Asymptotic properties of complete fluid systems

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Cergy-Pontoise, June 2010

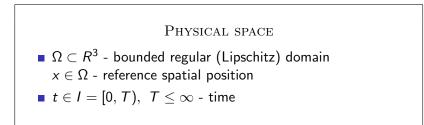
Motto:

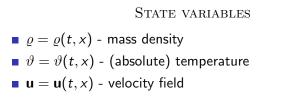
Die Energie der Welt ist constant; Die Entropie der Welt strebt einem Maximum zu

Rudolph Clausius, 1822-1888

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Physical background





Physical background

THERMODYNAMIC FUNCTIONS

•
$$p = p(\varrho, \vartheta)$$
 - pressure

•
$$e = e(\varrho, \vartheta)$$
 - (specific) internal energy

•
$$s = s(\varrho, \vartheta)$$
 - (specific) entropy

FUNDAMENTAL RELATION - GIBBS' EQUATION

$$De = \vartheta Ds - pD\left(rac{1}{arrho}
ight)$$

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└─ Time evolution - balance laws

The bulk relative motion of the fluid can cause only a small change in the statistical properties of the molecular motion when the characteristic time of the bulk motion is long compared with the characteristic time of the molecular motion

G.K. Batchelor, 1965

└─ Time evolution - balance laws

Balance Law

$$\int_{B} d(t_2, x) \, \mathrm{d}x - \int_{B} d(t_1, x) \, \mathrm{d}x$$
$$= -\int_{t_1}^{t_2} \int_{\partial B} \mathbf{F}(t, x) \cdot \mathbf{n}(x) \, \mathrm{d}S_x \, \mathrm{d}t + \int_{t_1}^{t_2} \int_{B} \mathbf{s}(t, x) \, \mathrm{d}x \, \mathrm{d}t$$

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└─ Time evolution - balance laws

INTEGRAL FORMULATION

$$V \equiv (t_1, t_2) \times B$$
$$\lim_{\varepsilon \to 0} \int_V [d(t, x); \mathbf{F}(t, x)] \cdot \nabla_{t, x} \varphi_{\varepsilon} \, \mathrm{d}x \, \mathrm{d}t = -\lim_{\varepsilon \to 0} \int_V s(t, x) \varphi_{\varepsilon} \, \mathrm{d}x \, \mathrm{d}t$$

 $\varphi_{\varepsilon} \in C_0^{\infty}(V), \ 0 \leq \varphi_{\varepsilon} \leq 1, \ \varphi_{\varepsilon}(x) = 1 \ \text{for } \operatorname{dist}[x; \partial \Omega] > \varepsilon$

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• *s* - (signed) measure on $[0, T] \times \overline{\Omega}$

└─ Time evolution - balance laws

ALTERNATIVE FORMULATION

$$\int_0^T \int_{\Omega} [d(t,x); \mathbf{F}(t,x)] \cdot \nabla_{t,x} \varphi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_{\Omega} s(t,x) \varphi \, \mathrm{d}x \, \mathrm{d}t$$

 $\varphi \in C_0^\infty((0, T) \times \Omega)$

BALANCE LAW IN DIFFERENTIAL FORM

$$\partial_t d + \operatorname{div}_x \mathbf{F} = s$$



 $\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = \mathbf{0}$

BALANCE OF MOMENTUM - NEWTON'S SECOND LAW

 $\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \operatorname{div}_x \mathbb{T} + \varrho \mathbf{f}$

STOKES' LAW

$$\mathbb{T} = \mathbb{S} - p\mathbb{I}$$

- $\blacksquare \ensuremath{\mathbb{T}}$ Cauchy stress
- S viscous stress
- **f** external force

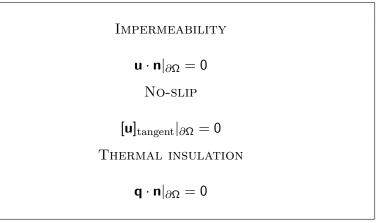
KINETIC ENERGY BALANCE

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 \right) + \operatorname{div}_x \left(\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \rho \right) \mathbf{u} - \mathbb{S} \cdot \mathbf{u} \right)$$
$$= \rho \operatorname{div}_x \mathbf{u} - \mathbb{S} : \nabla_x \mathbf{u} + \varrho \mathbf{f} \cdot \mathbf{u}$$
INTERNAL ENERGY BALANCE

 $\partial_t(\varrho e) + \operatorname{div}_x(\varrho e \mathbf{u}) + \operatorname{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - \rho \operatorname{div}_x \mathbf{u}$ Total energy balance

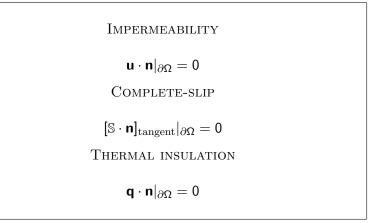
$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e \right) + \operatorname{div}_{\mathsf{x}} \left(\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + e + \rho \right) \mathbf{u} + \mathbf{q} - \mathbb{S} \cdot \mathbf{u} \right) = \varrho \mathbf{f} \cdot \mathbf{u}$$

ENERGETICALLY INSULATED BOUNDARY CONDITIONS, I

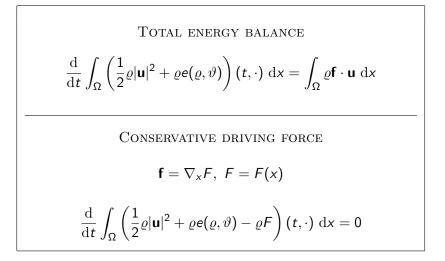


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ENERGETICALLY INSULATED BOUNDARY CONDITIONS, II



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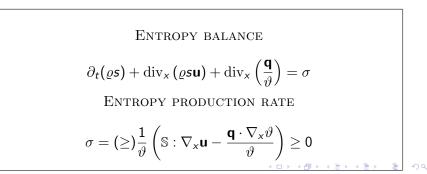


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INTERNAL ENERGY AND ENTROPY Internal energy balance

$$\partial_t(\varrho e) + \operatorname{div}_x(\varrho e \mathbf{u}) + \operatorname{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \operatorname{div}_x \mathbf{u}$$

Gibbs' equation
 $\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$



Navier-Stokes-Fourier system

NAVIER-STOKES-FOURIER SYSTEM - WEAK FORMULATION

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = \mathbf{0}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \rho = \operatorname{div}_x \mathbb{S} + \varrho \nabla_x F$$

$$\partial_t(\varrho s) + \operatorname{div}_x(\varrho s \mathbf{u}) + \operatorname{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma$$

$$\sigma \ge \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho \mathbf{e}(\varrho, \vartheta) - \varrho F\right)(t, \cdot) \, \mathrm{d}x = \mathbf{0}$$

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└─ Navier-Stokes-Fourier system

$$\sigma = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_{\mathsf{X}} \mathsf{u} - \frac{\mathsf{q} \cdot \nabla_{\mathsf{X}} \vartheta}{\vartheta} \right)$$



$$egin{aligned} 0 < \underline{arrho} \leq arrho(t,x) \leq \overline{arrho}, \ 0 < \underline{artheta} \leq artheta(t,x) \leq \overline{artheta} \ & |\mathbf{u}(t,x)| \leq U \ &
abla_x arrho \in L^2((0,T) imes \Omega; R^3) \end{aligned}$$

Navier-Stokes-Fourier system

CONSTITUTIVE EQUATIONS



$$\mathbb{S} = \mu \left(\nabla_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}}^{t} \mathbf{u} - \frac{2}{3} \operatorname{div}_{\mathbf{x}} \mathbf{u} \right) + \eta \operatorname{div}_{\mathbf{x}} \mathbf{u} \mathbb{I}$$
$$\mu > 0, \ \eta \ge 0$$

Fourier's law

$$\mathbf{q} = -\kappa \nabla_{\mathbf{x}} \vartheta$$

$$\kappa > 0$$

Mathematical issues

Well posedness

Jacques Hadamard, 1865 - 1963

- Existence. Given problem is solvable for any choice of (admissible) data
- Uniqueness. Solutions are uniquely determined by the data

Stability. Solutions depend continuously on the data

-Mathematical issues

Jacques-Luis Lions, 1928 - 2001

- Approximations. Given problem admits an approximation scheme that is solvable analytically and, possibly, numerically
- Uniform bounds. Approximate solutions possesses uniform bounds depending solely on the data

 Stability. The family of approximate solutions admits a limit representing a (generalized) solution of the given problem -Mathematical issues

- A priori bounds. Natural bounds imposed on *exact solutions* by the data
- (Weak) sequential stability. Closedness of the family of solutions bounded by a priori bounds in the framework of *weak formulation*.
- Consistency. Qualitative properties of solutions coincide with the expected ones.

A priori bounds, static states, thermodynamic stability

Equilibria - static states

$$oldsymbol{u}_{ ext{static}} \equiv 0, \ artheta_{ ext{static}} = \overline{artheta} > 0, \ arrho_{ ext{static}} = \widetilde{arrho}(x)$$

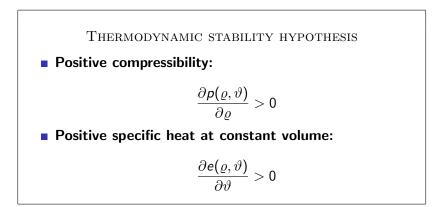
 $abla_x oldsymbol{p}(\widetilde{arrho}, \overline{artheta}) = \widetilde{arrho}
abla_x F \ ext{in } \Omega$

$$\liminf_{\varrho \to 0} \frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0 \text{ for any } \vartheta > 0 \Rightarrow \inf_{\Omega} \tilde{\varrho} > 0$$

$$F = P(\tilde{\varrho}, \overline{\vartheta}) + \text{const}, \ \frac{\partial P(\varrho, \overline{\vartheta})}{\partial \varrho} = \frac{1}{\varrho} \frac{\partial p(\varrho, \overline{\vartheta})}{\partial \varrho}$$

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A priori bounds, static states, thermodynamic stability



A priori bounds, static states, thermodynamic stability

HELMHOLTZ FUNCTION

$$H(\varrho,\vartheta) = \varrho e(\varrho,\vartheta) - \overline{\vartheta} \varrho s(\varrho,\vartheta)$$

$$\frac{\partial^2 H(\varrho,\overline{\vartheta})}{\partial \varrho^2} = \frac{1}{\varrho} \frac{\partial \rho(\varrho,\overline{\vartheta})}{\partial \varrho} > 0$$

• $\rho \mapsto H(\rho, \overline{\vartheta})$ is strictly convex

$$rac{\partial \mathcal{H}(arrho,artheta)}{\partial artheta} = rac{arrho}{artheta}(artheta-\overlineartheta)rac{\partial m{e}(arrho,artheta)}{\partial artheta}$$

• $\vartheta \mapsto H(\varrho, \vartheta)$ attains its strict local minimum at $\overline{\vartheta}$

A priori bounds, static states, thermodynamic stability

COERCIVITY OF HELMHOLTZ FUNCTION

$$egin{aligned} & \mathsf{H}(arrho,artheta) - rac{\partial \mathsf{H}(ilde{arrho},\overline{artheta})}{\partial arrho}(arrho- ilde{arrho}) - \mathsf{H}(ilde{arrho},\overline{artheta}) \ & \geq \mathsf{c}(\mathsf{B})\Big(|arrho- ilde{arrho}|^2 + |artheta-\overline{artheta}|^2\Big) \end{aligned}$$

provided ϱ, ϑ belong to a compact interval $B \subset (0, \infty)$

$$\geq \mathsf{c}(\mathsf{B}) \Big(1 + arrho \mathsf{e}(arrho, artheta) + arrho | \mathsf{s}(arrho, artheta) | \Big)$$

otherwise

as soon as $\tilde{\varrho}$, $\overline{\vartheta}$ belong to $\operatorname{int}[B]$

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A priori bounds, static states, thermodynamic stability

COROLLARY: PRINCIPLE OF MAXIMAL ENTROPY

$$\begin{split} \tilde{\varrho}, \ \overline{\vartheta} \text{ static state} &\Rightarrow \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} = F + \text{const} \Rightarrow \\ \int_{\Omega} \left(H(\varrho, \vartheta) - \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) - H(\tilde{\varrho}, \overline{\vartheta}) \right) \, \mathrm{d}x = \\ \int_{\Omega} \left(\left(\varrho e(\varrho, \vartheta) - \varrho F - \tilde{\varrho} e(\tilde{\varrho}, \overline{\vartheta}) + \tilde{\varrho} F \right) - \overline{\vartheta} \varrho s(\varrho, \vartheta) + \overline{\vartheta} \tilde{\varrho} s(\tilde{\varrho}, \overline{\vartheta}) \right) \, \mathrm{d}x \\ & \text{as soon as} \end{split}$$

$$\int_{\Omega} \varrho \, \mathrm{d} x = \int_{\Omega} \tilde{\varrho} \, \mathrm{d} x$$

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A priori bounds, static states, thermodynamic stability

PRINCIPLE OF MAXIMAL ENTROPY - CONCLUSION

- Given the total mass and energy, there is a unique static state $\tilde{\varrho}$, $\overline{\vartheta}$
- The static state *ρ̃*, *θ* maximizes the entropy among all admissible states *ρ*, *θ* with the same total mass and energy

A priori bounds, static states, thermodynamic stability

TOTAL DISSIPATION BALANCE

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho, \vartheta) - \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) - H(\tilde{\varrho}, \overline{\vartheta}) \right) \,\mathrm{d}x \\ + \overline{\vartheta} \int_{\Omega} \sigma \,\mathrm{d}x = 0 \\ \sigma \ge \frac{\mu}{\vartheta} \Big| \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \Big|^2 + \frac{\kappa}{\vartheta^2} |\nabla_x \vartheta|^2$$

A priori bounds, static states, thermodynamic stability

TECHNICAL HYPOTHESES IMPOSED ON CONSTITUTIVE RELATIONS

What is needed...

- integrability of all quantities in the weak formulation hypotheses of coercivity imposed on thermodynamic functions p, e, s
- bounds on the spatial gradients of u, θ the transport coefficients μ, κ depend on the temperature
- compactness of the temperature field on the "vacuum" zones - introducing radiation pressure

A priori bounds, static states, thermodynamic stability

THERMODYNAMIC FUNCTIONS Monoatomic gas:

$$p = \frac{2}{3} \varrho e \Rightarrow p = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right)$$

Third Law:

$$P(Z) \approx Z^{5/3}$$
 for $Z \to \infty$

Radiation pressure:

$$p(\varrho, artheta) = artheta^{5/2} P\left(rac{arrho}{artheta^{3/2}}
ight) + rac{a}{3}artheta^4$$

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A priori bounds, static states, thermodynamic stability

PRESSURE-ENERGY-ENTROPY

Pressure: $p(\varrho, \vartheta) = p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{2} \vartheta^4,$ Internal energy: $e(\varrho,\vartheta) = \frac{3}{2}\vartheta\left(\frac{\vartheta^{3/2}}{\varrho}\right)P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{\varrho}\vartheta^4$ Entropy: $s(\varrho, \vartheta) = S\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4}{3}\frac{a}{\varrho}\vartheta^3$

A priori bounds, static states, thermodynamic stability

TRANSPORT COEFFICIENTS

Shear viscosity:

$$0 < \underline{\mu}(1 + artheta^lpha) \leq \mu(artheta) \leq \overline{\mu}(1 + artheta^lpha), \,\, 1/2 \leq lpha \leq 1$$

Bulk viscosity:

$$0 \leq \eta(artheta) \leq \overline{\eta}(1 + artheta^lpha)$$

Heat conductivity:

$$0 < \overline{\kappa}(1 + \vartheta^3) \le \kappa(artheta) \le \overline{\kappa}(1 + artheta^3)$$

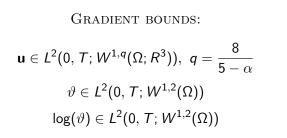
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A priori bounds, static states, thermodynamic stability

A priori bounds

UNIFORM-IN-TIME L^p -BOUNDS: $\sqrt{\varrho} \mathbf{u} \in L^{\infty}(0, T; L^2(\Omega; R^3))$ $\varrho \in L^{\infty}(0, T; L^{5/3}(\Omega))$ $\vartheta \in L^{\infty}(0, T; L^4(\Omega))$

A priori bounds, static states, thermodynamic stability



PRESSURE BOUNDS:

 $p(\varrho, \vartheta)\varrho^{\beta} \in L^{1}((0, T) \times \Omega)$ for a certain $\beta > 0$

Weak sequential stability

Weak sequential stability

$$\begin{split} \varrho_{\varepsilon} & \to \varrho \text{ weakly-}(*) \text{ in } L^{\infty}(0, T; L^{5/3}(\Omega)) \\ \\ \vartheta_{\varepsilon} & \to \vartheta \text{ weakly-}(*) \text{ in } L^{\infty}(0, T; L^{4}(\Omega)) \\ \\ \text{ and weakly in } L^{2}(0, T; W^{1,2}(\Omega)) \\ \\ \\ \mathbf{u}_{\varepsilon} & \to \mathbf{u} \text{ weakly in } L^{2}(0, T; W^{1,q}(\Omega; R^{3})) \end{split}$$

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Weak sequential stability

DIV-CURL LEMMA [F.Murat, L.Tartar, 1975]

Lemma

Let

 $\mathbf{v}_{\varepsilon} \rightarrow \mathbf{v}$ weakly in L^{p} , $\mathbf{w}_{\varepsilon} \rightarrow \mathbf{w}$ weakly in L^{q} ,

with

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}<1.$$

Let, moreover,

 $\operatorname{div}[\mathbf{v}_{\varepsilon}], \operatorname{curl}[\mathbf{w}_{\varepsilon}]$ be precompact in $W^{-1,s}$

Then

$$\mathbf{v}_{\varepsilon} \cdot \mathbf{w}_{\varepsilon} \rightarrow \mathbf{v} \cdot \mathbf{w}$$
 weakly in L^{r} .

Weak sequential stability

WEAK SEQUENTIAL STABILITY OF CONVECTIVE TERMS

$$\mathbf{v}_{\varepsilon} = [\varrho_{\varepsilon}, \varrho_{\varepsilon} \mathbf{u}_{\varepsilon}], \ \mathbf{w}_{\varepsilon} = [u_{\varepsilon}^{i}, 0, 0, 0], \ i = 1, 2, 3$$

Aubin-Lions argument (Div-Curl lemma) \Rightarrow

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u}$$

$$\overline{\varrho \mathbf{u} \otimes \mathbf{u}} = \varrho \mathbf{u} \otimes \mathbf{u}$$

$$\overline{\varrho s(\varrho, \vartheta)\vartheta} = \overline{\varrho s(\varrho, \vartheta)}\vartheta$$

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POINTWISE CONVERGENCE OF TEMPERATURE, I

GOAL: Use monotonicity of $s(\varrho, \vartheta)$ in ϑ to show $\int_0^T \int_\Omega \left(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta) \right) (\vartheta_\varepsilon - \vartheta) \, \mathrm{dx} \, \mathrm{dt} \to 0$ \Rightarrow $\|\vartheta_\varepsilon - \vartheta\|_{L^4} \to 0$

STEP 1: Aubin-Lions argument (Div-Curl lemma) \Rightarrow

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) (\vartheta_\varepsilon - \vartheta) \, \mathrm{d} x \, \mathrm{d} t \to 0$$

POINTWISE CONVERGENCE OF TEMPERATURE, II

STEP 2: Renormalized equation of continuity [DiPerna and P.-L. Lions, 1989]

$$\partial_t b(\varrho) + \operatorname{div}_x(b(\varrho)\mathbf{u}) + (b'(\varrho)\varrho - b(\varrho))\operatorname{div}_x\mathbf{u} = 0$$

STEP 3: Aubin-Lions argument (Div-Curl lemma):

$$\overline{b(\varrho)g(\vartheta)} = \overline{b(\varrho)} \ \overline{g(\vartheta)}$$

FUNDAMENTAL THEOREM ON YOUNG MEASURES [J.M Ball 1989, P.Pedregal 1997]

Theorem

Let $\mathbf{v}_{\varepsilon} : Q \subset \mathbb{R}^{N} \to \mathbb{R}^{M}$ be a sequence of vector fields bounded in $L^{1}(Q; \mathbb{R}^{M})$. Then there exists a subsequence (not relabeled) and a family of probability measures $\{\nu_{y}\}_{y \in Q}$ on \mathbb{R}^{M} such that: For any Carathéodory function $\Phi = \Phi(y, Z)$, yinQ, $Z \in \mathbb{R}^{M}$ such

that

$$\Phi(\cdot, \mathbf{v}_{\varepsilon})
ightarrow \overline{\Phi}$$
 weakly in $L^1(Q)$

we have

$$\overline{\Phi}(y) = \int_{R^M} \Phi(y,Z) \; \mathrm{d}
u_y(Z) \; \textit{for a.a. } y \in Q.$$

POINTWISE CONVERGENCE OF TEMPERATURE, III

STEP 4: Since we already know from STEP 3 that

$$\nu[\varrho_{\varepsilon}\vartheta_{\varepsilon}] = \nu[\varrho_{\varepsilon}] \otimes \nu[\vartheta_{\varepsilon}],$$

Fundamental theorem yields the desired conclusion

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta)(\vartheta_\varepsilon - \vartheta) \, \mathrm{d} x \, \mathrm{d} t \to 0$$

POINTWISE CONVERGENCE OF TEMPERATURE

$$\vartheta_{\varepsilon} \rightarrow \vartheta$$
 a.a. on $(0, T) \times \Omega$

POINTWISE CONVERGENCE OF DENSITY,I

STEP 1: Renormalized equation of continuity:

$$\partial_t(\rho \log(\rho)) + \operatorname{div}_x(\rho \log(\rho)\mathbf{u}) + \rho \operatorname{div}_x \mathbf{u} = 0$$

 $\partial_t(\overline{\varrho \log(\varrho)}) + \operatorname{div}_x(\overline{\varrho \log(\varrho)}\mathbf{u}) + \overline{\varrho \operatorname{div}_x \mathbf{u}} = 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho) \right) \, \mathrm{d}x = - \int_{\Omega} \left(\overline{\varrho \mathrm{div}_{x} \mathbf{u}} - \varrho \mathrm{div}_{x} \mathbf{u} \right) \, \mathrm{d}x$$

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POINTWISE CONVERGENCE OF DENSITY, II

STEP 2: Effective viscous pressure [P.-L.Lions, 1998]

$$\overline{p(\varrho, \vartheta)b(\varrho)} - \overline{p(\varrho, \vartheta)} \ \overline{b(\varrho)} = \overline{[\mathcal{R} : \mathbb{S}]b(\varrho)} - [\mathcal{R} : \mathbb{S}]\overline{b(\varrho)}$$
where
$$\mathcal{R}_{i,j} \equiv \partial_{x_i} \Delta^{-1} \partial_{x_j}$$

$$\mathcal{R}: \mathbb{S} = \mathcal{R}: \mathbb{S} - \left(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\right) \operatorname{div}_{x} \mathbf{u} + \left(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\right) \operatorname{div}_{x} \mathbf{u}$$

COMMUTATOR LEMMA[in the spirit of Coifman and Meyer]

Lemma

Let
$$w \in W^{1,r}(\mathbb{R}^N)$$
, $\mathbf{V} \in L^p(\mathbb{R}^N; \mathbb{R}^N)$ be given, where

$$1 < r < N, \ 1 < p < \infty, \ \frac{1}{r} + \frac{1}{p} - \frac{1}{N} < 1$$

The for any s satisfying

$$\frac{1}{r}+\frac{1}{p}-\frac{1}{N}<\frac{1}{s}<1$$

there exists $\beta > 0$ such that

$$\|\mathcal{R}[w\mathbf{V}] - w\mathcal{R}[\mathbf{V}]\|_{W^{eta,s}(R^N,R^N)} \leq c \|w\|_{W^{1,r}} \|\mathbf{V}\|_{L^p}.$$

POINTWISE CONVERGENCE OF DENSITY, III

STEP 3: Effective viscous pressure revisited:

$$0 \leq \overline{p(\varrho, \vartheta)\varrho} - \overline{p(\varrho, \vartheta)}\varrho = \left(\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)\right) \left(\overline{\varrho \operatorname{div}_{\mathsf{x}} \mathsf{u}} - \varrho \operatorname{div}_{\mathsf{x}} \mathsf{u}\right)$$

yielding

$$\varrho \log(\varrho) = \varrho \log(\varrho)$$

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POINTWISE CONVERGENCE OF DENSITY - GENERAL CASE, I

STEP 1: Renormalized equation of continuity:

$$\partial_t(\varrho L_k(\varrho)) + \operatorname{div}_x(\varrho L_k(\varrho)\mathbf{u}) + T_k(\varrho)\operatorname{div}_x\mathbf{u} = 0$$

$$\partial_t(\overline{\varrho L_k(\varrho)}) + \operatorname{div}_x(\overline{\varrho L_k(\varrho)}\mathbf{u}) + \overline{T_k(\varrho)\operatorname{div}_x\mathbf{u}} = 0$$

$$T_k(\varrho) = \min\{\varrho, k\}$$

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POINTWISE CONVERGENCE OF DENSITY - GENERAL CASE, II

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho) \right) \mathrm{d}x = \int_{\Omega} \left(T_k(\varrho) \mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)} \mathrm{div}_x \mathbf{u} \right) \mathrm{d}x \\ + \int_{\Omega} \left(\overline{T_k(\varrho)} \mathrm{div}_x \mathbf{u} - \overline{T_k(\varrho)} \mathrm{div}_x \mathbf{u} \right) \mathrm{d}x$$

STEP 2: Effective viscous flux revisited:

$$\overline{p(\varrho,\vartheta)T_{k}(\varrho)} - \overline{p(\varrho,\vartheta)} \overline{T_{k}(\varrho)}$$
$$= (\frac{4}{3}\mu(\vartheta) + \eta(\vartheta)) \Big(\overline{T_{k}(\varrho)\operatorname{div}_{x}\mathbf{u}} - \overline{T_{k}(\varrho)}\operatorname{div}_{x}\mathbf{u}\Big)$$
yielding

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OSCILLATIONS DEFECT MEASURE

$$\sup_{k\geq 1} \left[\limsup_{\varepsilon\to 0} \int_0^T \int_\Omega |T_k(\varrho_\varepsilon) - T_k(\varrho)|^q \, \mathrm{d}x \, \mathrm{d}t \right] < \infty$$
$$q = 5/3 + 1 = 8/3$$

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Boundedness of oscillations defect measure guarantees:

The limit functions *ρ*, **u** satisfy the renormalized equation of continuity

$$\int_{\Omega} \left(T_k(\varrho) \operatorname{div}_x \mathbf{u} - \overline{T_k(\varrho)} \operatorname{div}_x \mathbf{u} \right) \, \mathrm{d}x \to 0 \text{ for } k \to \infty$$

CONCLUSION - POINTWISE CONVERGENCE OF DENSITY

$$\overline{\varrho \log(\varrho)} = \lim_{k \to \infty} \overline{\varrho L_k(\varrho)} = \lim_{k \to \infty} \varrho L_k(\varrho) = \varrho \log(\varrho)$$

$$\varrho_{\varepsilon} \rightarrow \varrho$$
 a.a. on $(0, T) \times \Omega$

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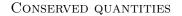
Long-time behavior

Conservative driving forces

Long-time behavior

Conservative driving forces

$$\mathbf{f}=\nabla_x F,\ F=F(x)$$



Total mass:

$$M = \int_{\Omega} \varrho \, \mathrm{d} x$$

Total energy:

$$E = \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right) \, \mathrm{d}x$$

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Long-time behavior

Conservative driving forces

STEP 1: Boundedness of total energy \Rightarrow boundedness of total entropy:

$$\mathcal{S}(t) = \int_\Omega arrho s(arrho, artheta)(t, \cdot) \, \mathrm{d} x \leq \mathcal{S}_\infty$$

STEP 2: Boundedness of total entropy \Rightarrow finite integral of the dissipation rate:

$$\int_{0}^{\infty} \int_{\Omega} \left(\frac{\mu}{2\vartheta} \left| \nabla_{x} \mathbf{u} + \nabla_{x}^{t} \mathbf{u} - \frac{2}{3} \operatorname{div}_{x} \mathbf{u} \right|^{2} + \frac{\kappa}{\vartheta^{2}} |\nabla_{x} \vartheta|^{2} \right) \, \mathrm{d}x \, \mathrm{d}t$$
$$\leq \sigma[(0, \infty) \times \overline{\Omega}] \, \mathrm{d}t < \infty$$

STEP 3: The velocity field **u** as well as the temperature gradient vanish in the asymptotic limit $t \to \infty \Rightarrow$ any solution tends to a uniquely determined *static state*

$$\tilde{\varrho} = \tilde{\varrho}(x), \ \overline{\vartheta} > 0$$

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Long-time behavior

└─ Conservative driving forces

STEP 4: Total dissipation balance:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho, \vartheta) - \frac{\partial H(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) - H(\tilde{\varrho}, \overline{\vartheta}) \right) \,\mathrm{d}x \\ + \overline{\vartheta} \int_{\Omega} \sigma \,\mathrm{d}x = 0$$

$$\int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \mathcal{H}(\varrho, \vartheta) - \frac{\partial \mathcal{H}(\tilde{\varrho}, \overline{\vartheta})}{\partial \varrho} (\varrho - \tilde{\varrho}) - \mathcal{H}(\tilde{\varrho}, \overline{\vartheta}) \right) \, \mathrm{d}x \to 0 \text{ as } t \to \infty$$

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Long-time behavior

Conservative driving forces

CONCLUSION:

LONG-TIME BEHAVIOR FOR CONSERVATIVE DRIVING FORCES

$$\mathbf{f}=\nabla_x F,\ F=F(x)$$

$$arrho(t,\cdot) o ilde{arrho}$$
 in $L^{5/3}(\Omega)$ as $t o\infty$

$$\vartheta(t,\cdot) o \overline{\vartheta}$$
 in $L^4(\Omega)$ as $t \to \infty$

$$(\varrho \mathbf{u})(t, \cdot) \rightarrow 0$$
 in $L^1(\Omega; R^3)$ as $t \rightarrow \infty$

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Long-time behavior

Conservative driving forces

Attractors

$$\begin{split} &\int_{\Omega} \varrho(t,\cdot) \, \mathrm{d} x > M, \ t > 0 \\ &\int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varrho F \right)(t,\cdot) \, \mathrm{d} x < E, \ t > 0 \\ &\int_{\Omega} \varrho s(\varrho, \vartheta)(t,\cdot) \, \mathrm{d} x > S_0, \ t > 0 \\ & \| \varrho(t,\cdot) - \tilde{\varrho} \|_{L^{5/3}(\Omega)} < \varepsilon \text{ for } t > T(\varepsilon) \\ & \| \vartheta(t,\cdot) - \overline{\vartheta} \|_{L^4(\Omega)} < \varepsilon \text{ for } t > T(\varepsilon) \\ & \| \varrho \mathbf{u}(t,\cdot) \|_{L^1(\Omega; R^3)} < \varepsilon \text{ for } t > T(\varepsilon) \end{split}$$

Long-time behavior

Conservative driving forces

UNIFORM DECAY OF DENSITY OSCILLATIONS

$$d(t) = \int_{\Omega} \Big(\overline{\varrho \log(\varrho)} - \varrho \log(\varrho)\Big)(t, \cdot) \, \mathrm{d}x$$

$\partial_t d(t) + \Psi(d(t)) \leq 0$

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Long-time behavior

└─Non-conservative driving forces

GENERAL TIME-DEPENDENT DRIVING FORCES

$$\mathbf{f} = \mathbf{f}(t, x), \ |\mathbf{f}(t, x)| \le \overline{F}$$

EITHER

$${oldsymbol E}(t)\equiv\int_\Omega \left(rac{1}{2}arrho|{f u}|^2+arrho {oldsymbol e}(arrho,artheta)
ight)(t,\cdot)~{
m d}x o\infty$$
 as $t o\infty$

OR

 $|E(t)| \leq E$ for a.a. t > 0

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Long-time behavior

└─ Non-conservative driving forces

In the case $E(t) \leq E$, each sequence of times $\tau_n \to \infty$ contains a subsequence such that $\mathbf{f}(\tau_n + \cdot, \cdot) \to \nabla_x F$ weakly-(*) in $L^{\infty}((0, 1) \times \Omega)$, where F = F(x) may depend on $\{\tau_n\}$

Long-time behavior

└─ Non-conservative driving forces

STEP 1: Assume that $E(\tau_n) < E$ for certain $\tau_n \to \infty \Rightarrow$ total entropy remains bounded \Rightarrow integral of entropy production bounded

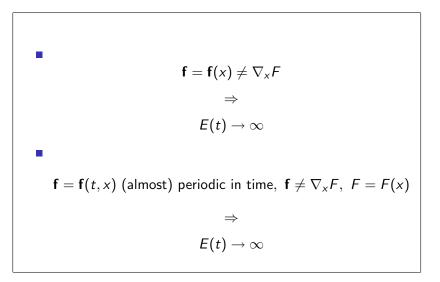
STEP 2: For $\tau_n \to \infty$ we have $\nabla_x p(\varrho, \vartheta) \approx \varrho \mathbf{f}$, $\vartheta \approx \overline{\vartheta}$, meaning, $\mathbf{f} \approx \nabla_x F$

STEP 3: The energy cannot "oscillate" since bounded entropy *static solutions* have bounded total energy

Long-time behavior

└─Non-conservative driving forces

COROLLARIES:



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Long-time behavior

└─Non-conservative driving forces

RAPIDLY OSCILLATING DRIVING FORCES

$$(\varrho \mathbf{u})(t, \cdot) \to 0 \text{ in } L^1(\Omega; R^3) \text{ as } t \to \infty$$

 $\varrho(t, \cdot) \to \overline{\varrho} \text{ in } L^{5/3}(\Omega) \text{ as } t \to \infty$
 $\vartheta(t, \cdot) \to \overline{\vartheta} \text{ in } L^4(\Omega) \text{ as } t \to \infty$

Singular limits

Motto:

However beautiful the strategy, you should occasionally look at the results

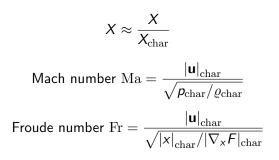
Sir Winston Churchill, 1874-1965

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Singular limits

└─Scaling and scaled equations

Singular limits



Incompressibility: $Ma \approx \varepsilon \rightarrow 0$ Stratification: $Fr \approx \varepsilon^{\alpha/2} \rightarrow 0$

Singular limits

└─Scaling and scaled equations

Scaled Navier-Stokes-Fourier system:

Sr $\partial_t \rho + \operatorname{div}_{\boldsymbol{X}}(\rho \mathbf{u}) = 0$

$$\operatorname{Sr} \partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\operatorname{Ma}^2} \nabla_x \boldsymbol{\rho} = \frac{1}{\operatorname{Re}} \operatorname{div}_x \mathbb{S} + \frac{1}{\operatorname{Fr}^2} \varrho \nabla_x \boldsymbol{F}$$

$$\operatorname{Sr} \partial_t(\varrho s) + \operatorname{div}_x(\varrho s \mathbf{u}) + \frac{1}{\operatorname{Pe}} \operatorname{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma$$
$$\sigma \ge \frac{1}{\vartheta} \left(\frac{\operatorname{Ma}^2}{\operatorname{Re}} \mathbb{S} : \nabla_x \mathbf{u} - \frac{1}{\operatorname{Pe}} \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left(\frac{\mathrm{Ma}^2}{2} \varrho |\mathbf{u}|^2 + \varrho \boldsymbol{e} - \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2} \varrho \boldsymbol{F} \right) = \mathbf{0}$$

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{0}, \ [\mathbb{S}\mathbf{n}]_{\mathrm{tan}} = \mathbf{0}$$

└─ Singular limits

└─Scaling and scaled equations

CHARACTERISTIC NUMBERS:

∎ S	YMBOL DEFINITION	■ NAME
Sr	$\mathrm{length}_{\mathrm{ref}}/(\mathrm{time}_{\mathrm{ref}}\mathrm{velocity}_{\mathrm{ref}})$	Strouhal number
Ma	$\rm velocity_{ref}/\sqrt{pressure_{ref}/density_{ref}}$	Mach number
Re	$density_{ref} velocity_{ref} length_{ref} / viscosity_{ref}$	Reynolds number
Fr	$\mathrm{velocity}_{\mathrm{ref}}/\sqrt{\mathrm{length}_{\mathrm{ref}}\mathrm{force}_{\mathrm{ref}}}$	Froude number
Pe /(te	${ m pressure}_{ m ref}$ heat conductivity ${ m ref}$)	$\operatorname{length}_{\operatorname{ref}}\operatorname{velocity}_{\operatorname{ref}}$ Péclet number

Singular limits

└─Scaling and scaled equations

LOW MACH NUMBER LIMIT - WEAK STRATIFICATION

$$Ma = \varepsilon$$
, $Fr = \sqrt{\varepsilon}$

STRATEGY:

- **I** Existence theory for the primitive Navier-Stokes-Fourier system
- 2 Uniform bounds independent of the singular parameter

- 3 Passage to the limit analysis of acoustic waves
- 4 Identification of the limit system

Singular limits

Scaling and scaled equations

Scaled Navier-Stokes-Fourier system

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0 \text{ in } (0, T) \times \Omega$$

 $\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \operatorname{div}_x \mathbb{S} + \frac{1}{\varepsilon} \varrho \nabla_x F \text{ in } (0, T) \times \Omega$$
$$[\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \operatorname{div}_x(\varrho s(\varrho, \vartheta) \mathbf{u}) + \operatorname{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma \text{ in } (0, T) \times \Omega$$
$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega} \left(\frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \varepsilon \varrho F\right) \, \mathrm{d}x = 0$$
$$\sigma \ge \frac{1}{\vartheta} \left(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right) \ge 0$$

└─ Singular limits

└─Scaling and scaled equations

TOTAL DISSIPATION BALANCE

$$\int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} \left(H(\varrho, \vartheta) - \partial_{\varrho} H(\tilde{\varrho}_{\varepsilon}, \overline{\vartheta})(\varrho - \tilde{\varrho}_{\varepsilon}) - H(\tilde{\varrho}_{\varepsilon}, \overline{\vartheta}) \right) \right) (\tau, \cdot) \, \mathrm{d}x \\ + \frac{\overline{\vartheta}}{\varepsilon^2} \int_0^{\tau} \int_{\Omega} \sigma \, \mathrm{d}x \, \mathrm{d}t = \\ \int_{\Omega} \left(\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \frac{1}{\varepsilon^2} \left(H(\varrho_0, \vartheta_0) - \partial_{\varrho} H(\tilde{\varrho}_{\varepsilon}, \overline{\vartheta})(\varrho_0 - \tilde{\varrho}_{\varepsilon}) - H(\tilde{\varrho}_{\varepsilon}, \overline{\vartheta}) \right) \right) \, \mathrm{d}x$$

$$\nabla_{x} p(\tilde{\varrho}_{\varepsilon}, \overline{\vartheta}) = \varepsilon \tilde{\varrho}_{\varepsilon} \nabla_{x} F, \ \int_{\Omega} \tilde{\varrho}_{\varepsilon} \ \mathrm{d}x = \int_{\Omega} \varrho_{0} \ \mathrm{d}x, \ \tilde{\varrho}_{\varepsilon} \approx \overline{\varrho}$$

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└─ Singular limits

└─Scaling and scaled equations

ILL-PREPARED INITIAL DATA

$$\begin{split} \varrho_0 &\approx \overline{\varrho} + \varepsilon \varrho_{0,\varepsilon}^{(1)}, \ \{\varrho_{0,\varepsilon}^{(1)}\}_{\varepsilon > 0} \text{ bounded in } L^1 \cap L^\infty(\Omega), \ \int_\Omega \varrho_{0,\varepsilon}^{(1)} \, \mathrm{d} x = 0 \\ \vartheta_0 &\approx \overline{\vartheta} + \varepsilon \vartheta_{0,\varepsilon}^{(1)}, \ \{\vartheta_{0,\varepsilon}^{(1)}\}_{\varepsilon > 0} \text{ bounded in } L^1 \cap L^\infty(\Omega), \ \int_\Omega \vartheta_{0,\varepsilon}^{(1)} \, \mathrm{d} x = 0 \\ \mathbf{u}_0 &\approx \mathbf{u}_{0,\varepsilon}, \ \{\mathbf{u}_{0,\varepsilon}\}_{\varepsilon > 0} \text{ bounded in } L^2(\Omega; R^3) \end{split}$$

<u>Singular</u> limits

Uniform bounds

UNIFORM BOUNDS

$$\begin{split} &\left\{\frac{\varrho_{\varepsilon}-\overline{\varrho}}{\varepsilon}\right\}_{\varepsilon>0} \text{ bounded in } L^{\infty}(0,T;L^{2}\oplus L^{q}(\Omega)), \ q<2\\ &\left\{\frac{\vartheta_{\varepsilon}-\overline{\vartheta}}{\varepsilon}\right\}_{\varepsilon>0} \text{ bounded in } L^{\infty}(0,T;L^{2}\oplus L^{q}(\Omega)), \ q<2\\ &\left\{\varrho_{\varepsilon}|\mathbf{u}_{\varepsilon}|^{2}\right\}_{\varepsilon>0} \text{ bounded in } L^{\infty}(0,T;L^{1}(\Omega))\\ &\left\{\frac{\sigma_{\varepsilon}}{\varepsilon^{2}}\right\}_{\varepsilon>0} \text{ bounded in } \mathcal{M}^{+}([0,T]\times\overline{\Omega}) \end{split}$$

$$\{ \nabla_{\mathsf{x}} \mathbf{u}_{\varepsilon} \}_{\varepsilon > 0} \text{ bounded in } L^{2}((0, T) \times \Omega; R^{3 \times 3})$$
$$\left\{ \frac{\nabla_{\mathsf{x}} \vartheta_{\varepsilon}}{\varepsilon} \right\}_{\varepsilon > 0} \text{ bounded in } L^{2}((0, T) \times \Omega; R^{3})$$

└─Singular limits

Uniform bounds

Convergence

$$\varrho_{\varepsilon} \rightarrow \overline{\varrho} \text{ in } L^{\infty}(0, T; L^{2} \oplus L^{q}(\Omega))$$

$$\vartheta_{\varepsilon} \to \overline{\vartheta}$$
 in $L^{\infty}(0, T; L^2 \oplus L^q(\Omega))$

$$\mathbf{u}_{\varepsilon} \rightarrow \mathbf{U}$$
 weakly in $L^{2}(0, T; W^{1,2}(\Omega; R^{3}))$

$$rac{artheta_arepsilon-artheta}{arepsilon}
ightarrow\Theta$$
 weakly in $L^2(0,\,T;\,W^{1,2}(\Omega))$

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Singular limits

└─ Target system

Oberbeck-Boussinesq system

 $\operatorname{div}_{x} \mathbf{U} = \mathbf{0}$

$$\overline{\varrho} \Big(\partial_t \mathbf{U} + \operatorname{div}_x (\mathbf{U} \otimes \mathbf{U}) \Big) + \nabla_x \Pi = \operatorname{div}_x \mathbb{S} + r \nabla_x F \text{ in } (0, T) \times \Omega$$
$$\mathbf{U} \cdot \mathbf{n}|_{\partial \Omega} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial \Omega} = 0$$

 $\overline{\varrho}c_{\rho}\left(\partial_{t}\Theta + \operatorname{div}_{x}(\Theta \mathbf{U})\right) - \operatorname{div}_{x}(G\mathbf{U}) - \operatorname{div}_{x}(\kappa \nabla_{x}\Theta) = 0 \text{ in } (0, T) \times \Omega$ $G = \beta F, \ \nabla_{x}\Theta \cdot \mathbf{n}|_{\partial\Omega} = 0$

 $r + \alpha \Theta = 0, \ \alpha > 0$

Singular limits

└─ Target system

AVAILABLE RESULTS

• Barotropic Navier-Stokes system - weak solutions, large time interval

P.-L.Lions, N. Masmoudi, J. Math. Pures Appl., 1998

B. Desjardins, E. Grenier, P.-L. Lions, N. Masmoudi, J. Math.

Pures Appl., 1999

B. Desjardins, E. Grenier, Royal Soc. London, 1999

• Navier-Stokes-Fourier system, strong solutions, short time interval

T. Alazard, Arch. Rational Mech. Anal., SIAM J. Math. Anal., 2006

Singular limits

Acoustic waves

LIGHTHILL'S ACOUSTIC EQUATION
$$(F = 0)$$

$$\begin{split} \varepsilon \partial_t Z_{\varepsilon} + \operatorname{div}_x \mathbf{V}_{\varepsilon} &= \varepsilon \operatorname{div}_x \mathbf{F}_{\varepsilon}^1 \\ \varepsilon \partial_t \mathbf{V}_{\varepsilon} + \omega \nabla_x Z_{\varepsilon} &= \varepsilon \left(\operatorname{div}_x \mathbb{F}_{\varepsilon}^2 + \nabla_x F_{\varepsilon}^3 + \frac{A}{\varepsilon^2 \omega} \nabla_x \Sigma_{\varepsilon} \right) \\ \mathbf{V}_{\varepsilon} \cdot \mathbf{n}|_{\partial \Omega} &= 0 \end{split}$$

$$Z_{\varepsilon} = \frac{\varrho_{\varepsilon} - \overline{\varrho}}{\varepsilon} + \frac{A}{\omega} \varrho_{\varepsilon} \left(\frac{s(\varrho_{\varepsilon}, \vartheta_{\varepsilon}) - s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} \right) + \frac{A}{\varepsilon \omega} \Sigma_{\varepsilon}, \ \mathbf{V}_{\varepsilon} = \varrho_{\varepsilon} \mathbf{u}_{\varepsilon}$$
$$< \Sigma_{\varepsilon}; \varphi > = < \sigma_{\varepsilon}; I[\varphi] >$$
$$I[\varphi](t, x) = \int_{0}^{t} \varphi(z, x) \ dz \text{ for any } \varphi \in L^{1}(0, T; C(\overline{\Omega}))$$

Singular limits

-Acoustic waves

HELMHOLTZ DECOMPOSITION

$$\mathbf{V}_{\varepsilon} = \mathbf{w}_{\varepsilon} + \nabla_{x} \Phi_{\varepsilon}, \ \operatorname{div}_{x}(\mathbf{w}_{\varepsilon}) = 0, \ \mathbf{w}_{\varepsilon} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

WAVE EQUATION

$$\varepsilon \partial_t Z_{\varepsilon} + \Delta_x \Phi_{\varepsilon} = \varepsilon G_{\varepsilon}^1 \text{ in } (0, T) \times \Omega$$
$$\varepsilon \partial_t \Phi_{\varepsilon} + \omega Z_{\varepsilon} = \varepsilon F_{\varepsilon}^2 \text{ in } (0, T) \times \Omega$$

 $\nabla_{x} \Phi_{\varepsilon} \cdot \mathbf{n}|_{\partial \Omega} = 0$

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Singular limits

Acoustic waves

LARGE DOMAINS

 $\Omega pprox \Omega_{arepsilon}$

 $\partial\Omega_{\varepsilon}=\Gamma_0\cup\Gamma_{\infty}^{\varepsilon}$ Γ_0 regular and compact

$$\varepsilon \operatorname{dist}[\Gamma_{\infty}^{\varepsilon}; \Gamma_{0}] \to \infty$$
 as $\varepsilon \to 0$

WAVE EQUATION REVISITED

$$\varepsilon \partial_t Z_{\varepsilon} + \Delta_x \Phi_{\varepsilon} = \varepsilon G_{\varepsilon}^1 \text{ in } (0, T) \times \Omega$$

$$\varepsilon \partial_t \Phi_{\varepsilon} + \omega Z_{\varepsilon} = \varepsilon F_{\varepsilon}^2 \text{ in } (0, T) \times \Omega$$

 $\nabla_{x} \Phi_{\varepsilon} \cdot \mathbf{n}|_{\partial \Omega} = 0$

Singular limits

—Acoustic waves

Abstract wave equation

$$\begin{split} \varepsilon \partial_t r_\varepsilon - \mathcal{A}[\Phi_\varepsilon] &= \varepsilon h_\varepsilon^1 \\ \varepsilon \partial_t \Phi_\varepsilon + r_\varepsilon &= \varepsilon h_\varepsilon^2 \end{split}$$

$$\mathcal{A}[\mathbf{v}] = -\omega \Delta_{\mathbf{x}} \mathbf{v}, \ \nabla_{\mathbf{x}} \mathbf{v} \cdot \mathbf{n}|_{\partial \Omega} = \mathbf{0}$$

A is a non-negative self-adjoint operator on the Hilbert space $L^2(\Omega)$

$$h_{\varepsilon}^1, h_{\varepsilon}^2 \in L^2(0, T; \mathcal{D}(G(A)))$$

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Duhamel's formula

DUHAMEL'S FORMULA

$$\begin{split} \Phi_{\varepsilon}(t,\cdot) &= \exp\left(\mathrm{i}\frac{t}{\varepsilon}\sqrt{A}\right) \left[\frac{1}{2}\Phi_{0,\varepsilon} + \frac{\mathrm{i}}{2\sqrt{A}}[r_{0,\varepsilon}]\right] \\ &+ \exp\left(-\mathrm{i}\frac{t}{\varepsilon}\sqrt{A}\right) \left[\frac{1}{2}\Phi_{0,\varepsilon} - \frac{\mathrm{i}}{2\sqrt{A}}[r_{0,\varepsilon}]\right] \\ &+ \int_{0}^{t}\exp\left(\mathrm{i}\frac{t-s}{\varepsilon}\sqrt{A}\right) \left[\frac{1}{2}h_{\varepsilon}^{2}(s) + \frac{\mathrm{i}}{2\sqrt{A}}[h_{\varepsilon}^{1}(s)]\right] \mathrm{d}s \\ &+ \int_{0}^{t}\exp\left(-\mathrm{i}\frac{t-s}{\varepsilon}\sqrt{A}\right) \left[\frac{1}{2}h_{\varepsilon}^{2}(s) - \frac{\mathrm{i}}{2\sqrt{A}}[h_{\varepsilon}^{1}(s)]\right] \mathrm{d}s \end{split}$$

Space-time estimates

Theorem [Kato, 1965]

Theorem

Let C be a closed densely defined linear operator and H a self-adjoint densely defined linear operator in a Hilbert space X. For $\lambda \notin R$, let $R_H[\lambda] = (H - \lambda Id)^{-1}$ denote the resolvent of H. Suppose that

$$\Gamma = \sup_{\lambda \notin R, \ v \in \mathcal{D}(C^*), \ \|v\|_X = 1} \|C \circ R_H[\lambda] \circ C^*[v]\|_X < \infty.$$

Then

$$\sup_{w \in X, \|w\|_X = 1} \frac{\pi}{2} \int_{-\infty}^{\infty} \|C \exp(-\mathrm{i} t \mathcal{H})[w]\|_X^2 \, \mathrm{d} t \leq \Gamma^2.$$

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Space-time estimates

Apply Kato's theorem to

$$egin{aligned} &X=L^2(\Omega)\ &H=\sqrt{A},\ C=arphi\circ\chi(A),\ \chi\in C^\infty_c(0,\infty),\ arphi\in C^\infty_0(\Omega)\ &<\chi(A)v;w>=\int_0^\infty\chi(\lambda)\ \mathrm{d}< P_\lambda v;w> \end{aligned}$$

$$\varphi \chi(A) \frac{1}{\sqrt{A} - \lambda} \chi(A) \varphi = \varphi \chi(A) \frac{\sqrt{A} + \lambda}{A - \lambda^2} \chi(A) \varphi$$

LIMITING ABSORPTION PRINCIPLE

$$\sup_{\mu \in \mathcal{C}, \ 0 < \alpha \leq \operatorname{Re}[\mu] \leq \beta < 1, \ 0 < |\operatorname{Im}[\mu]| < 1} \left\| \varphi \frac{1}{A - \mu} \varphi \right\|_{[L^2; L^2]} < c < \infty$$

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Space-time estimates

UNIFORM DECAY

$$\begin{split} &\int_{0}^{T}\int_{0}^{T}\left\|\varphi\chi(A)\exp\left(\mathrm{i}\frac{t-s}{\varepsilon}\sqrt{A}\right)\left[X_{\varepsilon}(s)\right]\right\|_{L^{2}(\Omega)}^{2}\,\mathrm{d}t\,\mathrm{d}s\\ &\leq \varepsilon\int_{0}^{T}\int_{0}^{\infty}\left\|\varphi\chi(A)\exp\left(\mathrm{i}t\sqrt{A}\right)\left[\exp\left(\mathrm{i}\frac{-s}{\varepsilon}\sqrt{A}\right)\left[X_{\varepsilon}(s)\right]\right]\right\|_{L^{2}(\Omega)}^{2}\,\mathrm{d}t\,\mathrm{d}s\\ &\leq \varepsilon c\|X_{\varepsilon}\|_{L^{2}(0,T;L^{2}(\Omega))}^{2} \end{split}$$

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└─ Space-time estimates

RAGE theorem

Theorem

Let X be a Hilbert space, $H : \mathcal{D}(H) \subset X \to X$ a self-adjoint operator, $C : X \to X$ a compact operator, and P_c the orthogonal projection onto H_c , where

$$X = H_c \oplus cl_X \Big\{ \operatorname{span} \{ w \in X \mid w \text{ an eigenvector of } H \} \Big\}.$$

Then

$$\left\|\frac{1}{\tau}\int_0^\tau \exp(-\mathrm{i}tH)CP_c\exp(\mathrm{i}tH)\,\mathrm{d}t\right\|_{\mathcal{L}(X)}\to 0\,\,\text{for}\,\,\tau\to\infty.$$

└─ Space-time estimates

Apply RAGE theorem to

 $X = L^2(\Omega), H = \sqrt{A}, C = \varphi^2 \chi(A)$, with $\varphi \in C_0^{\infty}(\Omega), \varphi \ge 0$, $\chi \in C_0^{\infty}(0, \infty), 0 \le \chi \le 1$:

$$\int_{0}^{T} \left\langle \exp\left(-\mathrm{i}\frac{t}{\varepsilon}\sqrt{A}\right)\varphi^{2}\chi(A)\exp\left(\mathrm{i}\frac{t}{\varepsilon}\sqrt{A}\right)X;Y\right\rangle \,\mathrm{d}t$$
$$=\varepsilon\int_{0}^{T/\varepsilon} \left\langle \exp\left(-\mathrm{i}t\sqrt{A}\right)\varphi^{2}\chi(A)\exp\left(\mathrm{i}t\sqrt{A}\right)X;Y\right\rangle \,\mathrm{d}t$$
$$\leq \omega(\varepsilon)\|X\|_{L^{2}(\Omega)}\|Y\|_{L^{2}(\Omega)},$$
ere $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$

where $\omega(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$

└─Space-time estimates

For Y = G(A)[X] we deduce that:

$$\int_{0}^{T} \left\| \varphi \chi(A) \exp\left(i \frac{t}{\varepsilon} \sqrt{A} \right) [X] \right\|_{L^{2}(\Omega)}^{2} dt$$
$$\leq \omega(\varepsilon) \|X\|_{L^{2}(\Omega)}^{2} \text{ for any } X \in L^{2}(\Omega)$$

 $\omega(\varepsilon) \rightarrow 0 \text{ for } \varepsilon \rightarrow 0$

Kato's theorem gives the same result with $\omega(\varepsilon) = \varepsilon$

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