_R g(x) \ \mathrm{d}x = 1 \text{ for any } n = 1, 2, \ldots.$$

Next, we observe that g_n *does not* converge weakly to 0. Indeed

$$\int_R g_n(x)\varphi(x) \ \mathrm{d}x = \int_{-1/n}^{1/n} g_n(x)\varphi(x) \ \mathrm{d}x \in \left[\min_{x\in[-1/n,1/n]} \varphi(x), \ \max_{x\in[-1/n,1/n]} \varphi(x) \right] \to \varphi(0)$$

as soon as φ is continuous. As a matter of fact, the limit object cannot be identified with any integral average, it is a *measure* - the Dirac mass δ_0 concentrated at 0.

2 Equations preventing oscillations, compactness and compensated compactness

We show that families of (hypothetical) solutions of certain problems cannot exhibit oscillations and/or concentrations.

2.1 Elliptic equations

A trivial examples of problems without oscillatory solutions are elliptic equations as

$$-\Delta_x u(x) = f(x) \text{ in a bounded domain } \Omega,\ u|_{\partial\Omega} = 0. \tag{2.1}$$

Indeed integrating by parts yields immediately

$$\int_\Omega |\nabla_x u|^2 \ \mathrm{d}x = \int_\Omega fu \ \mathrm{d}x \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}.$$

In view of the standard Poincaré inequality, we get

$$\|u\|_{L^2(\Omega)} \lesssim \|\nabla_x u\|_{L^2(\Omega)},$$

and, in accordance with the Rellich–Kondraschev theorem, the embedding

$$W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$$

is compact. Consequently, both oscillations and concentrations are prevented by constraint imposed by the elliptic equation (2.1).

2.2 Scalar conservation laws

A more subtle example of compactness is related to solution families of nonlinear scalar conservation laws. For the sake of simplicity, consider the Burgers equation

$$\partial_t u + \partial_x f(u) = 0,\ u(t,0) = u(t,1),\ t \in (0,T). \tag{2.2}$$

First observe that for *linear* f, the problem admits oscillatory solutions. Indeed any function of the form

$$u(t,x) = v(t-x),\ v(z+1) = v(z)$$

satisfies solves (2.2) with $f(u) = u$. General linear functions $f(u) = au$ can be handled by simple rescaling.

Equation (2.2) with a *non–linear* flux function f is also an iconic example of a problem for which solutions develop singularities in a finite time. Indeed consider the Cauchy problem

$$\partial_t u + u\partial_x u = 0,\ u(0,x) = u_0(x). \tag{2.3}$$

It can be checked by direct computation that solutions of (2.3) satisfy

$$u(t, x + tu_0(x)) = u_0(x),\ x \in R$$

Now consider the situation

$$u_0(x_1) > u_0(x_2) \text{ for some } x_1 < x_2$$

We have

$$u(t, x_1 + tu_0(x_1)) = u_0(x_1) \neq u_0(x_2) = u(t, x_2 + tu_0(x_2)),$$

however

$$x_1 + tu_0(x_1) = x_2 + tu_0(x_2) \text{ for } t = \frac{x_2 - x_1}{u_0(x_1) - u_0(x_2)} > 0.$$

Thus the solution u necessarily develops discontinuity - a shock type singularity - in a fimite time lap no matter how smooth and "small" the initial data are. Moreover, such a situation is "generic", it can be avoided only in the particular case

$$u_0(x_1) \leq u_0(x_2) \text{ whenever } x_1 \leq x_2$$

2.2.1 Non–linear case, entropies

For any continuously differentiable function S, we easily derive the identity

$$\partial_t S(u) + \partial_x F(u) = 0, \text{ where } F'(z) = f'(z)S'(z). \tag{2.4}$$

Although (2.4) can be formally derived from (2.2) on condition that the solutions is *smooth*, it may be viewed as an additional constraint imposed on the solution set. Convex S are called *entropies*, the associated function F_S is the entropy flux.

2.2.2 Maximum principle

Consider a convex function S such that

$$S(z) = \begin{cases} 0 \text{ for } z \leq L \\ \\ > 0 \text{ for } z > L \end{cases}$$

Integrating (2.4) over $(0, 1)$ and using periodicity of u we easily obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 S(u(t, x)) \ \mathrm{d}x = 0.$$

In particular

$$\int_0^1 S(u(t, x)) \ \mathrm{d}x = 0 \text{ for all } t \geq t_0 \text{ as long as } \int_0^1 S(u(t_0, x)) \ \mathrm{d}x = 0.$$

This can be reformulated as the *maximum principle*:

$$u(t, x) \leq L \text{ for all } t \geq t_0 \text{ as soon as } u(t_0, x) \leq L.$$

In a similar fashion we conclude

$$\inf_{y\in[0,1]} u(t_0,y) \leq u(t,x) \leq \sup_{y\in[0,1]} u(t_0,y) \text{ for all } t \geq t_0,\ x\in[0,1]. \tag{2.5}$$

Thus concentrations in the solution set of (2.2) can be controlled by the initial data. Here and hereafter in this section we therefore assume that solutions of (2.2) are uniformly bounded.

Finally, we note that the same can be derived under a weaker stipulation than (2.4), namely

$$\partial_t S(u) + \partial_x F(u) \leq 0, \text{ where } S \text{ is convex, } F'(z) = f'(z)S'(z). \tag{2.6}$$

2.2.3 Compensated compactness, Div–curl lemma

Our goal is to show that the family of constraints (2.4), or even (2.6) prevents oscillations in the family of solutions to (2.2) as long as f is a genuinely *non–linear* function. To this end, we recall the celebrated Div–Curl lemma of Murat and Tartar [14].

Lemma 2.1. *Let $\{\mathbf{U}_n\}_{n=1}^\infty$, $\{\mathbf{V}_n\}_{n=1}^\infty$ be two sequences of vector valued defined on a set $Q\subset R^N$ such that*

$$\mathbf{U}_n \to \mathbf{U} \text{ weakly in } L^p(Q;R^N),$$
$$\mathbf{V}_n \to \mathbf{V} \text{ weakly in } L^q(Q;R^N),$$

where

$$\frac{1}{p}+\frac{1}{q}<1.$$

In addition, let

$$\{\mathrm{div}\mathbf{U}_n\}_{n=1}^\infty \text{ be precompact in } W^{-1,s}(Q),$$
$$\{\mathbf{curl}\ \mathbf{V}_n\}_{n=1}^\infty \text{ be precompact in } W^{-1,s}(Q;R^{N\times N})$$

for some $s>1$.

Then

$$\mathbf{U}_n\cdot\mathbf{V}_n \to \mathbf{U}\cdot\mathbf{V} \text{ weakly in } L^r(Q),\ \frac{1}{r}=\frac{1}{p}+\frac{1}{q}.$$

We will not give the complete proof of this result but restrict ourselves to the situation

$$\mathrm{div}\mathbf{U}_n = 0,\ \mathbf{curl}\mathbf{V}_n = 0.$$

Moreover, as the result is local, we may assume that Q is a simply connected domain. Accordingly, we can write

$$\mathbf{V}_n = \nabla_x\Phi_n, \text{ where } \|\Phi_n\|_{L^q(Q)} + \|\nabla_x\Phi_n\|_{L^q(Q;R^N)} \leq c.$$

In particular, in accordance with the Rellich–Kondrachev compactness embedding theorem,

$$\Phi_n \to \Phi \text{ in } L^q(Q), \text{ where } \nabla_x\Phi = \mathbf{V}.$$

Now, for a compactly supported function φ we get

$$\int_Q \mathbf{V}_n\cdot\mathbf{U}_n\varphi\ \mathrm{d}x = \int_Q \nabla_x\Phi_n\cdot\mathbf{U}_n\varphi\ \mathrm{d}x = -\int_Q \Phi_n\mathbf{U}_n\nabla_x\varphi\ \mathrm{d}x \to -\int_Q \Phi\mathbf{U}\cdot\nabla_x\varphi\ \mathrm{d}x = \int_Q \mathbf{U}\cdot\mathbf{V}\varphi\ \mathrm{d}x$$

2.2.4 Compactness for genuinely nonlinear scalar conservation law

Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of solutions to (2.2) such that

$$|u_n(t,x)| \leq c \text{ for all } (t,x)$$

uniformly for $n \to \infty$.

Passing to suitable subsequences, we may suppose that

$$\begin{aligned} u_n &\to u \text{ weakly-(*) in } L^\infty, \\ S(u_n) &\to \overline{S(u)} \text{ weakly-(*) in } L^\infty, \\ F(u_n) &\to \overline{F(u)} \text{ weakly-(*) in } L^\infty \end{aligned}$$

for any convex entropy S with the corresponding flux F.

Seeing that $[S(u), F(u)]$ satisfy the equality (2.4), or even the inequality (2.6), we may use Div–Curl lemma to deduce

$$S_1(u_n)F_2(u_n) - F_1(u_n)S_2(u_n) \to \overline{S_1(u)}\ \overline{F_2(u)} - \overline{F_1(u)}\ \overline{S_2(u)} \tag{2.7}$$

passing again to subsequences as the case may be. Indeed we may apply Lemma 2.1, with $N = 2$,

$$\mathrm{div}_{t,x}[S_1, F_1] = \partial_t S_1 + \partial_x F_1,\ \mathrm{curl}_{t,x}[F_2, -S_2] = \partial_t S_2 + \partial_x F_2.$$

It turns out that validity of (2.7) for any convex entropies S_1, S_2 with the corresponding fluxes F_1, F_2 implies *strong convergence* of $\{u_n\}_{n=1}^{\infty}$ as soon as the flux f is a strictly convex function. Taking

$$S_1(u) = u,\ F_1(u) = f(u),\ S_2(u) = |u - U|,\ F_2(u) = \mathrm{sgn}(u - U)(f(u) - f(U)),\ U - \text{constant},$$

we deduce from (2.7)

$$\begin{aligned} &\lim_{n\to\infty} \Big[u_n \mathrm{sgn}(u_n - U)(f(u_n) - f(U)) - |u_n - U| f(u_n) \Big] \\ &\quad = \lim_{n\to\infty} u_n \lim_{n\to\infty} \Big[\mathrm{sgn}(u_n - U)(f(u_n) - f(U)) \Big] - \lim_{n\to\infty} |u_n - U| \lim_{n\to\infty} f(u_n), \end{aligned}$$

where the limits are understood in the weak sense. This relation can be rewritten as

$$\lim_{n\to\infty} \Big[|u_n - U| \Big(\overline{f(u)} - f(U) \Big) \Big] = (u - U)\overline{\mathrm{sgn}(u - U)(f(u) - f(U))} \tag{2.8}$$

Let $U = u(\tau, y)$, where (τ, y) is a Lebesgue point of u, specifically

$$\lim_{r\to 0} \frac{1}{|B_r(\tau,y)|} \int_{B_r(\tau,y)} |u - u(\tau,y)|\ \mathrm{d}x\, \mathrm{d}t = 0. \tag{2.9}$$

Now, relation (2.8) yields

$$\lim_{r\to 0}\frac{1}{|B_r(\tau,y)|}\int_{B_r(\tau,y)}\left[\lim_{n\to\infty}\left[|u_n-u(\tau,y)|\left(\overline{f(u)}-f(u(\tau,y))\right)\right]\right]\,\mathrm{d}x\,\mathrm{d}t$$
$$=\lim_{r\to 0}\frac{1}{|B_r(\tau,y)|}\int_{B_r(\tau,y)}(u-u(\tau,y))\overline{\mathrm{sgn}(u-u(\tau,y))(f(u)-f(u(\tau,y)))}\,\mathrm{d}x\,\mathrm{d}t=0.$$

Combining this with (2.9) we replace $u(\tau,y)$ by u in the integral on the left hand side to conclude

$$\lim_{r\to 0}\frac{1}{|B_r(\tau,y)|}\int_{B_r(\tau,y)}\left[\lim_{n\to\infty}|u_n-u|\left(\overline{f(u)}-f(u)\right)\right]\,\mathrm{d}x\,\mathrm{d}t=0$$

This relation can be interpreted that for any Lebesgue point of u

- either
$$\lim_{n\to\infty}|u_n-u|=0,$$
- or
$$\overline{f(u)}=f(u).$$

For strictly convex f this implies strong convergence of $\{u_n\}_{n=1}^{\infty}$.

3 Compressible Euler system

Our model problem is the compressible (barotropic) Euler system describing the time evolution of the density ϱ and the velocity $\mathbf{u}$ of a compressible inviscid fluid:

$$\begin{aligned}\partial_t\varrho+\mathrm{div}_x(\varrho\mathbf{u})&=0,\\ \partial_t(\varrho\mathbf{u})+\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla_x p(\varrho)&=0.\end{aligned}\tag{3.1}$$

For the sake of simplicity, we consider the periodic boundary conditions,

$$x\in\Omega=\left([-1,1]|_{\{-1,1\}}\right)^N.\tag{3.2}$$

The problem is supplemented by the initial conditions,

$$\varrho(0,\cdot)=\varrho_0,\ \ \mathbf{u}(0,\cdot)=\mathbf{u}_0.\tag{3.3}$$

3.1 Strong solutions

The compressible Euler system represents an iconic example of a non–linear conservation law. We briefly sketch ho to rewrite the Euler problem as a *symmetric hyperbolic system.* To this end, we make a simplifying but still physically relevant assumption that the pressure p is given by the isentropic constitutive relation $p(\varrho) = a\varrho^\gamma$, $\gamma > 1$. With the new choice of independent variables

$$r = \sqrt{\frac{2a\gamma}{\gamma - 1}}\varrho^{\frac{\gamma-1}{2}},$$

we may rewrite the Euler system in the form

$$\partial_t r + \mathbf{u} \cdot \nabla_x r + \frac{\gamma - 1}{2} r \mathrm{div}_x \mathbf{u} = 0 \tag{3.4}$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla_x \mathbf{u} + \frac{\gamma - 1}{2} r \nabla_x r = 0. \tag{3.5}$$

To obtain suitable *a priori* estimates, we differentiate the equations $([N/2] + 1)-$times with respect to the $x-$variable and then multiply (3.4) on $\partial_x^\alpha \varrho$, and (3.5) on $\partial_x^\alpha \mathbf{u}$, $\alpha = [N/2] + 1$. We obtain the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\partial_x^\alpha \varrho|^2 + |\partial_x^\alpha \mathbf{u}|^2 \ \ \mathrm{d}x \leq c + \left(\int_\Omega |\partial_x^\alpha \varrho|^2 + |\partial_x^\alpha \mathbf{u}|^2 \ \ \mathrm{d}x \right)^M, \ M > 0$$

from we deduce local–in–time boundedness of the Sobolev norm

$$\|\partial_x^\alpha \varrho\|_{L^2(\Omega)} + \|\partial_x^\alpha \mathbf{u}\|^2_{L^2(\Omega)} .$$

This operation requires the embedding relation

$$W^{\alpha,2} \hookrightarrow C \text{ for } \alpha > \frac{N}{2}.$$

As a consequence, the system (3.1–3.3) is locally well–posed. We report the following result, see e.g. Majda [12], Kleinerman and Majda [9].

Theorem 3.1. *Let $p \in C^\infty(0,\infty)$, $p'(\varrho) > 0$ for $\varrho > 0$, and*

$$\varrho_0 > 0, \varrho_0 \in W^{m,2}(\Omega), \ \mathbf{u}_0 \in W^{m,2}(\Omega; R^N) \textit{ for } m > \left[\frac{N}{2}\right] + 1.$$

Then there exists a positive time $T > 0$ such that problem (3.1–3.3) admits a strong solution $[\varrho, \mathbf{u}]$ in $(0,T) \times \Omega$, unique in the class

$$\varrho \in C([0,T]; W^{m,2}(\Omega)), \ \partial_t \varrho \in C([0,T]; W^{m-1,2}(\Omega)),$$
$$\mathbf{u} \in C([0,T]; W^{m,2}(\Omega; R^N)), \ \partial_t \mathbf{u} \in C([0,T]; W^{m-1,2}(\Omega; R^N)).$$

3.1.1 Possible blow-up of smooth solutions

Solutions of non-linear problems similar to the compressible Euler system (3.1) may develop singularities in a finite time no matter how smooth and/or small the data are. Note that "small" here means sufficiently close to the equilibrium state $\varrho = \text{const}$, $\mathbf{u} = 0$.

The classical example of blow up is provided by the 1-D Burgers equation

$$\partial_t V + V \partial_x V = 0,\ V(0) = V_0. \tag{3.6}$$

Indeed it is easy to observe that smooth solutions must satisfy

$$V(t, x + V_0(x)t) = V_0(x),\ t \geq 0. \tag{3.7}$$

Obviously, if $V_0(x_1) > V_0(x_2)$ for some $x_1 < x_2$, we get

$$x = x_1 + tV_0(x_1) = x_2 + tV_0(x_2) \text{ for } t = \frac{x_2 - x_1}{V_0(x_1) - V_0(x_2)} > 0;\ \text{whence } V(t,x) = \begin{cases} V_0(x_1) \\ V_0(x_2) \end{cases}$$

Now we show how this construction can be adapted to the compressible Euler system. We restrict ourselves to the case of one space dimension, where (3.1) can be rewritten in the Lagrangian (mass) coordinates as the so–called $p-$system , namely

$$\partial_t s - \partial_x v \ = \ 0 \tag{3.8}$$

$$\partial_t v - \partial_x P(s) \ = \ 0,\ P' > 0. \tag{3.9}$$

Indeed introducing new independent variables

$$[t, x] \mapsto [t, y(t,x) = \int_{-\infty}^{x} \varrho(t,z)\ \mathrm{d}z].$$

and the Lagrangian velocity $v = v(t, y)$ satisfying

$$v\left(t, \int_{-\infty}^{x} \varrho(t,z)\ \mathrm{d}z\right) = u(t,x);$$

whence

$$\partial_t v - \varrho u \partial_y v = \partial_t u,\ \varrho \partial_y v = \partial_x u,$$

and the resulting system of equations reads

$$\partial_t U - \partial_y v = 0, \tag{3.10}$$

and

$$\partial_t v + \partial_y p\left(\frac{1}{U}\right) = 0 \tag{3.11}$$

where $U = \frac{1}{\varrho}$ is the *specific volume*. Problem (3.10), (3.11) is called *p-system* and obviously can be recast in the form (3.8), (3.9).

The system (3.8), (3.9) can be written in terms of the so-called Riemann invariants, namely

$$\begin{aligned} \sqrt{P'(s)}\partial_t s - \sqrt{P'(s)}\partial_x v &= 0 \\ \partial_t v - \sqrt{P'(s)}\sqrt{P'(s)}\partial_x P(s) &= 0. \end{aligned}$$

Introducing $Z(s) = \int_0^s \sqrt{P'(z)}\ \mathrm{s}z$ we get

$$\begin{aligned} \partial_t Z - A(Z)\partial_x v &= 0 \\ \partial_t v - A(Z)\partial_x Z &= 0, \end{aligned}$$

or

$$\begin{aligned} \partial_t R_1 - A\left(\frac{R_1 + R_2}{2}\right)\partial_x R_1 &= 0 \\ \partial_t R_2 + A\left(\frac{R_1 + R_2}{2}\right)\partial_x R_2 &= 0, \end{aligned}$$

where we have introduced the Riemann invariants

$$R_1 = v + z,\ R_2 = Z - v.$$

Setting either R_1 or R_2 constant, we end up with a non-linear scalar equation of type (3.6).

We conclude that solutions of the compressible Euler system (3.1) may, and indeed do, develop singularities for a generic class of smooth initial data.

3.2 Weak solutions

To "resolve" the problem of global–in–time existence of solutions to the Euler system, we introduce the class of *weak solutions* The weak formulation of problem (3.1–3.3) reads:

- $$\left[\int_\Omega \varrho\varphi\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi]\ \mathrm{d}x\,\mathrm{d}t \tag{3.12}$$

  for any $0 \le \tau \le T$, and any $\varphi \in C^1([0,T]\times\Omega)$;

- $$\left[\int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho)\mathrm{div}_x\boldsymbol{\varphi}]\ \mathrm{d}x\,\mathrm{d}t \tag{3.13}$$

  for any $0 \le \tau \le T$, and any $\varphi \in C^1([0,T]\times\Omega; R^N)$.

As we shall see, however, this class is to large to ensure both existence and uniqueness of weak solutions, even if supplemented by suitable compatibility conditions. As a matter of fact, examples of non–uniqueness were known long time ago, see e.g. the monograph by Smoller Smoller [13]. To identify the physically admissible solutions, we introduce the energy

$$e(\varrho, \mathbf{m}) = e_{\rm kin}(\varrho, \mathbf{u}) + e_{\rm int}(\varrho),\ e_{\rm kin}(\varrho, \mathbf{m}) = \frac{1}{2}\varrho|\mathbf{u}|^2,\ e_{\rm int}(\varrho) = P(\varrho),\ P'(\varrho)\varrho - P(\varrho) = p(\varrho).$$

It is not difficult to check that (smooth) solutions to the Euler system (3.12), (3.13) satisfy the total energy balance

$$\partial_t e(\varrho, \mathbf{u}) + {\rm div}_x \left[(e(\varrho, \mathbf{u}) + p(\varrho))\, \mathbf{u} \right] \leq 0, \tag{3.14}$$

or at least its integrated form,

$$\frac{\rm d}{{\rm d}t} E(t) \leq 0,\ E \equiv \int_Q \left[\frac{1}{2}[\varrho |\mathbf{u}|^2 + P(\varrho) \right] \, {\rm d}x. \tag{3.15}$$

3.2.1 A short excursion in the world of weak solutions

As we have observed solutions of the compressible Euler system develop singularities – shock waves – in finite time no matter how smooth or small the initial data are. Accordingly, the concept of weak (distributional) solution (3.12), (3.13) has been introduced to study global–in–time behavior. The existence of weak solutions in the simplified monodimensional geometry has been established for a rather general class of initial data, see Chen and Perepelitsa [2], DiPerna [7], Lions, Perthame, and Souganidis [10], among others. More recently, the theory of convex integration has been used to show existence of weak solutions for $N = 2, 3$ again for a rather vast class of data, see Chiodaroli [3], De Lellis and Székelyhidi [5], Luo, Xie, and Xin [11]. A typical product of the theory reads as follows, see Chiodaroli [3], EF [8]:

Theorem 3.2. *Let $p \in C^3(0, \infty)$ and let $T > 0$ be given. Let the initial data belong to the class*

$$\varrho_0 \in C^3(\Omega),\ \varrho_0 > 0,\ \mathbf{u}_0 \in C^3(\Omega; R^N),\ \Omega = \mathcal{T}^N,\ N = 2, 3.$$

Then the compressible Euler system (3.1–3.3) admits infinitely many *weak solutions emanating from the initial state $[\varrho_0, \mathbf{u}_0]$ and belonging to the class*

$$\varrho \in L^\infty((0, T) \times \Omega),\ \varrho > 0,\ \mathbf{u} \in L^\infty((0, T) \times \Omega; R^N).$$

There is a refined version of the above result that shows the weak solutions may exhibit really "unexpected" behavior, see Abbatiello, EF [1].

Theorem 3.3. *Let $\{\tau_n\}_{n=1}^\infty \subset (0, T)$ be an arbitrary (dense) countable family of times. Let the initial data belong to the class*

$$\varrho_0 \in C(\Omega),\ \varrho_0 > 0,\ \mathbf{u}_0 \in C(\Omega; R^N),\ {\rm div}_x \mathbf{u}_0 \in C(\Omega), \Omega = \mathcal{T}^N,\ N = 2, 3.$$

Then the compressible Euler system (3.1–3.3) admits infinitely many *weak solutions emanating from the initial state* $[\varrho_0, \mathbf{u}_0]$ *such that the mapping*

$$t \mapsto [\varrho(t, \cdot), \mathbf{m} = \varrho \mathbf{u}(t, \cdot)]$$

is not strongly L^1*-continuous at any of the times* τ_n*.*

Uniqueness and stability with respect to the initial data in the framework of weak solutions is a more delicate issue. Apparently, the Euler system is ill–posed in the class of weak solutions and explicit examples of multiple solutions emanating from the same initial state have been constructed. The *admissible solutions* satisfy, in addition to (3.12), (3.13). Although the notion of admissible weak solution is definitely stronger than the mere weak solution, the following result holds, see Chiodaroli [3]:

Theorem 3.4. *Let* $\Omega = \mathcal{T}^N$*,* $N = 2, 3$*, and let* $p \in C^3(0, \infty)$*. Let*

$$\varrho_0 \in C^3(\Omega),\ \ \varrho_0 > 0 \ \text{in}\ \Omega,$$

be a given density distribution.

Then there exist $T > 0$ *and* u_0*,*

$$u_0 \in L^\infty(\Omega; R^N),$$

such that the corresponding initial-value problem for the Euler system admits infinitely many admissible weak solutions in $(0, T) \times \Omega$*.*

The locality in time in the above result has been removed by Luo, Xie, and Xin [11].

In the light of the above results it may still seem that one could save the game by taking smooth initial data and working in the class of admissible solutions for the Euler system. Unfortunately, even in this case the presence of “wild solutions” cannot be avoided, see Chiodaroli, DeLellis, Kreml [4].

Theorem 3.5. *Let* $\Omega = R^2$*,* $N = 2, 3$*,* $p \in C^3(0, \infty)$*.*

Then there exist such initial data

$$\varrho_0 \in W^{1,\infty}(\Omega),\ \ \mathbf{u}_0 \in W^{1,\infty}(\Omega; R^2),\ \ \varrho_0 > 0 \ \text{uniformly in}\ R^2$$

such that the corresponding initial-value problem for the Euler system admits infinitely many admissible weak solutions in $(0, \infty) \times \Omega$*.*

4 Oscillatory solutions to the compressible Euler system

In the light of the ill–posedness results listed in the previous part, the Euler system indeed admits oscillatory solutions at least in higher space dimension setting. We examine this phenomena in the following section. To begin, we introduce the momentum $\mathbf{m} = \varrho\mathbf{u}$ and rewrite the system in the conservative form:

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0, \tag{4.1}$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0. \tag{4.2}$$

4.1 Reformulation via Helmholtz decomposition

We start by introducing the momentum $\mathbf{m}$ along with its Helmholtz decomposition

$$\mathbf{m} = \mathbf{v} + \nabla_x \Phi, \ \mathrm{div}_x \mathbf{v} = 0, \ \int_\Omega \Phi \ \mathrm{d}x = 0.$$

Accordingly, the equation of continuity reads

$$\partial_t \varrho + \Delta_x \Phi = 0, \ \varrho(0, \cdot) = \varrho_0, \tag{4.3}$$

while the momentum equation takes the form

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Phi) \otimes (\mathbf{v} + \nabla_x \Phi)}{\varrho} \right) + \nabla_x \left(\partial_t \Phi + p(\varrho) \right) = 0, \ \mathbf{v}(0, \cdot) = \mathbf{v}_0. \tag{4.4}$$

4.1.1 Density ansatz

At this stage we can choosed and *arbitrary* (smooth) density profile $\varrho(t, x)$, $t \in [0, T]$, $x \in \Omega$ satisfying only

$$\varrho(0, \cdot) = \varrho_0, \ \partial_t \varrho(0, \cdot) = -\Delta_x \Phi_0, \ \varrho_0 \mathbf{u}_0 = \mathbf{v}_0 + \nabla_x \Phi_0,$$

and

$$0 < \underline{\varrho} \leq \varrho(t, x) \leq \overline{\varrho}, \ t \in [0, T], \ x \in \Omega.$$

Now, the potential Φ is uniquely determined by the elliptic problem

$$\Delta_x \Phi = -\partial_t \varrho.$$

4.1.2 Reformulation as abstract "Euler" system

From now on, we shall assume that $\mathbf{h} = \nabla_x \Phi$ and $H = \partial_t \Phi + p(\varrho)$ are given functions and look for solution of (4.4) that reads:

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{h}) \otimes (\mathbf{v} + \mathbf{h})}{\varrho} - \frac{1}{N} \frac{|\mathbf{v} + \mathbf{h}|^2}{\varrho} \mathbb{I} \right) + \nabla_x \left(H + \frac{1}{N} \frac{|\mathbf{v} + \mathbf{h}|^2}{\varrho} \right) = 0, \tag{4.5}$$

$$\mathrm{div}_x \mathbf{v} = 0, \tag{4.6}$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0. \tag{4.7}$$

4.1.3 Kinetic energy

We will look for solutions with prescribed kinetic energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = E,$$

where $E = E(t,x)$ is a given non–negative function. Going back to (4.5) we get

$$H + \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = H + \frac{2}{N}E;$$

whence the choice

$$E = \Lambda - \frac{N}{2}H,$$

where Λ is constant, reduces (4.5) to the "pressureless" equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v}+\mathbf{h})\otimes(\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho}\mathbb{I} \right) = 0.$$

Finally, we impose the impermeability conditions

$$\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

4.2 Abstract problem

For given functions h, $\varrho \geq 0$, and $E \geq 0$, our goal is to find a weak solution for the problem:

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v}+\mathbf{h})\otimes(\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho}\mathbb{I} \right) = 0, \tag{4.8}$$

$$\mathrm{div}_x \mathbf{v} = 0, \tag{4.9}$$

$$\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{4.10}$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0, \tag{4.11}$$

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = E. \tag{4.12}$$

The weak formulation of (4.8–4.12) reads:

- $$\mathbf{v} \in L^\infty((0,T)\times\Omega)\cap C_{\mathrm{weak}}([0,T];L^2(\Omega;R^N)),\ \ \mathbf{v}(0,\cdot) = \mathbf{v}_0; \tag{4.13}$$

- $$\int_0^T \int_\Omega \mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + \left(\frac{(\mathbf{v}+\mathbf{h}) \otimes (\mathbf{v}+\mathbf{h})}{\varrho} - \frac{1}{N} \frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t = 0 \tag{4.14}$$

  for all $\boldsymbol{\varphi} \in C^1_c((0,T) \times \Omega)$;

- $$\int_0^T \int_\Omega \mathbf{v} \cdot \nabla_x \varphi \ \mathrm{d}x \, \mathrm{d}t = 0 \tag{4.15}$$

  for any $\varphi \in C^1([0,T] \times \overline{\Omega})$;

- $$\frac{1}{2} \frac{|\mathbf{v}+\mathbf{h}|^2}{\varrho} = E \text{ a.a. in } (0,T) \times \Omega. \tag{4.16}$$

5 Oscillatory lemma

We reproduce the basic tool of the $L^\infty-$approach in fluid mechanics called *oscillatory lemma* due to DeLellis and Székelyhidi [5].

5.1 Formulation with constant coefficients

We consider a very particular case of problem (4.8–4.12) with a specific choice of parameters: $\Omega = [-1,1]^N$, $N = 2,3$, $\mathbf{h} = 0$, $\varrho = 1$, $E > 0$ - a positive constant. In addition, we consider the "do nothing" boundary conditions, meaning there is no restriction imposed on the test functions on $\partial\Omega$. The relevant weak formulation reads:

- $$\mathbf{v} \in L^\infty((0,T) \times \Omega) \cap C_{\rm weak}([0,T]; L^2(\Omega; R^N)), \ \mathbf{v}(0,\cdot) = \mathbf{v}_0; \tag{5.1}$$

- $$\int_0^T \int_\Omega \mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + \left(\mathbf{v} \otimes \mathbf{v} - \frac{1}{N} |\mathbf{v}|^2 \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t = 0 \tag{5.2}$$

  for all $\boldsymbol{\varphi} \in C^1_c([0,T] \times R^N)$;

- $$\int_0^T \int_\Omega \mathbf{v} \cdot \nabla_x \varphi \ \mathrm{d}x \, \mathrm{d}t = 0 \tag{5.3}$$

  for any $\varphi \in C^1_c([0,T] \times R^N)$;

- $$\frac{1}{2} |\mathbf{v}|^2 = E \text{ a.a. in } (0,T) \times \Omega. \tag{5.4}$$

5.2 Subsolutions

We introduce a the set X of subsolutions $[\mathbf{v}, \mathbb{U}]$ satisfying

- $$\mathbf{v} \in C^1([0,T] \times \overline{\Omega}; R^N),\ \mathbb{U} \in C^1\left([0,T) \times \overline{\Omega}; R^{N\times N}_{\mathrm{sym},0}\right); \tag{5.5}$$

- $$\int_0^T \int_\Omega \mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + \mathbb{U} : \nabla_x \boldsymbol{\varphi}\ \mathrm{d}x\, \mathrm{d}t = 0,\ \mathbf{v}(0,\cdot) = \mathbf{v}_0 \tag{5.6}$$

  for all $\boldsymbol{\varphi} \in C^1_c([0,T] \times R^N)$;

- $$\int_0^T \int_\Omega \mathbf{v} \cdot \nabla_x \varphi\ \mathrm{d}x\, \mathrm{d}t = 0 \tag{5.7}$$

  for any $\varphi \in C^1_c([0,T] \times R^N)$;

- $$\frac{1}{2}|\mathbf{v}|^2 \leq \frac{N}{2}\lambda_{\max}\left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U}\right] < E \text{ in } [0,T] \times \overline{\Omega}. \tag{5.8}$$

5.3 Oscillatory lemma

The basic result reads as follows:

Lemma 5.1. *Let $Q = (-1,1) \times (-1,1)^N$, $N = 2,3$, and let $\mathbf{h} \in R^N$, $\mathbb{V} \in R^{N\times N}_{\mathrm{sym},0}$, $E > 0$ be constant satisfying*

$$\frac{1}{2}|\mathbf{h}|^2 \leq \frac{N}{2}\lambda_{\max}\left[\mathbf{h} \otimes \mathbf{h} - \mathbb{V}\right] < E \leq \overline{E}.$$

Then there are sequences

$$\mathbf{v}_n \in C^\infty_c(Q; R^N),\ \mathbb{U}_n \in C^\infty_c(Q; R^N),\ n = 1,2,\ldots$$

such that

$$\partial_t \mathbf{v}_n + \mathrm{div}_x \mathbb{U}_n = 0,\ \mathrm{div}_x \mathbf{v}_n = 0,$$

$$\frac{1}{2}|\mathbf{h} + \mathbf{v}_n|^2 \leq \frac{N}{2}\lambda_{\max}\left[(\mathbf{h} + \mathbf{v}_n) \otimes (\mathbf{h} + \mathbf{v}_n) - \mathbb{V} - \mathbb{U}_n\right] < E,\ n = 1,2,\ldots, \tag{5.9}$$

$$\mathbf{v}_n \to 0 \text{ weakly in } L^2(Q; R^N) \text{ as } n \to \infty,$$

and

$$\liminf_{n\to\infty} \frac{1}{|Q|} \int_Q |\mathbf{v}_n|^2\ \mathrm{d}y \geq c(\overline{E}) \left(E - \frac{1}{2}|\mathbf{h}|^2\right)^2. \tag{5.10}$$

Remark 5.2. As the constant fields $\mathbf{h}$ and $\mathbb{V}$ obviously solve the exact problem, the oscillatory lemma asserts the existence of compactly supported perturbation such that:

- the perturbed solution remain in the set of subsolutions;
- the oscillatory increment of the L^2-norm of the field $\mathbf{v}_n$ expressed through (5.10) "shifts" the subsolution to the neighborhood of the boundary of the set

$$\frac{1}{2}|\mathbf{v}|^2 = \frac{N}{2}\lambda_{\max}\left[\mathbf{v}\otimes\mathbf{v} - \mathbb{U}\right] = E.$$

The rest of this section is devoted to the proof of Lemma 5.1. We follow the arguments of De Lellis and Székelyhidi [5].

5.3.1 Geometric setting

We introduce the set

$$\mathfrak{C}(E) = \left\{ \mathbf{h}\in R^N,\ \mathbb{V}\in R^{N\times N}_{\mathrm{sym},0} \ \Big|\ \frac{N}{2}\lambda_{\max}\left[\mathbf{h}\otimes\mathbf{h}-\mathbb{V}\right] < E \right\}.$$

We list several facts that can be verified by direct computation:

-
$$[\mathbf{h},\mathbb{V}] \mapsto \lambda_{\max}\left[\mathbf{h}\otimes\mathbf{h}-\mathbb{V}\right] \tag{5.11}$$
is a convex function on $R^N\times R^{N\times N}_{\mathrm{sym},0}$;
-
$$\frac{N}{2}\lambda_{\max}\left[\mathbf{h}\otimes\mathbf{h}-\mathbb{V}\right] \geq \frac{1}{2}|\mathbf{h}|^2 \tag{5.12}$$
$$\frac{N-1}{2}\lambda_{\max}\left[\mathbf{h}\otimes\mathbf{h}-\mathbb{V}\right] \geq \frac{1}{2}\|\mathbb{V}\|^2 \tag{5.13}$$
for any $\mathbf{v}\in R^N$, $\mathbb{V}\in R^{N\times N}_{\mathrm{sym},0}$, moreover,
$$\frac{N}{2}\lambda_{\max}\left[\mathbf{h}\otimes\mathbf{h}-\mathbb{V}\right] = \frac{1}{2}|\mathbf{h}|^2 \ \Leftrightarrow\ \mathbf{h}\otimes\mathbf{h} - \frac{1}{N}|\mathbf{h}|^2\mathbb{I} = \mathbb{V}. \tag{5.14}$$

Indeed $\mathfrak{C}(E)$ is convex as the function

$$[\mathbf{v},\mathbb{V}] \mapsto (\mathbf{h}\otimes\mathbf{h}) : (\xi\otimes\xi) - \mathbb{V} : (\xi\otimes\xi) = |\left\langle \mathbf{h};\xi\right\rangle|^2 - \mathbb{V} : (\xi\otimes\xi)$$

is convex for any fixed $\xi\in R^N$, $|\xi| = 1$ and therefore its supremum over all ξ is convex. The remaining relations (5.12–5.14) can be checked by direct computation.

In accordance with (5.11–5.13), the set $\mathfrak{C}(E)$ is an bounded open convex set that is non–void as soon as $E > 0$. Moreover, it can be shown that

$$\mathcal{E}[\mathfrak{C}(E)] = \left\{ \mathbf{h}\in R^N,\ \mathbb{V} = \mathbf{h}\otimes\mathbf{h} - \frac{1}{N}|\mathbf{h}|^2\mathbb{I} \ \Big|\ \frac{1}{2}|\mathbf{h}|^2 = E \right\}$$

contains the set of all extreme points of the closure of the set $\mathfrak{C}(E)$. To see this, it is enough to consider the case $E = 1$. Supposing it is not the case, we find a vector $\mathbf{h} \in R^N$ such that

$$\frac{1}{2}|\mathbf{h}|^2 = 1, \ \frac{N}{2}\lambda_{\max}\left\{\mathbf{h}\otimes\mathbf{h} - \mathbb{V}\right\} < 1.$$

Moreover, rotating the coordinate system, we may assume that

$$\mathbf{h}\otimes\mathbf{h} - \mathbb{V} = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3], \ \lambda_1 \geq \lambda_2 \geq \lambda_3,$$

where $\lambda_1 \leq \frac{1}{N}$. Now, if $\mathbf{h} = \sum_{i=1}^N h_i e_i$, we consider

$$\overline{\mathbf{h}} = e_3, \ \overline{\mathbb{V}} = \sum_{i=1}^{N-1} h_i \left(e_i \otimes e_3 + e_3 \otimes e_i\right).$$

By direct computation, we show that

$$(\mathbf{h} + \xi\overline{\mathbf{h}}) \otimes (\mathbf{h} + \xi\overline{\mathbf{h}}) - \mathbb{V} - \xi\overline{\mathbb{V}} \in \mathfrak{C}(1)$$

for ξ small enough.

By Krein–Milman theorem

$$\mathfrak{C}(E) = \mathrm{conv}[\mathcal{E}[\mathfrak{C}(E)]]. \tag{5.15}$$

Our next goal is to show that for any $[\mathbf{h}, \mathbb{V}]$ lying in the interior of $\mathfrak{C}(E)$, there is a sufficiently long segment parallel to a difference of two boundary elements with the center at $[\mathbf{h}, \mathbb{V}]$.

Lemma 5.3. *Let $E > 0$ and $[\mathbf{h}, \mathbb{V}] \in \mathfrak{C}(E)$.*

Then there exists $\mathbf{a}$, $\mathbf{b}$ enjoying the following properties:

- $$\frac{1}{2}|\mathbf{a}|^2 = \frac{1}{2}|\mathbf{b}|^2 = E, \ |\mathbf{a} \pm \mathbf{b}| > 0; \tag{5.16}$$

- *there is $L > 0$ such that*

$$[\mathbf{h} + \lambda(\mathbf{a} - \mathbf{b}), \mathbb{V} + \lambda\left(\mathbf{a}\otimes\mathbf{a} - \mathbf{b}\otimes\mathbf{b}\right)] \in \mathfrak{C}(A),$$

$$\mathrm{dist}\left[\ [\mathbf{h} + \lambda(\mathbf{a} - \mathbf{b}), \mathbb{V} + \lambda\left(\mathbf{a}\otimes\mathbf{a} - \mathbf{b}\otimes\mathbf{b}\right)]\,; \partial\mathfrak{C}(A)\ \right] \geq \frac{1}{2}\mathrm{dist}\left[\ [\mathbf{h}, \mathbb{V}]\,; \partial\mathfrak{C}(A)\ \right] \tag{5.17}$$

for all $-L \leq \lambda \leq L$;

- $$L|\mathbf{a} - \mathbf{b}| \geq C(N)\frac{1}{\sqrt{E}}\left(E - \frac{1}{2}|\mathbf{h}|^2\right). \tag{5.18}$$

Proof. The dimension of the space $R^N \times R^{N\times N}_{\mathrm{sym},0}$ is

$$n = \frac{N(N+3)}{2} - 1.$$

As the set $\mathfrak{C}(A)$ is convex with the extreme points given by (5.15), Caratheodory's theorem yields

$$[\mathbf{h}, \mathbb{V}] = \sum_{i=1}^{n} \alpha_i \left[\mathbf{a}_i, \mathbf{a}_i \otimes \mathbf{a}_i - \frac{1}{N} E\mathbb{I} \right],\ \frac{1}{2}|\mathbf{a}_i| = E,\ \sum_{i=1}^{n} \alpha_i = 1,\ \alpha_1 \geq \alpha_2 \dots \alpha_n \geq 0.$$

Moreover, shifting $[\mathbf{h}, \mathbb{V}]$ slightly as the case may be, we may assume

$$|\mathbf{a}_i \pm \mathbf{a}_j| > 0 \text{ whenever } i \neq j.$$

Now, consider the segment

$$[\mathbf{h} + \lambda[\mathbf{a}_j - \mathbf{a}_1], \mathbf{V} + \lambda[\mathbf{a}_j \otimes \mathbf{a}_j - \mathbf{a}_1 \otimes \mathbf{a}_1]]\ \ \lambda \in [-\alpha_j, \alpha_j],\ j > 1. \tag{5.19}$$

Since

$$[\mathbf{h}, \mathbb{V}] = \sum_{i=1}^{n} \alpha_i \left[\mathbf{a}_i, \mathbf{a}_i \otimes \mathbf{a}_i - \frac{1}{N} E\mathbb{I} \right],$$

and $\alpha_1 \geq \alpha_j$, $j > 1$, the endpoints of the segment represent a convex combination of $[\mathbf{a}_i, \mathbf{a}_i \otimes \mathbf{a}_i - \frac{1}{N} E\mathbb{I}]$; whence they belong to $\mathfrak{C}(A)$, and, in view of convexity of this set, the whole segment (5.19) is contained in $\mathfrak{C}(A)$.

Finally, we chose $\mathbf{a}_j$, $j > 1$ such that

$$\alpha_j |\mathbf{a}_j - \mathbf{a}_1| \geq \alpha_k |\mathbf{a}_k - \mathbf{a}_1| \text{ for all } k > 1, \tag{5.20}$$

and set

$$L = \frac{1}{2}\alpha_j,\ \ \mathbf{a} = \mathbf{a}_1,\ \ \mathbf{b} = \mathbf{a}_j.$$

One can easily check, using convexity, that (5.17) holds. Thus it remains to verify (5.18). To see (5.18) we realize that

$$\mathbf{h} = \sum_{i=1}^{n} \alpha_i \mathbf{a}_i;$$

whence, by virtue of (5.20),

$$|\mathbf{h} - \mathbf{a}| \leq \left| \sum_{i=2}^{n} \alpha_i (\mathbf{a}_i - \mathbf{a}) \leq n \right| \leq n\alpha_j |\mathbf{a}_j - \mathbf{a}_1| = 2nL|\mathbf{a} - \mathbf{b}|.$$

Finally,

$$\begin{aligned} 2E - |\mathbf{h}|^2 = \left(\sqrt{2E} + |\mathbf{h}|\right)\left(\sqrt{2E} - |\mathbf{h}|\right) &\leq 2\sqrt{2E}\left(\sqrt{2E} - |\mathbf{h}|\right) \\ &= 2\sqrt{2E}\left(|\mathbf{a}_1| - |\mathbf{h}|\right) \leq 4\sqrt{2E} nL|\mathbf{a} - \mathbf{b}|. \end{aligned}$$

□

5.3.2 PDE setting

Our goal is to solve the system of PDE's in the form

$$\mathrm{div}_x \mathbf{v} = 0,\ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0$$

that can be recast in a concise form as

$$\mathrm{DIV}_{t,x} \left[\begin{array}{cc} 0 & \mathbf{v} \\ \mathbf{v} & \mathbb{U} \end{array} \right] = 0,\ \mathrm{DIV}_{t,x} \mathbf{V} = \partial_t V_0 + \partial_{x_1} V_1 + \cdots + \partial_{x_N} V_N.$$

Following De Lellis and Székelyhidi [5], we look for a differential operator

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial) \in R^{(N+1)\times(N+1)}_{0,\mathrm{sym}},\ \partial = (\partial_t, \partial_{x_1}, \dots, \partial_{x_n})$$

enjoying the following properties:

- if $\phi \in C^\infty_c(R \times R^N)$, then

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial)[\phi] = \left[\begin{array}{cc} 0 & \mathbf{w} \\ \mathbf{w} & \mathbb{H} \end{array} \right] \text{ satisfies } \partial_t \mathbf{w} + \mathrm{div}_x \mathbb{H} = 0,\ \mathrm{div}_x \mathbf{w} = 0; \tag{5.21}$$

- there exists a vector $\eta_{\mathbf{a},\mathbf{b}} \in R^{N+1}$ such that

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial)[\psi([t,\mathbf{x}]\cdot \eta_{\mathbf{a},\mathbf{b}})] = \psi'''([t,\mathbf{x}]\cdot\eta_{\mathbf{a},\mathbf{b}}) \left[\begin{array}{cc} 0 & \mathbf{a}-\mathbf{b} \\ \mathbf{a}-\mathbf{b} & \mathbf{a}\otimes\mathbf{a} - \mathbf{b}\otimes\mathbf{b} \end{array} \right] \tag{5.22}$$

  for any $\psi \in C^\infty(R)$.

It is more convenient to work in the Fourier variables, and, accordingly, to look for the mapping

$$\xi = [\xi_0, \xi_1, \dots, \xi_n] \mapsto \mathbb{A}_{\mathbf{a},\mathbf{b}}(\xi) \in R^{(N+1)\times(N+1)}_{0,\mathrm{sym}}.$$

It was shown in [5] that the ansatz:

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\xi) = \frac{1}{2}\left((\mathbb{R}\cdot\xi) \otimes (\mathbb{Q}(\xi)\cdot\xi) + (\mathbb{Q}(\xi)\cdot\xi)\otimes(\mathbb{R}\cdot\xi) \right) \tag{5.23}$$

where

$$\mathbb{Q} = \xi\otimes e_0 - e_0 \otimes \xi,\ \mathbb{R} = ([0,\mathbf{a}]\otimes[0,\mathbf{b}]) - ([0,\mathbf{b}]\otimes[0,\mathbf{a}]),$$

and

$$e_0 = [1,0,\dots,0],\ \mathbf{a},\mathbf{b}\in R^N,\ \frac{1}{2}|\mathbf{a}|^2 = \frac{1}{2}|\mathbf{b}|^2 = E > 0,\ |\mathbf{a}\pm\mathbf{b}| > 0,$$

indeed gives rise to (5.21), (5.22), with

$$\eta_{\mathbf{a},\mathbf{b}} = -\frac{1}{(|\mathbf{a}||\mathbf{b}| + \mathbf{a}\cdot\mathbf{b})^{2/3}} \Big[[0,\mathbf{a}] + [0,\mathbf{b}] - (|\mathbf{a}||\mathbf{b}| + \mathbf{a}\cdot\mathbf{b})\, e_0 \Big]. \tag{5.24}$$

To illuminate the main ideas, we restrict ourselves to the case $N = 2$. Moreover, we look for $\mathbb{A}_{\mathbf{a},\mathbf{b}}$ being a polynomial with constant coefficients. Accordingly,

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\xi) = \begin{bmatrix} 0 & A^{0,1}(\xi) & A^{0,2}(\xi) \\ A^{1,0}(\xi) & A^{1,1}(\xi) & A^{1,2}(\xi) \\ A^{2,0}(\xi) & A^{2,1}(\xi) & A^{2,2}(\xi) \end{bmatrix}$$

Due to the symmetry requirement, it is enough to determine the coefficients

$$A^{1,1}(\xi) = -A^{2,2}(\xi),\ A^{0,1}(\xi) = A^{1,0}(\xi),\ A^{0,2}(\xi) = A^{2,0}(\xi),\ \text{and } A^{1,2}(\xi) = A^{2,1}(\xi).$$

Moreover, introducing

$$B = \frac{A^{0,1}}{A^{1,1}},\ C = \frac{A^{0,2}}{A^{1,1}},\ D = \frac{A^{1,2}}{A^{1,1}}$$

we obtain

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\xi) = \begin{bmatrix} 0 & B(\xi) & C(\xi) \\ B(\xi) & 1 & D(\xi) \\ C(\xi) & D(\xi) & -1 \end{bmatrix} A^{1,1}(\xi).$$

In view of (5.2), we obtain a linear system

$$\xi_1 B + \xi_2 C = 0,\ \xi_0 B + \xi_1 + \xi_2 D = 0,\ \xi_0 C + \xi_1 D - \xi_2 = 0$$

that admits a (unique) solution for $\xi_0\xi_1\xi_2 \neq 0$, namely,

$$B(\xi) = \frac{1}{2}\frac{\xi_2^2 - \xi_1^2}{\xi_0\xi_1} - \frac{\xi_2^2}{\xi_0\xi_1},\ C(\xi) = \frac{\xi_2}{\xi_0} - \frac{1}{2}\frac{\xi_2^2 - \xi_1^2}{\xi_0\xi_2},\ D(\xi) = \frac{1}{2}\frac{\xi_2^2 - \xi_1^2}{\xi_1\xi_2}$$

Consequently, the only coefficient depending on $\mathbf{a}$, $\mathbf{b}$ is

$$A^{1,1}_{\mathbf{a},\mathbf{b}}(\xi) = A_{\mathbf{a},\mathbf{b}}\ \xi_0\xi_1\xi_2;$$

whence

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\xi) = A_{\mathbf{a},\mathbf{b}} \begin{bmatrix} 0 & -\frac{\xi_2}{2}(\xi_1^2 + \xi_2^2) & \frac{\xi_1}{2}(\xi_1^2 + \xi_2^2) \\ -\frac{\xi_2}{2}(\xi_1^2 + \xi_2^2) & \xi_0\xi_1\xi_2 & \frac{\xi_0}{2}(\xi_2^2 - \xi_1^2) \\ \frac{\xi_1}{2}(\xi_1^2 + \xi_2^2) & \frac{\xi_0}{2}(\xi_2^2 - \xi_1^2) & -\xi_0\xi_1\xi_2 \end{bmatrix}. \tag{5.25}$$

This is the only form of the operator $\mathbb{A}_{\mathbf{a},\mathbf{b}}$ compatible with (5.21).

Finally, we compute

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial)\psi\left([t,x]\cdot\eta\right) = A_{\mathbf{a},\mathbf{b}}\psi'''\left([t,x]\cdot\eta\right) \begin{bmatrix} 0 & -\frac{\eta_2}{2}(\eta_1^2 + \eta_2^2) & \frac{\eta_1}{2}(\eta_1^2 + \eta_2^2) \\ -\frac{\eta_2}{2}(\eta_1^2 + \eta_2^2) & \eta_0\eta_1\eta_2 & \frac{\eta_0}{2}(\eta_2^2 - \eta_1^2) \\ \frac{\eta_1}{2}(\eta_1^2 + \eta_2^2) & \frac{\eta_0}{2}(\eta_2^2 - \eta_1^2) & -\eta_0\eta_1\eta_2 \end{bmatrix}$$

To comply with (5.22), we have to find $A = A_{\mathbf{a},\mathbf{b}}$ and $\eta = \eta_{\mathbf{a},\mathbf{b}}$ so that

$$\frac{A}{2}(\eta_1^2 + \eta_2^2)\,[-\eta_2, \eta_1] = \mathbf{a} - \mathbf{b},$$
$$A\eta_0 \left[\begin{array}{cc} \eta_1\eta_2 & \frac{1}{2}(\eta_2^2 - \eta_1^2) \\ \frac{1}{2}(\eta_2^2 - \eta_1^2) & -\eta_1\eta_2 \end{array} \right] = \mathbf{a}\otimes\mathbf{a} - \mathbf{b}\otimes\mathbf{b},\ \frac{1}{2}|\mathbf{a}|^2 = \frac{1}{2}|\mathbf{b}|^2 = E.$$

Seeing that the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are orthogonal, we can choose

$$[\eta_1, \eta_2] = \Lambda(\mathbf{a} + \mathbf{b})$$

and comptute Λ, A and η_0 in accordance with (5.24).

5.3.3 Proof of oscillatory lemma

We are ready to prove Lemma 5.1.

Step 1
Given $[\mathbf{h}, \mathbb{V}] \in \mathfrak{C}(E)$, we identify the vectors $\mathbf{a}$, $\mathbf{b}$ satisfying (5.16–5.18).

Step 2
$\mathbf{a}$, $\mathbf{b}$ being fixed, we consider the operator $\mathbb{A}_{\mathbf{a},\mathbf{b}}$ and the vector $\eta_{\mathbf{a},\mathbf{b}}$ as in (5.23–5.24).

Step 3
Let $\varphi \in C_c^\infty(Q)$ such that

$$0 \leq \varphi \leq 1,\ \varphi(t,x) = 1 \text{ whenever } -\frac{1}{2} \leq t \leq \frac{1}{2},\ -\frac{1}{2} \leq x_j \leq \frac{1}{2},\ j = 1, \dots, N.$$

The vectors $\mathbf{v}_n$, $\mathbb{U}_n$ can be taken in the form

$$\left[\begin{array}{cc} 0 & \mathbf{v}_n \\ \mathbf{v}_n & \mathbb{U}_n \end{array} \right] = \mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial) \left[\varphi \frac{L}{n^3} \cos\left(n[t,x]\cdot\eta_{\mathbf{a},\mathbf{b}}\right) \right],$$

where L is the constant form (5.17), (5.18). In accordance with (5.21), the functions $\mathbf{v}_n$, $\mathbb{U}_n$ satisfy

$$\partial_t \mathbf{v}_n + \mathrm{div}_x \mathbb{U}_n = 0,\ \mathrm{div}_x \mathbf{v}_n = 0$$

and are compactly supported in Q. Moreover, in view of (5.22) and the fact that $\mathbb{A}_{\mathbf{a},\mathbf{b}}$ is a differential operator of third order, we get

$$\mathbb{A}_{\mathbf{a},\mathbf{b}}(\partial) \left[\varphi \frac{L}{n^3} \cos\left(n[t,x]\cdot\eta_{\mathbf{a},\mathbf{b}}\right) \right] = \varphi \sin\left(n[t,x]\cdot\eta_{\mathbf{a},\mathbf{b}}\right) L \left[\begin{array}{cc} 0 & (\mathbf{a}-\mathbf{b}) \\ (\mathbf{a}-\mathbf{b}) & \mathbf{a}\otimes\mathbf{a} - \mathbf{b}\otimes\mathbf{b} \end{array} \right] + \frac{1}{n} R_n$$

with $|R_n|$ uniformly bounded for $n \to \infty$. This yields (5.9) for n large enough, and, in view of (5.18), relation (5.10) follows. We have proved Lemma 5.1.

5.4 Continuous basic functions

We conclude this part by extending validity of the oscillatory lemma to continuous functions $\mathbf{h} : Q \to R^N$ and $\mathbb{V} : Q \to R^{N\times N}_{0,\mathrm{sym}}$. To this end we first observe that, by simple time and space rescaling, that Lemma 5.1 holds for any

$$Q = (t_1, t_2) \times \Pi_{i=1}^N (a_i, b_i).$$

Next, as the oscillating functions have compact support, the conclusion of Lemma 5.1 remains valid if

$$Q = \cup_{j=1}^m \overline{Q}^j,\ Q^j = (t_1^j, t_2^j) \times \Pi_{i=1}^N (a_i^j, b_i^j),\ Q^j \cap Q^k = \emptyset \text{ whenever } j \neq k,$$

and the basic functions $\mathbf{h}$, $\mathbb{V}$ are constant on Q^j.

Finally, approximating continuous function by a piecewise constant one we obtain:

Lemma 5.4. *Let $Q = (0, T) \times \Omega$, $N = 2, 3$, and let $\mathbf{h} \in C(\overline{Q}; R^N)$, $\mathbb{V} \in C(\overline{Q}; R^{N\times N}_{\mathrm{sym},0})$, $E \in C(\overline{Q})$ be continuous functions satisfying*

$$\frac{1}{2}|\mathbf{h}|^2 \leq \frac{N}{2}\lambda_{\max}\left[\mathbf{h}\otimes\mathbf{h} - \mathbb{V}\right] < E \leq \overline{E} \ \text{in}\ \overline{Q}.$$

Then there are sequences

$$\mathbf{v}_n \in C^\infty_c(Q; R^N),\ \mathbb{U}_n \in C^\infty_c(Q; R^N),\ n = 1, 2, \dots$$

such that

$$\partial_t \mathbf{v}_n + \mathrm{div}_x \mathbb{U}_n = 0,\ \mathrm{div}_x \mathbf{v}_n = 0,$$

$$\frac{1}{2}|\mathbf{h} + \mathbf{v}_n|^2 \leq \frac{N}{2}\lambda_{\max}\left[(\mathbf{h}+\mathbf{v}_n)\otimes(\mathbf{h}+\mathbf{v}_n) - \mathbb{V} - \mathbb{U}_n\right] < E,\ n = 1, 2, \dots,$$

$$\mathbf{v}_n \to 0 \ \text{weakly in}\ L^2(Q; R^N) \ \text{as}\ n \to \infty,$$

and

$$\liminf_{n\to\infty} \int_Q |\mathbf{v}_n|^2 \ \mathrm{d}y \geq c(\overline{E}) \int_Q \left(E - \frac{1}{2}|\mathbf{h}|^2\right)^2 \ \mathrm{d}y$$

6 Ill posedness of the Euler system in the space dimension $N = 2, 3$

As a corollary of the existence of oscillatory subsolutions we show that the compressible Euler system is basically ill posed in the class of weak solutions. The basic result in this direction concerns the "pressureless" incompressible Euler system:

$$\begin{aligned} \mathrm{div}_x \mathbf{v} = 0, \\ \partial_t \mathbf{v} + \mathrm{div}_x \left(\mathbf{v}\otimes\mathbf{v} - \frac{1}{N}|\mathbf{v}|^2\right) = 0, \end{aligned} \tag{6.1}$$

with the boundary conditions

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{6.2}$$

and prescribed energy

$$\frac{1}{2}|\mathbf{v}|^2 = E \text{ a.a. in } (0,T) \times \Omega. \tag{6.3}$$

Note that if $E > 0$ is constant, then the equation of continuity reduces to

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\mathbf{v} \otimes \mathbf{v} \right) = 0;$$

whence $\mathbf{v}$ solves the *compressible* Euler system (3.1) with $\varrho \equiv 1$!

6.1 Infinitely many weak solutions to problem (6.1–6.3)

Our goal will be to show that given a continuously differentiable solenoidal field $\mathbf{v}_0$, problem (6.1–6.3) possesses infinitely many weak solutions starting from $\mathbf{v}_0$ provided $E > 0$ is chosen large enough. The main tool is the oscillatory lemma proved in the previous section.

6.1.1 The set of subsolutions

We consider the set of subsolutions introduced in (5.5–5.8) with *constant* energy $E > 0$. Let the initial velocity $\mathbf{v}_0$ be a smooth vector field, $\mathrm{div}_x \mathbf{v}_0 = 0$, $\mathbf{v}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$. Taking $\mathbf{v} \equiv \mathbf{v}_0$ and $\mathbb{U} \equiv= 0$ we observe easily that (5.5–5.7) are satisfied. In addition, (5.8) holds provided E is large enough. We conclude that the subsolution set is non–empty.

Next, we observe that for E bounded, the set of function (subsolutions) satisfying (5.5–5.7) is **(i)** bounded in $L^\infty((0,T) \times \Omega)$, **(ii)** a subset of $C_{\mathrm{weak}}([0,T]; L^2(\Omega; R^N))$. Consequently, it is metrizable by the metrics

$$d(\mathbf{u}, \mathbf{v}) = \sum_{k=1}^{\infty} \frac{1}{k^2} \frac{| < \mathbf{u} - \mathbf{v}; \phi_k > |}{1 + | < \mathbf{u} - \mathbf{v}; \phi_k > |}, \ \phi_k \text{ a basis of } L^2(\Omega; R^N).$$

We consider the topological metric space X - the completion of the set of subsolutions with respect to the metrics d. It is easy to check that:

- for any $\mathbf{v} \in X$, there is $\mathbb{U} \in L^\infty((0,T) \times \Omega; R^{N \times N}_{0,\mathrm{sym}})$ such that (5.5–5.7) are satisfied;
- $$\frac{1}{2}|\mathbf{v}|^2 \leq \frac{N}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right] \leq E. \tag{6.4}$$

Note that (6.4) follows from (5.8) and the fact that

$$(\mathbf{v}, \mathbb{U}) \mapsto \frac{N}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right]$$

is convex.

6.1.2 A Baire category argument

We introduce a concave functional

$$I(\mathbf{v}) = \int_0^T \int_\Omega \left(E - \frac{|\mathbf{v}|^2}{2} \right) \ \mathrm{d}x \, \mathrm{d}t$$

defined on the space X. As X is a complete metric space, the points of continuity of I form a residual set, in particular they are dense in X.

Our final goal is to show that if $\mathbf{v}$ is a point of continuity of I, then $I(\mathbf{v}) = 0$, or, equivalently,

$$\frac{1}{2}|\mathbf{v}|^2 = E \text{ a.a. in } (0,T) \times \Omega, \ \mathbb{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{N}|\mathbf{v}|^2 \mathbb{I}.$$

Suppose that $I(\mathbf{v}) > 0$ for some $\mathbf{v} \in X$. Then there is a sequence of subsolutions $\{\mathbf{v}_n\}_{n=1}^\infty$, with the corresponding fluxes $\{\mathbb{U}_n\}_{n=1}^\infty$ such that:

$$\mathbf{v}_n \to \mathbf{v} \text{ in } X, \ I(\mathbf{v}_n) \to I(\mathbf{v}) \geq \delta > 0 \text{ as } n \to \infty.$$

By virtue of Oscillatory Lemma (Lemma 5.4), we may construct a sequence

$$\mathbf{v}_{m,n} \to \mathbf{v}_n \text{ in } X \text{ as } m \to \infty$$

satisfying

$$\liminf_{m \to \infty} \int_0^T \int_\Omega \frac{1}{2} \frac{|\mathbf{v}_n + \mathbf{v}_{m,n}|^2}{2} \ \mathrm{d}x \, \mathrm{d}t \geq \int_0^T \int_\Omega \frac{|\mathbf{v}_n|^2}{2} \ \mathrm{d}x \, \mathrm{d}t + c_1(\Omega, E) \int_0^T \int_\Omega \left(E - \frac{1}{2}|\mathbf{v}_n|^2 \right)^2 \ \mathrm{d}x \, \mathrm{d}t$$
$$\geq \int_0^T \int_\Omega \frac{|\mathbf{v}_n|^2}{2} \ \mathrm{d}x \, \mathrm{d}t + c_2(\Omega, E) I(\mathbf{v}_n)^2 \geq \int_0^T \int_\Omega \frac{|\mathbf{v}_n|^2}{2} \ \mathrm{d}x \, \mathrm{d}t + c_3(\Omega, E)\delta^2.$$

Consequently, if $\delta > 0$, we can construct a sequence $\{\mathbf{w}_n\}_{n=1}^\infty$,

$$\mathbf{w}_n \to \mathbf{v} \text{ in } X, \ \limsup_{n \to \infty} I(\mathbf{w}_m) < I(\mathbf{v}),$$

meaning $\mathbf{v}$ cannot be a point of continuity of I.

As each point of continuity of I represent a weak solution of (6.1), we obtain the following result.

Theorem 6.1. *Let $\Omega \subset R^N$, $N = 2, 3$ be a bounded smooth domain. Let $\mathbf{v}_0 \in C^1(\overline{\Omega}; R^N)$ be given such that*

$$\mathrm{div}_x \mathbf{v}_0 = 0, \ \mathbf{v}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Then there exists $E_0 > 0$ such that for any $E \geq E_0$, the problem (6.1–6.3) admits inifinitely many weak solutions satisfying

$$\frac{1}{2}|\mathbf{v}|^2 = E \text{ for a.a. } (t,x) \in (0,T) \times \Omega. \tag{6.5}$$

Note carefully that solutions $\mathbf{v}$ obtained in Theorem 6.1 solve the compressible Euler system (3.1 – 3.3) with constant density $\varrho \equiv 1$.

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