Partial differential equations describing the motion of compressible, viscous, and heat conducting fluids

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Main topics addressed: Weak vs strong solutions

Lecture I

1. Field equations describing the motion of a compressible and/or heat conducting fluid
2. Strong vs weak solutions
3. Dissipative solutions
4. Conditional regularity in the viscous case
5. Weak solutions in the inviscid case
6. Method of convex integration

Lecture II

1. Singular limits for a compressible, viscous and/or heat conducting fluid equations
2. Low Mach number limit
3. High Reynolds/Peclet number limit
4. Methods based on relative entropy
5. Propagation of acoustic waves
6. Other effects: stratification

Main topics addressed: Rotating fluids

Lecture III

1. Compressible viscous fluid description in the rotating frame
2. Low Mach number limit
3. High Reynolds/Peclet number limit
4. High Rossby number limit
5. Poincaré waves
6. Analysis of oscillations in particular geometries

Field equations

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

ϱ . . . . mass density
$\mathbf{u}$ . . . . velocity field

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x \mathbb{T} + \varrho \mathbf{f}$$

$\mathbb{T} = \mathbb{S} - p\mathbb{I}$ . . . . Cauchy stress
$\mathbb{S}$ . . . . viscous stress tensor
$\mathbf{f}$ . . . . external force
p . . . . pressure

Internal energy balance

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{T} : \nabla_x \mathbf{u}$$

e . . . . specific internal energy
$\mathbf{q}$ . . . . internal energy flux

Constitutive relations

Newton's law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \ \mu, \eta \geq 0$$

Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta, \ \ \kappa \geq 0$$

Gibbs' equation

$$\vartheta D s(\varrho, \vartheta) = D e(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

s .... (specific) entropy

Thermodynamic stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

First Law of Thermodynamics

$$\partial_t \left[\varrho \left(\frac{1}{2} |\mathbf{u}|^2 + e \right) \right] + \mathrm{div}_x \left[\varrho \left(\frac{1}{2} |\mathbf{u}|^2 + e \right) \mathbf{u} \right] = \mathrm{div}_x (\mathbb{T}\mathbf{u}) + \varrho \mathbf{f} \cdot \mathbf{u}$$

Second Law of Thermodynamics

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \sigma$$

σ .... entropy production rate

$$\sigma = (\geq) \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \geq 0$$

Energetically closed systems

No flux boundary conditions

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ or } \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Impermeability

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

No slip vs complete slip

$$[\mathbf{u}]_{\rm tan}|_{\partial\Omega} = 0 \text{ or } [\mathbb{S} \cdot \mathbf{n}]_{\rm tan} = 0$$

Total dissipation balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho\,(e - \Theta s)\right]\ \mathrm{d}x + \Theta \int_\Omega \sigma\ \mathrm{d}x = \int_\Omega \varrho \mathbf{f}\cdot\mathbf{u}\ \mathrm{d}x,\ \ \Theta > 0$$

Ballistic free energy

$$H_\Theta(\varrho,\vartheta) = \varrho\Big(e(\varrho,\vartheta) - \Theta s(\varrho,\vartheta)\Big)$$

Coercivity of the ballistic free energy

$\varrho \mapsto H_\Theta(\varrho,\Theta)$ strictly convex

$\vartheta \mapsto H_\Theta(\varrho,\vartheta)$ decreasing for $\vartheta < \Theta$ and increasing for $\vartheta > \Theta$

Local well posedness

Initial data

$$\varrho(0,\cdot) = \varrho_0 > 0, \ \vartheta(0,\cdot) = \vartheta_0 > 0, \ \mathbf{u}(0,\cdot) = \mathbf{u}_0$$

Regularity

$$\varrho, \ \vartheta, \ \mathbf{u} \in W^{m,2}, \ m \geq 3$$

Local existence for viscous fluids - Navier-Stokes-Fourier system

A. Valli, W.Zajaczkowski [1982] - local existence for large data,
A.Matsumura, T.Nishida [1980,1983] - global existence for small data

Local existence for ideal (inviscid) fluids - Euler-Fourier system

T. Alazard [2006], D. Serre [2008]- local existence for large data

Several "equivalent" forms of energy balance

Internal energy balance

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \boxed{\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}} - \boxed{p \mathrm{div}_x \mathbf{u}}$$

Entropy production

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) \boxed{=} \frac{1}{\vartheta}\left(\boxed{\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right)$$

Total energy balance

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right) + \mathrm{div}_x\left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right)\mathbf{u} + p\mathbf{u}\right] + \mathrm{div}_x \mathbf{q}$$

$$= -\boxed{\mathrm{div}_x(\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u})}$$

Second law - entropy inequality

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) \geq \frac{1}{\vartheta}\left(\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right)$$

First law - total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e\right] \ \mathrm{d}x = \int_\Omega \varrho \mathbf{f} \cdot \mathbf{u} \ \mathrm{d}x$$

Conservative driving force

$$\mathbf{f} = \nabla_x F, \ F = F(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e - \varrho F\right] \ \mathrm{d}x = 0$$

Relative entropy (energy)

Relative entropy functional

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \;\middle|\; r, \Theta, \mathbf{U}\right)$$

$$= \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r,\Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta) \right) \, \mathrm{d}x$$

Ballistic free energy

$$H_\Theta(\varrho, \vartheta) = \varrho\Big(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta) \Big)$$

Coercivity of the ballistic free energy

$\varrho \mapsto H_\Theta(\varrho, \Theta)$ strictly convex

$\vartheta \mapsto H_\Theta(\varrho, \vartheta)$ decreasing for $\vartheta < \Theta$ and increasing for $\vartheta > \Theta$

Relative entropy inequality

$$\left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Remainder $(\mathbf{f} \equiv 0)$

$$\boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})}$$

$$= \int_\Omega \Big(\varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) + \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \Big) \ \mathrm{d}x$$

$$+ \int_\Omega \Big[\Big(p(r, \Theta) - p(\varrho, \vartheta) \Big) \mathrm{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \Big] \ \mathrm{d}x$$

$$- \int_\Omega \Big(\varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \partial_t \Theta + \varrho \Big(s(\varrho, \vartheta) - s(r, \Theta) \Big) \mathbf{u} \cdot \nabla_x \Theta$$

$$+ \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \Big) \ \mathrm{d}x$$

$$+ \int_\Omega \frac{r - \varrho}{r} \Big(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \Big) \ \mathrm{d}x$$

Weak solutions - state-of-art

Global existence in the viscous case

Global-in-time weak dissipative solutions of the **Navier-Stokes-Fourier system** exist for any finite energy initial data (under some hypotheses imposed on constitutive relations)

Compatibility

Regular weak solutions are strong solutions

Weak $\Rightarrow$ dissipative

Weak solutions satisfy the relative entropy inequality

Weak-strong uniqueness

Weak (dissipative) and strong solutions emanating from the same (regular) initial data coincide as long as the latter exists. The strong solutions are unique in the class of weak solutions

Conditional regularity

Sufficient condition for regularity

Suppose that a dissipative weak solution to the Navier-Stokes-Fourier system emanating from regular initial data satisfies

$$\|\nabla_x \mathbf{u}\|_{L^\infty((0,T)\times\Omega)} < \infty.$$

Then the solution is regular in $(0, T)$.

Previously cited results are contained in joint publications with Bum Ja Jin [Muan], A. Novotný [Toulon], Y. Sun [Nanjing]

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\left[\partial_t(\varrho \vartheta) + \mathrm{div}_x(\varrho \vartheta \mathbf{u})\right] - \Delta \vartheta = -\varrho \vartheta \mathrm{div}_x \mathbf{u}$$

System supplemented with *spatially periodic* boundary conditions

Initial data

$$\varrho_0,\ \vartheta_0,\ \mathbf{u}_0 \in C^3,\ \varrho_0 > 0,\ \vartheta_0 > 0$$

Global existence

For any (smooth) initial data ϱ_0, ϑ_0, $\mathbf{u}_0$ the Euler-Fourier system admits infinitely many weak solutions on a given time interval $(0, T)$

Regularity class

$$\varrho \in C^2,\ \partial_t \vartheta,\ \nabla_x^2 \vartheta \ \in L^p \text{ for any } 1 \leq p < \infty$$

$$\mathbf{u} \in C_{\rm weak}([0, T]; L^2) \cap L^\infty,\ {\rm div}_x \mathbf{u} \in C^1$$

Joint results with E.Chiodaroli (Zurich) and O.Kreml (Prague)

Results of DeLellis and Shékelyhidi for the Euler system

Incompressible Euler system

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x (\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Reformulation

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0, \ \mathbb{U} = R^{3\times 3}_{\mathrm{sym},0}$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}(T, \cdot) = \mathbf{v}_0$$

$$\mathbb{U} = \mathbf{v} \otimes \mathbf{v} - \frac{1}{3}|\mathbf{v}|^2 \mathbb{I}, \ \Pi = -\frac{1}{3}|\mathbf{v}|^2$$

Prescribed energy

$$\frac{1}{2}|\mathbf{v}|^2(t, \cdot) = e(t, \cdot), \ t \in (0, T)$$

Construction via convex integration

The space of subsolutions

$$X_0 = \Big\{ \mathbf{v} \in C_{\rm weak}([0,T]; L^2) \ \Big| \ \mathbf{v}(0,\cdot) = \mathbf{v}(T,\cdot) = \mathbf{v}_0,$$

$$\mathrm{div}_x \mathbf{v} = 0, \ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0, \ \mathbf{v}, \ \mathbb{U} \text{ smooth in } (0,T)$$

$$\frac{3}{2} \lambda_{\rm max} \left[\mathbf{v} \times \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathbb{I} - \mathbb{U} \right] < e - \frac{1}{2} |\mathbf{v}|^2 \text{ in } (0,T) \Big\}$$

$$X = \mathrm{closure}_{C_{\rm weak}([0,T];L^2)} X_0$$

Observations

1. $e = \frac{1}{2} |\mathbf{v}|^2 \Rightarrow \mathbf{v} \times \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathbb{I} = \mathbb{U}$
2. e bounded $\Rightarrow \mathbf{v}, \ \mathbb{U}$ bounded in terms of e

Existence of subsolutions

The space X_0 is non-empty. Take $\mathbf{v}_0 \in C^1$, $\mathrm{div}_x \mathbf{v}_0 = 0$, $\mathbb{U} \equiv 0$, *e large enough*

Oscillatory lemma

For any $\mathbf{v} \in X_0$, there exists a sequence $\{\mathbf{w}_n\}_{n=1}^{\infty}$ of smooth functions compactly supported in $(0, T)$ such that $\mathbf{v} + \mathbf{w}_n \in X_0$,

$$\mathbf{w}_n \to 0 \text{ in } C_{\mathrm{weak}}([0, T]; L^2)$$

$$\liminf_{n\to\infty} \int_0^T \|\mathbf{w}_n\|_{L^2}^2 \, \mathrm{d}t \geq c\left(\|e\|_{L^\infty}\right) \int_0^T \int \left(e - \frac{1}{2}|v|^2\right)^2 \, \mathrm{d}t$$

Observations

1. Oscillatory lemma is "scale" invariant, therefore extendable to "variable coefficients"
2. $$\liminf_{n\to\infty} \int_0^T \|\mathbf{w}_n\|_{L^2}^2 \, \mathrm{d}t = \liminf_{n\to\infty} \int_0^T \left(\|\mathbf{v} + \mathbf{w}_n\|_{L^2}^2 - \|\mathbf{v}\|_{L^2}^2\right) \, \mathrm{d}t$$

Application of the convex integration method

Ansatz

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi, \ \ \mathrm{div}_x \mathbf{v} = 0$$

Equations

$$\partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x (\partial_t \Psi + \varrho \vartheta) = 0$$

$$\frac{3}{2} \left(\partial_t (\varrho \vartheta) + \mathrm{div}_x \Big(\vartheta (\mathbf{v} + \nabla_x \Psi) \Big) \right) - \Delta \vartheta = -\varrho \vartheta \mathrm{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

“Energy”

$$e = e[\mathbf{v}] = \chi(t) - \frac{3}{2} \varrho \vartheta [\mathbf{v}] - \frac{3}{2} \partial_t \Psi$$

Convex integration applied to Euler-Fourier system

I. Separation of the density

Fix the function ϱ and the potential Ψ to satisfy the equation of continuity

$$\partial_t \varrho + \Delta \Psi = 0$$

II. Temperature

Given ϱ, Ψ, and $\mathbf{v}$ solve

$$\frac{3}{2}\left(\partial_t(\varrho\vartheta) + \mathrm{div}_x\Big(\vartheta(\mathbf{v} + \nabla_x\Psi)\Big)\right) - \Delta\vartheta = -\varrho\vartheta\mathrm{div}_x\left(\frac{\mathbf{v} + \nabla_x\Psi}{\varrho}\right)$$

to obtain $\vartheta = \vartheta[\mathbf{v}]$ determined uniquely by $\mathbf{v}$.
Use the entropy equation to observe that $\|\vartheta\|_{L^\infty}$ is bounded *independently* of $\mathbf{v}$

III. Energy

Set

$$e[\mathbf{v}] = \chi(t) - \frac{3}{2}\varrho\vartheta[\mathbf{v}] - \frac{3}{2}\partial_t \Psi$$

and observe, using the parabolic regularity theory, that $\mathbf{v} \mapsto e$ is a compact functional in X_0

IV. Subsolutions

Define a space of subsolutions

$$X_0 = \Big\{ \mathbf{v} \in C_{\rm weak}([0,T]; L^2) \ \Big| \ \mathbf{v}(0,\cdot) = \mathbf{v}(T,\cdot) = \mathbf{v}_0,$$

$$\mathrm{div}_x \mathbf{v} = 0, \ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0, \ \mathbf{v}, \ \mathbb{U} \text{ smooth in } (0,T)$$

$$\frac{3}{2}\lambda_{\rm max}\left[\frac{(\mathbf{v} + \nabla_x \Psi) \times (\mathbf{v} + \nabla_x \Psi)}{\varrho} - \frac{1}{3\varrho}|\mathbf{v} + \nabla_x \Psi|^2 \mathbb{I} - \mathbb{U} \right]$$

$$< e[\mathbf{v}] - \frac{1}{2\varrho}|\mathbf{v} + \nabla_x \Psi|^2 \text{ in } (0,T) \Big\}$$

V. Oscillatory lemma

Show a "variable coefficients" variant of the oscillatory lemma replacing

$$\mathbf{v} \approx \frac{\mathbf{v} + \nabla_x \Psi}{\sqrt{\varrho}}$$

Dissipative solutions

Dissipative solutions are weak solutions of the Euler-Fourier system satisfying, in addition, the relative entropy inequality. A dissipative solution coincides with the strong solution emanating from the same initial data (weak-strong uniqueness) as long as the latter exists.

Initial data

$$\varrho_0 \in C^2, \vartheta_0 \in C^2, \ \varrho_0 > 0, \ \vartheta_0 > 0$$

Infinitely many dissipative weak solutions

For any regular initial data ϱ_0, ϑ_0, there exists a velocity field $\mathbf{u}_0$ such that the Euler-Fourier problem admits infinitely many dissipative weak solutions in $(0, T)$

Lecture II

1. Singular limits for a compressible, viscous and/or heat conducting fluid equations
2. Low Mach number limit
3. High Reynolds/Peclet number limit
4. Methods based on relative entropy
5. Propagation of acoustic waves
6. Other effects: stratification

Scaled Navier-Stokes-Fourier system

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Balance of momentum

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2}} \nabla_x p(\varrho, \vartheta) = \boxed{\varepsilon^a} \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0$$

Entropy production

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta) \mathbf{u}) + \boxed{\varepsilon^b} \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right)$$

$$= \frac{1}{\vartheta} \left(\boxed{\varepsilon^{2+a}} \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \boxed{\varepsilon^b} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right)$$

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta, \ \kappa > 0$$

Boundary conditions

Complete slip condition

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

No flux

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Far-field conditions

$$\mathbf{u} \to 0, \ \varrho \to \overline{\varrho} > 0, \ \vartheta \to \overline{\vartheta} > 0 \text{ as } |x| \to \infty$$

Scaled relative entropy

Relative entropy functional

$$\mathcal{E}_\varepsilon\left(\varrho, \vartheta, \mathbf{u} \,\middle|\, r, \Theta, \mathbf{U}\right)$$

$$= \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + \frac{1}{\varepsilon^2}\left(H_\Theta(\varrho, \vartheta) - \frac{\partial H_\Theta(r, \Theta)}{\partial \varrho}(\varrho - r) - H_\Theta(r, \Theta) \right) \right] \, \mathrm{d}x$$

Ballistic free energy

$$H_\Theta(\varrho, \vartheta) = \varrho\Big(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta) \Big)$$

Coercivity of the ballistic free energy

$\varrho \mapsto H_\Theta(\varrho, \Theta)$ strictly convex

$\vartheta \mapsto H_\Theta(\varrho, \vartheta)$ decreasing for $\vartheta < \Theta$ and increasing for $\vartheta > \Theta$

Scaled relative entropy inequality

Relative entropy inequality

$$\left[\mathcal{E}_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau}$$

$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta} \left(\varepsilon^\alpha \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \varepsilon^{\beta - 2} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R}_\varepsilon(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \ \mathrm{d}t$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Uniform bounds

The uniform bounds independent of ε are obtained by means of the choice

$$r = \overline{\varrho},\ \Theta = \overline{\vartheta},\ \mathbf{U} = 0$$

in the relative entropy inequality

Uniform bounds for ill-prepared data

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\|_{L^2 + L^q(\Omega)} \leq c \text{ for some } 1 < q < 2$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|_{L^2(\Omega)} \leq c,$$

$$\mathrm{ess} \sup_{t \in (0,T)} \| \sqrt{\varrho} \mathbf{u}_\varepsilon \|_{L^2(\Omega; R^3)} \leq c$$

Prepared data

Ill-prepared data

$$\varrho(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\vartheta(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \vartheta^{(1)}_{0,\varepsilon} \to \vartheta^{(1)}_0 \text{ in } L^2(\Omega) \text{ and weakly-(*) in } L^\infty(\Omega)$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3),\ \mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0] \in W^{k,2}(\Omega; R^3),\ k > \frac{5}{2}$$

Hypotheses

$$b > 0, \ 0 < a < \frac{10}{3}$$

Convergence

$$\mathrm{ess}\sup_{t \in (0,T)} \| \varrho_\varepsilon(t,\cdot) - \overline{\varrho} \|_{L^2 + L^q(\Omega)} \leq \varepsilon c$$

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \sqrt{\overline{\varrho}} \ \mathbf{v} \text{ in } \boxed{L^\infty_{\rm loc}((0,T]; L^2_{\rm loc}(\Omega; R^3))}$$

and weakly-(*) in $L^\infty(0,T; L^2(\Omega; R^3))$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to T \text{ in } \boxed{L^\infty_{\rm loc}((0,T]; L^s_{\rm loc}(\Omega; R^3)), \ 1 \leq s < 2},$$

and weakly-(*) in $L^\infty(0,T; L^2(\Omega))$

Target system

incompressibility

$$\mathrm{div}_x \mathbf{v} = 0, \ \ \mathbf{v} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

Euler system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0$$

Temperature transport

$$\partial_t T + \mathbf{v} \cdot \nabla_x T = 0$$

Basic assumption

The incompressible Euler system possesses a strong solution $\mathbf{v}$ *on a time interval* $(0, T_{\rm max})$ *for the initial data* $\mathbf{v}_0 = \mathbf{H}[\mathbf{u}_0]$.

Acoustic equation

$$\varepsilon \partial_t \boxed{\left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right)} + \mathrm{div}_x (\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$$

$$\varepsilon \partial_t (\varrho_\varepsilon \mathbf{u}_\varepsilon) + \nabla_x \left(\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} \right) = \varepsilon \mathbf{f}_1$$

Transport equation

$$\partial_t \left(\overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right)$$

$$+ \mathrm{div}_x \left[\left(\overline{\varrho} \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}} + \overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \boxed{\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}} \right) \mathbf{u}_\varepsilon \right] = \varepsilon f_2$$

Another application of the relative entropy inequality

Take

$$r_\varepsilon = \overline{\varrho} + \varepsilon R_\varepsilon, \ \Theta_\varepsilon = \overline{\vartheta} + \varepsilon \mathcal{T}_\varepsilon, \ \mathbf{U}_\varepsilon = \mathbf{v} + \nabla_x \Phi_\varepsilon$$

as test functions in the relative entropy inequality

Acoustic equation

$$\varepsilon \partial_t (\alpha R_\varepsilon + \beta \mathcal{T}_\varepsilon) + \omega \Delta \Phi_\varepsilon = 0$$

$$\varepsilon \partial_t \nabla_x \Phi_\varepsilon + \nabla_x (\alpha R_\varepsilon + \beta \mathcal{T}_\varepsilon) = 0$$

Transport equation

$$\partial_t (\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon) + \mathbf{U}_\varepsilon \cdot \nabla_x (\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon) + (\delta \mathcal{T}_\varepsilon - \beta R_\varepsilon) \mathrm{div}_x \mathbf{U}_\varepsilon = 0$$

Wave equation

$$\varepsilon \partial_t Z + \Delta \Phi = 0, \ \varepsilon \partial_t \Phi + Z = 0,$$

Neumann boundary condition

$$\nabla_x \Phi \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

Initial conditions

$$\Phi(0\cdot) = \Phi_0, \ \nabla_x \Phi_0 \approx \mathbf{H}^{\perp}[\mathbf{u}_0]$$

$$Z(0,\cdot) = Z_0 \approx \alpha \varrho_0^{(1)} + \beta \vartheta_0^{(1)}$$

Solution formula

Acoustic potential

$$\Phi(t,\cdot) = \frac{1}{2}\exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[\Phi_0 - \frac{\mathrm{i}}{\sqrt{-\Delta_N}}Z_0\right] + \frac{1}{2}\exp\left(-\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[\Phi_0 + \frac{\mathrm{i}}{\sqrt{-\Delta_N}}Z_0\right]$$

Time derivative

$$Z(t,\cdot) = \frac{1}{2}\exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[\mathrm{i}\sqrt{-\Delta_N}[\Phi_0] + Z_0\right] + \frac{1}{2}\exp\left(-\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)\left[-\mathrm{i}\sqrt{-\Delta_N}[\Phi_0] + Z_0\right]$$

Strichartz estimates for the flat Laplacean

Strichartz estimates

$$\int_{-\infty}^{\infty} \left\| \exp\left(\pm \mathrm{i}\sqrt{-\Delta} t \right) [h] \right\|_{L^q(R^3)}^p \,\mathrm{d}t \leq \|h\|_{H^{1,2}(R^3)}^p$$

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty$$

Local energy decay

$$\int_{-\infty}^{\infty} \left\| \chi \exp\left(\pm \mathrm{i}\sqrt{-\Delta} t \right) [h] \right\|_{H^{\alpha,2}(R^3)}^2 \,\mathrm{d}t \leq c(\chi) \|h\|_{H^{\alpha,2}(R^3)}^2$$

$$\alpha \leq \frac{3}{2}, \ \chi \in C_c^\infty(R^3)$$

Limiting absorption principle

The cut-off resolvent operator

$$(1+|x|^2)^{-s/2} \circ [-\Delta_N - \mu \pm \mathrm{i}\delta]^{-1} \circ (1+|x|^2)^{-s/2}, \ \delta > 0, \ s > 1$$

can be extended as a bounded linear operator on $L^2(\Omega)$ *for* $\delta \to 0$ *and* μ *belonging to compact subintervals of* $(0, \infty)$.

Kato's theorem

Theorem

Let A be a closed densely defined linear operator and H a self-adjoint densely defined linear operator in a Hilbert space X. For $\lambda \notin R$, let $R_H[\lambda] = (H - \lambda \mathrm{Id})^{-1}$ denote the resolvent of H. Suppose that

$$\Gamma = \sup_{\lambda \notin R,\ v \in \mathcal{D}(A^*),\ \|v\|_X = 1} \|A \circ R_H[\lambda] \circ A^*[v]\|_X < \infty.$$

Then

$$\sup_{w \in X,\ \|w\|_X = 1} \frac{\pi}{2} \int_{-\infty}^{\infty} \|A \exp(-\mathrm{i}tH)[w]\|_X^2 \ \mathrm{d}t \leq \Gamma^2.$$

Frequency localized energy decay

$$\int_{-\infty}^{\infty} \left\| \chi G(\sqrt{-\Delta_N}) \exp\left(\pm \mathrm{i} \sqrt{-\Delta_N} t \right) [h] \right\|_{H^{\alpha,2}(\Omega)}^2 \ \mathrm{d}t \leq c(\chi) \|h\|_{L^2(\Omega)}^2$$

$$\chi \in C_c^\infty(\Omega),\ G \in C_c^\infty(0, \infty)$$

Admissible domains

Limiting absorption principle

The operator Δ_N satisfies the limiting absorption principle in Ω

Strichartz estimates on "larger" domain

There is a domain such that $D \cap \{|x| > R\} = \Omega \cap \{|x| > R\}$ and Δ_N satisfies the Strichartz estimates in D

Local decay on "larger" domain

The operator Δ_N satisfies the local energy decay estimates in D

Frequency localized Strichartz estimates

$$\int_{-\infty}^{\infty} \left\| G(-\Delta_N) \exp\left(\pm \mathrm{i} \sqrt{-\Delta_N} t \right) [h] \right\|_{L^q(\Omega)}^p \leq c(G) \|h\|_{L^2(\Omega)}^p$$

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty, \ G \in C_c^\infty(0, \infty)$$

Lecture III

1. Compressible viscous fluid description in the rotating frame
2. Low Mach number limit
3. High Reynolds/Peclet number limit
4. High Rossby number limit
5. Poincaré waves
6. Analysis of oscillations in particular geometries

Scaled Navier-Stokes system

Continuity equation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon}} \varrho \mathbf{f} \times \mathbf{u} + \boxed{\frac{1}{\varepsilon^{2m}}} \nabla_x p(\varrho)$$

$$= \boxed{\varepsilon^{\alpha}} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \boxed{\frac{1}{\varepsilon^{2n}}} \varrho \nabla_x G$$

Newtonian viscous stress

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \ \mu > 0$$

f-plane approximation

$$\mathbf{f} = [0, 0, 1], \ \ \nabla_x G = [0, 0, -1]$$

Spatial domain and boundary conditions

Infinite slab

$$\Omega = R^2 \times (0,1)$$

Complete slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = u_3|_{\partial\Omega} = 0, \ [\mathbb{S} \cdot \mathbf{n}]_{\tan}|_{\partial\Omega} = 0$$

Far field conditions

$$\varrho \to \tilde{\varrho}_\varepsilon, \ \mathbf{u} \to 0 \text{ as } |x| \to \infty$$

Static density distribution

$$\nabla_x p(\tilde{\varrho}_\varepsilon) = \varepsilon^{2(m-n)} \tilde{\varrho}_\varepsilon \nabla_x G, \ \tilde{\varrho}_\varepsilon \to 1 \text{ as } \varepsilon \to 0$$

Singular limits

Low Mach number

Mach number $\approx \varepsilon^m$:

$$\text{compressible} \ \to \text{incompressible}$$

Low Rossby number

Rossby number $\approx \varepsilon$:

$$\text{3D flow} \ \to \text{2D flow}$$

High Reynolds number

Reynolds number $\approx \varepsilon^{-\alpha}$:

$$\text{viscous (Navier-Stokes)} \ \to \text{inviscid (Euler)}$$

Low stratification

$$\frac{m}{2} > n \geq 1$$

Uniform bounds

Energy inequality

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\varepsilon^{2m}} \left(H(\varrho) - H'(\tilde\varrho_\varepsilon)(\varrho - \tilde\varrho_\varepsilon) - H(\tilde\varrho_\varepsilon) \right) \right] (\tau, \cdot) \ \mathrm{d}x$$

$$+\varepsilon^\alpha \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_\Omega \left[\frac{1}{2}\varrho_{0,\varepsilon}|\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^{2m}} \left(H(\varrho_{0,\varepsilon}) - H'(\tilde\varrho_\varepsilon)(\varrho_{0,\varepsilon} - \tilde\varrho_\varepsilon) - H(\tilde\varrho_\varepsilon) \right) \right] \ \mathrm{d}x$$

$$H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z, \ p(\varrho) \approx a\varrho^\gamma, \ \gamma > \frac{3}{2}$$

Ill-prepared initial data

$$\varrho_{0,\varepsilon} = \tilde\varrho_\varepsilon + \varepsilon^m \varrho^{(1)}_{0,\varepsilon}, \ \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \text{ in } L^2(\Omega), \ \|\varrho^{(1)}_{0,\varepsilon}\|_{L^\infty} \leq c,$$

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2(\Omega; R^3)$$

Limit system

Limit density deviation

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon(t,\cdot) - 1\|_{L^\gamma_{\rm loc}(\Omega)} \leq \varepsilon^m c$$

Limit velocity

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \mathbf{v} \left\{ \begin{array}{l} \text{weakly-(*) in } L^\infty(0,T;L^2(\Omega;R^3)), \\ \boxed{\text{strongly in } L^1_{\rm loc}((0,T)\times\Omega;R^3)}, \end{array} \right.$$

Euler system

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla_x \mathbf{v} + \nabla_x \Pi = 0 \text{ in } (0,T)\times R^2$$

$$\mathbf{v}_0 = \mathbf{H}\left[\int_0^1 \mathbf{u}_0 \, {\rm d}x_3\right]$$

Relative entropy

$$\mathcal{E}_\varepsilon \left[\varrho, \mathbf{u} \middle| r, \mathbf{U} \right]$$

$$= \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + \frac{1}{\varepsilon^{2m}} \Big(H(\varrho) - H'(r)(\varrho - r) - H(r) \Big) \right] \ \mathrm{d}x$$

Relative entropy inequality

$$\mathcal{E}_\varepsilon \left(\varrho, \mathbf{u} \ \middle| \ r, \mathbf{U} \right) (\tau) + \varepsilon^\alpha \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) \Big) : \Big(\nabla_x \mathbf{u} - \nabla_x \mathbf{U} \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \mathcal{E}_\varepsilon \left(\varrho_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \ \middle| \ r(0,\cdot), \mathbf{U}(0,\cdot) \right) + \int_0^\tau \int_\Omega \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t$$

Test functions

$$r > 0, \ \ \mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \ (r - \tilde{\varrho}_\varepsilon) \to 0, \ \ \mathbf{U} \to 0 \text{ as } |x| \to \infty$$

Remainder

$$\int_0^\tau \int_\Omega \mathcal{R}(\varrho, \mathbf{u}, r, \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t$$

$$= \int_0^\tau \int_\Omega \varrho \left(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}\right) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t$$

$$+\varepsilon^\alpha \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t + \frac{1}{\varepsilon} \int_0^\tau \int_\Omega \varrho (\mathbf{f} \times \mathbf{u}) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \frac{1}{\varepsilon^{2m}} \int_0^\tau \int_\Omega \Big[(r - \varrho) \partial_t H'(r) + \nabla_x \left(H'(r) - H'(\tilde{\varrho}_\varepsilon) \right) \cdot (r \mathbf{U} - \varrho \mathbf{u}) \Big] \ \mathrm{d}x \ \mathrm{d}t$$

$$- \frac{1}{\varepsilon^{2m}} \int_0^\tau \int_\Omega \mathrm{div}_x \mathbf{U} \Big(p(\varrho) - p(r) \Big) \ \mathrm{d}x \ \mathrm{d}t + \frac{1}{\varepsilon^{2n}} \int_0^\tau \int_\Omega (\varrho - r) \nabla_x G \cdot \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t$$

Reformulation

Decomposition

$$r_\varepsilon = \frac{\varrho_\varepsilon - 1}{\varepsilon^m} = q_\varepsilon + s_\varepsilon, \ \ \varrho_\varepsilon \mathbf{u}_\varepsilon = \mathbf{v}_\varepsilon + \mathbf{V}_\varepsilon$$

$[q_\varepsilon, \mathbf{v}_\varepsilon]$ .... non-oscillatory component

$[s_\varepsilon, \mathbf{V}_\varepsilon]$ .... oscillatory component

"Acoustic analogy" - Poincaré waves

$$\varepsilon^m \partial_t \left[\frac{\varrho_\varepsilon - 1}{\varepsilon^m} \right] + \mathrm{div}_x [\varrho_\varepsilon \mathbf{u}_\varepsilon] = 0$$

$$\varepsilon^m \partial_t [\varrho_\varepsilon \mathbf{u}_\varepsilon] + \varepsilon^{m-1} \mathbf{f} \times [\varrho_\varepsilon \mathbf{u}_\varepsilon] + \nabla_x \left[\frac{\varrho_\varepsilon - 1}{\varepsilon^m} \right] = \varepsilon \mathbf{f}_\varepsilon$$

Test function ansatz

Density deviation

$$r = \tilde{\varrho}_\varepsilon + \varepsilon^m (q_\varepsilon + s_\varepsilon)$$

Velocity decomposition

$$\mathbf{U} = \mathbf{v}_\varepsilon + \mathbf{V}_\varepsilon$$

Initial data

$$\varrho^{(1)}_{0,\varepsilon} = (q_\varepsilon + s_\varepsilon)(0,\cdot), \ \ \mathbf{u}_{0,\varepsilon} = (\mathbf{v}_\varepsilon + \mathbf{V}_\varepsilon)(0,\cdot)$$

Diagnostic equation

$$\omega \mathbf{f} \times \mathbf{v}_\varepsilon + \nabla_x q_\varepsilon = 0, \ \omega = \varepsilon^{m-1}$$

$$\omega \mathrm{curl} \mathbf{v} = -\Delta q_\varepsilon$$

Perturbed Euler system

$$\partial_t \left(\Delta q_\varepsilon - \omega^2 q_\varepsilon \right) - \frac{1}{\omega} \nabla^t q_\varepsilon \cdot \nabla \left(\Delta q_\varepsilon - \omega^2 q_\varepsilon \right) = 0$$

Initial data

$$\left(\Delta q_\varepsilon - \omega^2 q_\varepsilon \right)(0, \cdot) = \omega \mathrm{curl} \left[\int_0^1 \mathbf{u}_{0,\varepsilon} \ \mathrm{d}x_3 \right] - \omega^2 \int_0^1 \varrho_{0,\varepsilon} \ \mathrm{d}x_3$$

Oscillatory - vanishing part

Poincaré waves

$$\varepsilon^m \partial_t s_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = 0$$

$$\varepsilon^m \partial_t \mathbf{V}_\varepsilon + \omega \mathbf{f} \times \mathbf{V}_\varepsilon + \nabla_x s_\varepsilon = 0, \ \omega = \varepsilon^{m-1}$$

Antisymmetric acoustic propagator

$$\mathcal{B}(\omega) : \left[\begin{array}{c} s \\ \mathbf{V} \end{array} \right] \mapsto \left[\begin{array}{c} \mathrm{div}_x \mathbf{V} \\ \omega \mathbf{f} \times \mathbf{V} + \nabla_x s \end{array} \right].$$

Poincaré waves

$$\varepsilon^m \partial_t \left[\begin{array}{c} s_\varepsilon(\xi, k, \omega) \\ \mathbf{V}_\varepsilon(\xi, k, \omega) \end{array} \right] = \mathrm{i}\mathcal{A}(\xi, k, \omega) \left[\begin{array}{c} s_\varepsilon(\xi, k, \omega) \\ \mathbf{V}_\varepsilon(\xi, k, \omega) \end{array} \right]$$

Hermitian matrix

$$\mathrm{i}\mathcal{B}(\omega) \approx \mathcal{A}(\xi, k, \omega) = \left[\begin{array}{cccc} 0 & \xi_1 & \xi_2 & k \\ \xi_1 & 0 & \omega \mathrm{i} & 0 \\ \xi_2 & -\omega \mathrm{i} & 0 & 0 \\ k & 0 & 0 & 0 \end{array} \right].$$

Eigenvalues

$$\lambda_{1,2}(\xi, k, \omega) = \pm \left[\frac{\omega^2 + |\xi|^2 + k^2 + \sqrt{(\omega^2 + |\xi|^2 + k^2)^2 - 4\omega^2 k^2}}{2} \right]^{1/2}$$

$$\lambda_{3,4}(\xi, k, \omega) = \pm \left[\frac{\omega^2 + |\xi|^2 + k^2 - \sqrt{(\omega^2 + |\xi|^2 + k^2)^2 - 4\omega^2 k^2}}{2} \right]^{1/2}$$

Frequency cut-off

$$k \text{ fixed}, \ \psi \in C_c^\infty(0,\infty), \ 0 \leq \psi \leq 1$$

$$Z(\tau, x_h, k, \omega) = \mathcal{F}^{-1}_{\xi \to x_h} \left[\exp\Big(\pm \mathrm{i}\lambda_j(|\xi|, k, \omega)\tau \Big) \psi(|\xi|) \hat{h}(\xi) \right], \ \tau = t/\varepsilon^m$$

Fourier transform of radially symmetric function

$$\|Z(\tau t, \cdot, k, \omega)\|_{L^\infty(R^2)}$$

$$\leq \left\| \mathcal{F}^{-1}_{\xi \to x_h} \left[\exp\Big(\pm \mathrm{i}\lambda_j(|\xi|, k, \omega)\tau \Big) \psi(|\xi|) \right] \right\|_{L^\infty(R^2)} \|h\|_{L^1(R^2)}$$

$$\mathcal{F}^{-1}_{\xi \to x_h} \left[\exp\Big(\pm \mathrm{i}\lambda_j(|\xi|, k, \omega)\tau \Big) \psi(|\xi|) \right] (x_h)$$

$$= \int_0^\infty \exp\Big(\pm \mathrm{i}\lambda_j(r, k, \omega)\tau \Big) \psi(r) r J_0(r|x_h|) \ \mathrm{d}r,$$

Lemma

Let $\Lambda = \Lambda(z)$ be a smooth function away from the origin,

$$\partial_z \Lambda(z) \text{ monotone}, \ |\partial_z \Lambda(z)| \geq \Lambda_0 > 0$$

for all $z \in [a, b]$, $0 < a < b < \infty$. Let Φ be a smooth function on $[a, b]$. Then

$$\left| \int_a^b \exp\left(\mathrm{i} \Lambda(z) \tau \right) \Phi(z) \ \mathrm{d}z \right| \leq c \frac{1}{\tau \Lambda_0} \left[|\Phi(b)| + \int_a^b |\partial_z \Phi(z)| \ \mathrm{d}z \right],$$

where c is an absolute constant [*independent*] *of the specific shape Λ and Φ.*

Decay estimates

$L^p - L^q$ estimates

$$\|Z(\tau,\cdot,k,\omega)\|_{L^p(R^2)} \le c(\psi,p,k)\max\left\{\frac{1}{\omega\tau^{1-\beta/2}};\frac{1}{\tau^{\beta/2}}\right\}^{1-\frac{2}{p}}\|h\|_{L^{p'}(R^2)}$$

$$\text{for } p\ge 2,\ \frac{1}{p}+\frac{1}{p'}=1,\ \beta>0,\ \lambda_j\ne 0.$$

Scaling

$$\omega\approx\varepsilon^{m-1},\ \tau\approx t/\varepsilon^m$$

Dispersive decay

$$\left\|Z\left(\frac{t}{\varepsilon^m},\cdot,k,\omega\right)\right\|_{L^p(R^2)} \le c\,\varepsilon^{\frac{1}{2}-\frac{1}{p}}\max\left\{\frac{1}{t^{1-1/2m}};\frac{1}{t^{1/2m}}\right\}^{1-\frac{2}{p}}\|h\|_{L^{p'}(R^2)}$$