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**Liftings and dilations of commuting
systems of linear mappings
on vector spaces**

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LIFTINGS AND DILATIONS OF COMMUTING SYSTEMS OF LINEAR MAPPINGS ON VECTOR SPACES

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ABSTRACT. We show that each set of commuting linear mappings on a vector space has a lifting consisting of commuting injective mappings and a dilation consisting of commuting bijective mappings.

1. INTRODUCTION

The dilation theory of Hilbert space operators is an important part of operator theory. The most important result is that each Hilbert space contraction has a unitary dilation, i.e., if T is a contraction acting on a Hilbert space H , then there exists a Hilbert space $K \supset H$ and a unitary operator $U \in B(K)$ such that $T^n = P_H U^n|_H$ for all $n \geq 0$, where P_H is the orthogonal projection onto H .

A closely related result is the existence of isometrical liftings of Hilbert space contractions. For each contraction $T \in B(H)$ there exists a Hilbert space $L \supset H$ and an isometry $V \in B(L)$ such that $T^* = V^*|_H$. Clearly V is an isometrical dilation of T .

The isometrical and unitary dilations proved to be a very useful tool in operator theory with many applications in various situations.

Apart from the dilations of single operators, dilations of commuting tuples have also been considered intensely. By the Ando theorem, every pair of commuting Hilbert space contractions has a dilation consisting of two commuting unitary operators. For more than two contractions this is not true in general, but there are many results giving dilations in important particular situations.

The standard reference for the dilation theory is the monograph of B. Sz.-Nagy and C. Foias [4].

In a recent paper [1], Bhat, De and Rakshit proved analogous dilation results for mappings on sets. The dilations in the category of vector spaces and linear mapping among them was considered in [2] and [3].

The main result of [2] was that each linear mapping on a vector space has a bijective dilation. In [3], it was proved an Ando type result and shown that each pair of commuting linear mapping has a dilation consisting of two commuting injective linear mapping.

In the present paper we improve these results and show that any set (finite or infinite) of mutually commuting linear mappings on a vector space has a commuting lifting consisting of injective mappings, and a dilation consisting of bijective mappings. This gives also an answer to some questions posed in [3]

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2. LIFTINGS OF LINEAR MAPPINGS

We use the following notations.

Notations. Let J be a set and $\mathcal{T} = \{T_j : j \in J\}$ a system of mutually commuting linear mappings on a vector space X . Let Y be a vector space, $X \subset Y$, and let $\mathcal{S} = \{S_j : j \in J\}$ be a system of mutually commuting linear mappings on Y . We say that \mathcal{S} is an extension of \mathcal{T} if $T_j = S_j|_X$ for all $j \in J$.

Let $P : Y \rightarrow Y$ be a projection onto X (i.e., P is a linear mapping such that $P^2 = P$ and $PY = X$). Then \mathcal{S} is an extension of \mathcal{T} if and only if each S_j has the form

$$S_j = \begin{pmatrix} T_j & * \\ 0 & * \end{pmatrix}$$

in the decomposition $Y = X \oplus (I - P)Y$.

We say that \mathcal{S} is a lifting of \mathcal{T} if there exists a projection P onto X such that each S_j has the form

$$S_j = \begin{pmatrix} T_j & 0 \\ * & * \end{pmatrix}$$

in the same decomposition.

We say that \mathcal{S} is a power dilation of \mathcal{T} if there exists a projection P onto X such that

$$S_{j_k} \cdots S_{j_2} S_{j_1} = \begin{pmatrix} T_{j_k} \cdots T_{j_1} & * \\ * & * \end{pmatrix}$$

for all $k \in \mathbb{N}$ and $j_1, \dots, j_k \in J$. Equivalently, $T_{j_k} \cdots T_{j_1} = PS_{j_k} \cdots S_{j_1}|_X$ for all $j_1, \dots, j_k \in J$.

Clearly if \mathcal{S} is either an extension or a lifting of \mathcal{T} , then \mathcal{S} is a power dilation of \mathcal{T} .

We show first that each commuting system of linear mappings has a lifting consisting of mutually commuting injective linear mappings.

We start with the following lemma.

Lemma 1. Let $\mathcal{T} = \{T_j : j \in J\}$ be a system of mutually commuting linear mappings on a vector space X . Let $j_0 \in J$. Then there exists a vector space $Y \supset X$ and a system $\mathcal{S} = \{S_j : j \in J\}$ of mutually commuting linear mappings on Y such that

- (i) \mathcal{S} is a lifting of \mathcal{T} ;
- (ii) S_{j_0} is injective;
- (iii) if $j \in J$ and T_j is injective then S_j is injective.

Proof. Let Y be the vector space of all sequences (x_0, x_1, x_2, \dots) of elements of X with finite support.

Define the linear mapping $S_{j_0} : Y \rightarrow Y$ by

$$S_{j_0}(x_0, x_1, \dots) = (T_{j_0}x_0, x_0, x_1, x_2, \dots).$$

For $j \in J$, $j \neq j_0$ define mapping $S_j : Y \rightarrow Y$ by

$$S_j(x_0, x_1, \dots) = (T_jx_0, T_jx_1, T_jx_2, \dots).$$

Clearly $S_j S_{j'} = S_{j'} S_j$ for all $j, j' \in J$, $j \neq j_0 \neq j'$.

Let $j \in J$, $j \neq j_0$. Then

$$S_{j_0} S_j(x_0, x_1, \dots) = S_{j_0}(T_jx_0, T_jx_1, T_jx_2, \dots) = (T_{j_0}T_jx_0, T_jx_0, T_jx_1, \dots)$$

and

$$S_j S_{j_0}(x_0, x_1, \dots) = S_j(T_{j_0}x_0, x_0, x_1, \dots) = (T_j T_{j_0}x_0, T_j x_0, T_j x_1, \dots).$$

Since $T_j T_{j_0} = T_{j_0} T_j$, we have $S_j S_{j_0} = S_{j_0} S_j$ and the mappings S_j ($j \in J$) are mutually commuting.

If we identify each vector $x \in X$ with the sequence $(x, 0, 0, \dots) \in Y$, then $X \subset Y$. Let $P : Y \rightarrow Y$ be the projection defined by $S(x_0, x_1, x_2, \dots) = (x, 0, 0, \dots)$. Then clearly each S_j ($j \in J$) is a lifting of T_j with respect to this projection.

Let $j \in J$, $j \neq j_0$ and let T_j be injective. Then clearly S_j is injective.

Suppose that $S_{j_0}(x_0, x_1, \dots) = (0, 0, \dots)$. Since $S_{j_0}(x_0, x_1, \dots) = (T_{j_0}x_0, x_0, x_1, \dots)$, we have $x_i = 0$ for all $i \geq 0$. So S_{j_0} is injective.

This finishes the proof.

Corollary 2. Let $\mathcal{T} = \{T_j : j \in J\}$ be a system of mutually commuting linear mappings on a vector space X . Then there exists a vector space $Y \supset X$ and a system $\mathcal{S} = \{S_j : j \in J\}$ of mutually commuting injective linear mappings on Y such that \mathcal{S} is a lifting of \mathcal{T} .

Proof. If the set J is finite, then we can apply the previous lemma finitely many times, in each step increasing the number of injective mappings. If the set J is countable then the required lifting can be constructed by induction.

If the set J is uncountable then we can proceed similarly, using the transfinite induction. We may assume that the set J is well ordered, $J = \{j_\alpha : \alpha < \beta\}$ for some ordinal number β . By transfinite induction we can construct vector spaces Y_α ($\alpha \leq \beta$) such that $Y_0 = X$ and $Y_\alpha \subset Y_{\alpha'}$ whenever $\alpha < \alpha' \leq \beta$, and commuting systems $\mathcal{S}^{(\alpha)} = \{S_\gamma^{(\alpha)} : \gamma < \beta\}$ on Y_α such that the mappings $S_\gamma^{(\alpha)}$ are injective for all $\gamma < \alpha$ and $\mathcal{S}^{(\alpha')}$ is a lifting of $\mathcal{S}^{(\alpha)}$ whenever $\alpha < \alpha' \leq \beta$.

Setting $Y = Y_\beta$ and $\mathcal{S} = \mathcal{S}^{(\beta)}$, we get the required lifting of \mathcal{T} consisting of injective mappings.

3. DILATIONS OF LINEAR MAPPINGS

Next we show that any commuting system of injective mappings may be extended to a system consisting of bijective mappings.

Lemma 3. Let $\mathcal{T} = \{T_j : j \in J\}$ be a system of mutually commuting injective linear mappings on a vector space X . Let $j_0 \in J$. Then there exists a vector space $Y \supset X$ and a system $\mathcal{S} = \{S_j : j \in J\}$ of mutually commuting injective linear mappings on Y such that

- (i) \mathcal{S} is an extension of \mathcal{T} ;
- (i) S_{j_0} is a bijection;
- (ii) if $j \in J$ and T_j is a bijection then S_j is a bijection.

Proof. Let Z be the vector space of all sequences (x_0, x_1, x_2, \dots) of elements of X with finite support.

Define the linear mapping $V_{j_0} : Z \rightarrow Z$ by

$$V_{j_0}(x_0, x_1, \dots) = (T_{j_0}x_0 + x_1, x_2, x_3, \dots).$$

For $j \in J$, $j \neq j_0$ define mappings $V_j : Z \rightarrow Z$ by

$$V_j(x_0, x_1, \dots) = (T_j x_0, T_j x_1, T_j x_2, \dots).$$

Clearly $V_j V_{j'} = V_{j'} V_j$ for all $j, j' \in J$, $j \neq j_0 \neq j'$.

Let $j \in J$, $j \neq j_0$. Then

$$V_j V_{j_0}(x_0, x_1, \dots) = V_j(T_{j_0}x_0 + x_1, x_2, x_3, \dots) = (T_j T_{j_0}x_0 + T_j x_1, T_j x_2, T_j x_3, \dots)$$

and

$$V_{j_0} V_j(x_0, x_1, \dots) = V_{j_0}(T_j x_0, T_j x_1, T_j x_2, \dots) = (T_{j_0} T_j x_0 + T_j x_1, T_j x_2, \dots).$$

So $V_j V_{j_0} = V_{j_0} V_j$ and the mappings V_j ($j \in J$) are mutually commuting.

Let Z_0 be the subspace of Z formed by all sequences $(x_0, x_1, \dots) \in Z$ such that there exists $k \in \mathbb{N}$ with $x_i = 0$ for all $i > k$ and $\sum_{i=0}^k T_{j_0}^{k-i} x_i = 0$.

Clearly $V_j Z_0 \subset Z_0$ for all $j \in J$, $j \neq j_0$.

If $(x_0, x_1, \dots) \in Z_0$, i.e., $\sum_{i=0}^k T_{j_0}^{k-i} x_i = 0$ for some k , then $V_{j_0}(x_0, x_1, \dots) = (T_{j_0}x_0 + x_1, x_2, \dots)$ and

$$T_{j_0}^{k-1}(T_{j_0}x_0 + x_1) + \sum_{i=2}^k T_{j_0}^{k-i} x_i = 0,$$

and so $V_{j_0}(x_0, x_1, \dots) \in Z_0$. Hence $V_{j_0} Z_0 \subset Z_0$.

Let $Y = Z/Z_0$ and let $S_j : Y \rightarrow Y$ be the quotient mappings induced by V_j ($j \in J$). Clearly the mappings S_j ($j \in J$) are mutually commuting.

Let us identify a vector $x \in X$ with the class $(x, 0, 0, \dots) + Z_0 \in Y$. Note that if $x \neq 0$ then $(x, 0, 0, \dots) \notin Z_0$ since T_{j_0} is injective. With this identification X will become a subspace of Y and each S_j ($j \in J$) is an extension of T_j .

The mapping V_{j_0} is surjective, and so is S_{j_0} .

If $j \in J$ and T_j is surjective, then V_j is surjective, and so is S_j .

It remains to show that the mappings S_j are injective.

Let $j \in J$, $j \neq j_0$ and let $S_j((x_0, x_1, \dots) + Z_0) = Z_0$. Then $S_j(x_0, x_1, \dots) = (T_j x_0, T_j x_1, \dots) \in Z_0$, and so

$$\sum_{i=0}^k T_{j_0}^{k-i} T_j x_i = 0$$

for some $k \in \mathbb{N}$ with $x_i = 0$ for all $i > k$. Since T_j is injective and commutes with T_{j_0} , we have $\sum_{i=0}^k T_{j_0}^{k-i} x_i = 0$, and so $(x_0, x_1, \dots) \in Z_0$. Hence S_j is injective.

Let $S_{j_0}((x_0, x_1, \dots) + Z_0) = Z_0$, i.e., $S_{j_0}(x_0, x_1, \dots) = (T_{j_0}x_0 + x_1, x_2, \dots) \in Z_0$. Then

$$\sum_{i=0}^{k+1} T_{j_0}^{k+1-i} x_i = T_{j_0}^k(T_{j_0}x_0 + x_1) + \sum_{i=2}^k T_{j_0}^{k-i+1} x_i = 0$$

for all k large enough. So $(x_0, x_1, \dots) \in Z_0$ and S_{j_0} is injective.

This finishes the proof.

Corollary 4. Let $\mathcal{T} = \{T_j : j \in J\}$ be a system of mutually commuting injective linear mappings on a vector space X . Then there exists a vector space $Y \supset X$ and a system $\mathcal{S} = \{S_j : j \in J\}$ of mutually commuting bijective linear mappings on Y such that \mathcal{S} is an extension of \mathcal{T} .

Proof. As in the proof of Corollary 2 we can use the previous lemma and construct the required extension \mathcal{S} by transfinite induction.

Corollary 5. Let $\mathcal{T} = \{T_j : j \in J\}$ be a system of mutually commuting linear mappings on a vector space X . Then there exists a vector space $Y \supset X$ and a

system $\mathcal{S} = \{S_j : j \in J\}$ of mutually commuting bijective linear mappings on Y such that \mathcal{S} is a power dilation of \mathcal{T} (with respect to some projection $P : Y \rightarrow Y$ onto X).

Proof. By Corollary 2, there exists a vector space $Z \supset X$ and a commuting system $\mathcal{V} = \{V_j : j \in J\}$ of injective mappings on Z , which is a lifting of \mathcal{T} .

By Corollary 4, it is possible to extend the system \mathcal{V} to a commuting system $\mathcal{S} = \{S_j : j \in J\}$ on a space $Y \supset Z$, which consists of bijective mappings.

Clearly \mathcal{S} is a power dilation of \mathcal{T} .

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