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Dedicated to the Memory of Professor Antonín Novotný

Abstract

As an extension of the recent work of Novotný et al. [17], we study the dissipative weak solutions to a compressible two-fluid model system describing the time evolution of two fluid flows sharing the same velocity field in multi-dimensional spaces. We prove the existence of dissipative weak solutions alternatively via a finite volume approximation. Further, we apply the weak–strong uniqueness principle to show the convergence of the finite volume approximation towards the strong solution on the lifespan of the latter.

Keywords: two-fluid model, dissipative weak solutions, finite volume method, convergence
Mathematics Subject Classification: 76T17; 35D30; 76M12; 65N22

Contents

1 Introduction

In this paper, we are concerned with a compressible two-fluid model, whose governing equations take the form:

$$\begin{cases} \partial_t \varrho + \mathrm{div}_x(\varrho \boldsymbol{u}) = 0, \\ \partial_t n + \mathrm{div}_x(n \boldsymbol{u}) = 0, \\ \partial_t(\mathfrak{r}\boldsymbol{u}) + \mathrm{div}_x(\mathfrak{r}\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p(\varrho, n) = \mathrm{div}_x \mathbb{S}(\nabla_x \boldsymbol{u}). \end{cases} \tag{1.1}$$

Here, $t \in (0,T)$ and $x \in \Omega$ represent the time variable and space variable, respectively; Ω is a bounded smooth domain in $\mathbb{R}^d$, $d = 2,3$. We denote by $\varrho = \varrho(t,x)$ and $n = n(t,x)$ the densities of two different fluids, $\mathfrak{r} = \varrho + n$ the total density of the mixture, $\boldsymbol{u} = \boldsymbol{u}(t,x) \in \mathbb{R}^d$ the velocity field of the mixture, $p = p(\varrho, n)$ the scalar pressure. The pressure is a nonlinear function of the two densities given by

$$p(\varrho, n) = \varrho^\gamma + n^\alpha, \tag{1.2}$$

where $\gamma, \alpha > 1$ are the adiabatic exponents. Further, $\mathbb{S} = \mathbb{S}(\nabla_x \boldsymbol{u})$ stands for the Newtonian viscous stress tensor

$$\mathbb{S}(\nabla_x \boldsymbol{u}) = \mu \left(\nabla_x \boldsymbol{u} + \nabla_x^t \boldsymbol{u} - \frac{2}{d} \mathrm{div}_x \boldsymbol{u}\, \mathbb{I} \right) + \lambda \mathrm{div}_x \boldsymbol{u}\, \mathbb{I}, \tag{1.3}$$

where $\mu > 0$ and $\lambda \geq 0$ are the shear viscosity coefficient and bulk viscosity coefficient, respectively.

System (1.1) is formally closed with

$$\begin{cases} \text{either periodic boundary conditions with} & \Omega = \mathcal{T}^d = \left([0,1]_{\{0,1\}}\right)^d, \\ \text{or non-slip boundary conditions} & \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{0}, \end{cases} \tag{1.4}$$

as well as the initial conditions:

$$(\varrho, n, \mathfrak{r}\boldsymbol{u})|_{t=0} = (\varrho_0, n_0, \mathbf{m}_0). \tag{1.5}$$

When $n = 0$, system (1.1) reduces to the compressible Navier-Stokes system. The existence of global weak solutions with finite energy initial data was first obtained by Lions [21] for any $\gamma \geq \frac{9}{5}$ ($d = 3$), and refined by Feireisl et al. [10] for any $\gamma > \frac{3}{2}$. When $n \neq 0$, the corresponding mathematical results are not so fruitful. In case of $d = 1$, Evje et al. [5] showed the existence of weak solutions to the liquid-gas two-fluid model (see also [6]), meaning the pressure takes the form $p = p(\varrho, n) = -b(\varrho, n) + \sqrt{b^2(\varrho, n) + c(n)}$, with $b(\varrho, n), c(n)$ being linear functions of ϱ, n. In the case of $d = 2$, Yao et al. [28] established the existence and asymptotic behavior of global weak solutions to the liquid-gas two-fluid model. For $d = 3$ we refer to Bresch et al. [3] for the existence of weak solutions to the compressible two-fluid model with algebraic pressure closure, which is extended to more general equation of state by Novotný et al. [25]. We refer to the monographs [2, 14] for more discussions on other physically relevant two-fluid models. By developing the new method of variable reduction, Vasseur et al. [26] proved the existence of global weak solutions to (1.1)–(1.5) for $d = 3$ with some conditions between the initial densities or adiabatic parameters. These constraints are further relaxed by Wen [27] to any $\gamma \geq \frac{9}{5}, \alpha \geq \frac{9}{5}$ without any domination conditions.

Very recently, Novotný et al. [17] studied the existence of dissipative weak (DW) solutions for the two-fluid system for all adiabatic parameter $\gamma, \alpha > 1$ as well as the dissipative weak–strong uniqueness principle. We emphasize that this work shall play an important role in the numerical analysis, especially the convergence analysis, of numerical solutions of the system (1.1). Let us point out that the convergence analysis of numerical solutions to compressible viscous fluids has been a challenging problem in the past decades. Pioneered from the convergence of (a suitable subsequence of) a mixed finite volume (FV)–finite element (FE) approximation towards a weak solution in the work of Karper [18], more recent works have been done via the weak–strong uniqueness principle in the class of dissipative measure-valued solutions, see Feireisl et al. [7, 8, 23] and the monograph [9] for an overview.

The idea of the current paper falls into the approach of Feireisl et al [7, 8, 9] with a slight adaptation to the class of DW solutions to the two-fluid system. We first recall the recent concept of DW solutions to the system by incorporating the effects of oscillations and concentrations hidden in some nonlinear terms. Then, we extend the finite volume method studied in [8] to the two-fluid model and prove the existence of DW solutions by passing to the limit of the finite volume approximation. Finally, applying the weak–strong uniqueness principle we also derive the convergence of the finite volume approximation to the classical solution on the lifespan of the latter.

Our work can be view as an extension to Novotný et al. [17] in two folds. First, we provide an alternative proof for the existence of DW solutions via a numerical construction. Second, we show the application of the weak–strong uniqueness argument in the convergence analysis of numerical methods.

1.1 Dissipative weak solutions

Before giving a definition to dissipative weak solutions, we introduce some notations. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and Q_T be the time-space cylinder $(0,T) \times \Omega$. $\mathcal{M}(\overline{\Omega})$ signifies the space of signed Borel measures over $\overline{\Omega}$. $\mathcal{M}^+(\overline{\Omega})$ is the subspace of $\mathcal{M}(\overline{\Omega})$ representing non-negative measures. $A \lesssim B$ means $A \leq CB$ for generic positive constants C changing from line to line. The space of $d \times d$ matrices are denoted by $\mathbb{R}^{d\times d}$, while its subspace $\mathbb{R}^{d\times d}_{sym}$ represents symmetric ones.

We are now ready to introduce the concept of dissipative weak solutions to (1.1)–(1.5).

Definition 1.1 (Dissipative weak solution)**.** *A triple $(\varrho, n, \boldsymbol{u})$ is said to be a dissipative weak solution to the compressible two-fluid model (1.1)–(1.5) in Q_T provided that*

- *Regularities class*

$$\begin{cases} \varrho, n \geq 0 \ a.e. \ \text{in } Q_T, \\ \varrho \in L^\infty(0,T; L^\gamma(\Omega)), \ n \in L^\infty(0,T; L^\alpha(\Omega)), \\ \boldsymbol{u} \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^d)), \\ \sqrt{\mathfrak{r}}\, \boldsymbol{u} \in L^\infty(0,T; L^2(\Omega; \mathbb{R}^d)); \end{cases} \tag{1.6}$$

- *The continuity equation for ϱ*

$$\int_0^\tau \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \boldsymbol{u} \cdot \nabla_x \varphi \Big) \mathrm{d}x \mathrm{d}t = \left[\int_\Omega \varrho \varphi \mathrm{d}x \right]_{t=0}^{t=\tau} \tag{1.7}$$

  for any $\varphi \in C^1(\overline{Q_T})$;

- *The continuity equation for n*

$$\int_0^\tau \int_\Omega \Big(n \partial_t \varphi + n \boldsymbol{u} \cdot \nabla_x \varphi \Big) \mathrm{d}x \mathrm{d}t = \left[\int_\Omega n \varphi \mathrm{d}x \right]_{t=0}^{t=\tau} \tag{1.8}$$

  for any $\varphi \in C^1(\overline{Q_T})$;

- *Balance of momentum*

$$\int_0^\tau \int_\Omega \Big(\mathfrak{r}\boldsymbol{u} \cdot \partial_t \boldsymbol{\varphi} + \mathfrak{r}\boldsymbol{u} \otimes \boldsymbol{u} : \nabla_x \boldsymbol{\varphi} + p(\varrho, n)\mathrm{div}_x \boldsymbol{\varphi} - \mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{\varphi} \Big) \mathrm{d}x\mathrm{d}t$$
$$+ \int_0^\tau \int_{\overline{\Omega}} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mu_c(t)\mathrm{d}t = \left[\int_\Omega \mathfrak{r}\boldsymbol{u} \cdot \boldsymbol{\varphi} \mathrm{d}x \right]_{t=0}^{t=\tau} \tag{1.9}$$

for any $\boldsymbol{\varphi} \in C_c^1([0,T] \times \Omega; \mathbb{R}^d)$ *and certain* $\mu_c \in L^\infty(0,T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^{d\times d}_{sym}))$*;*

- *Balance of total energy*

$$\int_\Omega \left(\frac{1}{2}\mathfrak{r}|\boldsymbol{u}|^2 + P(\varrho, n) \right) \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{u} \mathrm{d}x\mathrm{d}t$$
$$+ \int_{\overline{\Omega}} \mathrm{d}\mathcal{D}(\tau) \le \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0 + n_0} + P(\varrho_0, n_0) \right) \mathrm{d}x \tag{1.10}$$

for a.e. $\tau \in (0,T)$ *and certain* $\mathcal{D} \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}))$*; Here, we denoted by* $P(\varrho, n)$ *the potential energy*

$$P(\varrho, n) = H(\varrho) + G(n), \quad \text{with } H(\varrho) := \frac{1}{\gamma - 1}\varrho^\gamma, \quad G(\alpha) := \frac{1}{\alpha - 1} n^\alpha;$$

- *Compatibility condition*

$$|\mu_c(\tau)| \lesssim \mathcal{D}(\tau) \tag{1.11}$$

for a.e. $\tau \in (0,T)$.

Remark 1.1. *We now make some comments about the definition. The regularity class (1.6) mainly comes from the energy inequality. In (1.9), the measure* μ_c *represents the concentration and oscillation phenomena resulting from the nonlinearities* $\mathfrak{r}\boldsymbol{u} \otimes \boldsymbol{u}$, ϱ^γ, n^α*. In (1.10), the non-negative measure* $\mathcal{D}$ *signifies the defect of energy dissipation. Furthermore, the two measures are interrelated by (1.11), which is generally satisfied for suitable approximating scheme. In case of periodic boundary conditions, one defines* $\boldsymbol{u} \in L^2(0,T; W^{1,2}(\mathcal{T}^d; \mathbb{R}^d))$ *and other items remain basically unchanged.*

1.2 Main theorem

Before stating the main result of this paper, we recall the weak–strong uniqueness principle related to the stability of strong solutions within the framework of dissipative weak solutions.

Proposition 1.1. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded smooth domain and the pressure* $p(\varrho, n)$ *be subject to (1.2) with* $\gamma > 1$, $\alpha > 1$*. Assume that* $(\tilde{\varrho}, \tilde{n}, \tilde{\boldsymbol{u}})$ *is a strong solution to (1.1)–(1.5) emanating from* $(\tilde{\varrho}_0, \tilde{n}_0, \tilde{\boldsymbol{u}}_0)$ *and belonging to*

$$\begin{cases} \inf\limits_{Q_T} \tilde{\varrho} > 0, \quad \inf\limits_{Q_T} \tilde{n} > 0, \\ (\tilde{\varrho}, \tilde{n}) \in C^1(\overline{Q_T}; \mathbb{R}^2), \quad \tilde{\boldsymbol{u}} \in C^1(\overline{Q_T}; \mathbb{R}^d), \\ \mathrm{div}_x \mathbb{S}(\nabla_x \tilde{\boldsymbol{u}}) \in C(\overline{Q_T}; \mathbb{R}^d). \end{cases} \tag{1.12}$$

Let $(\varrho, n, \boldsymbol{u})$ *be a dissipative weak solution to (1.1)–(1.5) with initial data* $(\tilde{\varrho}_0, \tilde{n}_0, (\tilde{\varrho}_0 + \tilde{n}_0)\tilde{\boldsymbol{u}}_0)$ *in the sense of Definition 1.1. Then*

$$\varrho = \tilde{\varrho}, \;\; n = \tilde{n}, \;\; \boldsymbol{u} = \tilde{\boldsymbol{u}} \text{ in } Q_T,$$
$$\mu_c = \mathbf{0}, \quad \mathcal{D} = 0.$$

Remark 1.2. *In a very recent work of Novotný et al. [17], the authors considered a mixture of two non-interacting compressible fluids filling a bounded domain with general non zero inflow/outfow boundary conditions and established the weak–strong uniqueness principle. To make the paper self-contained, we shall present the proof of Proposition 1.1 in the Appendix A.*

Our main theorem is concerned with the existence of dissipative weak solutions, which is accomplished by a finite volume method. We may also interpret it as the convergence proof of the finite volume approximation thanks to the weak–strong uniqueness principle stated above. More specifically, the theorem reads:

Theorem 1.1. *Let $\gamma > 1$, $\alpha > 1$, and $(\varrho_h, n_h, \boldsymbol{u}_h)$ be a solution to the finite volume method defined in Definition 2.1 with discretization parameters $\Delta t \approx h \in (0,1)$, and the initial data (1.5) be subject to*

$$\varrho_0 > 0,\ n_0 > 0,\ \varrho_0 \in L^\infty(\Omega),\ n_0 \in L^\infty(\Omega), \boldsymbol{u}_0 \in L^\infty(\Omega;\mathbb{R}^d).$$

Then we have the following convergence results:

1. *There exists a subsequence of $(\varrho_h, n_h, \boldsymbol{u}_h)$, not relabelled, such that*

$$\begin{aligned} &\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)),\\ &n_h \to n \text{ weakly-(*) in } L^\infty(0,T;L^\alpha(\Omega)),\\ &\boldsymbol{u}_h \to \boldsymbol{u} \text{ weakly in } L^2(Q_T;\mathbb{R}^d), \end{aligned} \tag{1.13}$$

   where $(\varrho, n, \boldsymbol{u})$ represents a dissipative weak solution of the two-fluid system (1.1)–(1.5) in the sense of Definition 1.1.

2. *In addition, suppose that the two-fluid system (1.1)–(1.5) admits a strong solution in the class (1.12). Then the convergence in (1.13) is strong and unconditional (for the whole sequence). Moreover, the limit quantity $(\varrho, n, \boldsymbol{u})$ coincides with the strong solution.*

Remark 1.3. *We give several remarks on strong solutions to the two-fluid system (1.1)–(1.5). Roughly speaking, the results are in the same line with "mono-fluid". When the initial densities contain no vacuum states, local existence and uniqueness of strong solutions may be proved by the classical iteration method and the Schauder fixed point theorem, see [15, 24]; while the global existence and uniqueness of strong solutions may be obtained when the initial densities are sufficiently close to non-vacuum constant states, see [22]. When the initial densities allow vacuum, one also has local well-posedness, similar to [4].*

The rest of this article is structured as follows. The existence of dissipative weak solutions is given in Section 2 through a finite volume approximation scheme. Section 3 is the conclusion. In the appendix, we present the proof of weak–strong uniqueness principle by the relative energy inequality.

2 Proof of the main results

In this section, we aim to construct a dissipative weak solution in the sense of Definition 1.1 via a numerical approach. The by-product can be view as the convergence analysis of the numerical approximation.

As discussed in the introduction, the two-fluid system (1.1)–(1.5) degenerate to the classical barotropic Naiver-Stokes system when either of the two fluids disappears ($\varrho = 0$ or $n = 0$). Here, we may select any numerical method from [7, 8, 23] and further extend it for the approximation of the two-fluid system. As an example, we adapt the fundamental finite volume method [8] to the approximation of the two-fluid system, and leave the analysis of other methods mentioned above to the reader as an exercise. For simplicity, we consider the no-slip boundary condition in (1.4) and point out that the upcoming analysis holds also for periodic boundary conditions.

In the following subsections, we shall introduce the FV method, discuss the stability and consistency of the FV approximation and finally show the convergence towards a dissipative weak solution as well as a strong solution.

2.1 A finite volume method

In this subsection we propose a FV method for the approximation of two fluid system (1.1)–(1.5). To begin, we approximate the physical domain Ω by a family of computational domains Ω_h in the following way.

Mesh. Let Ω_h be a uniform mesh discretization of Ω consists of squares (if $d = 2$) or cubes ($d = 3$) with the following notations:

- We denote by $h = \max_{1\leq i\leq d} h_i$ the size of the mesh discretization, where h_i is the uniform length of any arbitrary cell in the i^{th}-direction. Moreover, we assume $h \in (0, h_0)$ for some $h_0 < 1$.
- We denote by $\mathcal{E}$ the set of all faces, $\mathcal{E}_{\text{ext}}$ the set of all faces on the boundary, $\mathcal{E}_{\text{int}} = \mathcal{E}\backslash\mathcal{E}_{\text{ext}}$ the set of all interior faces, and by $\mathcal{E}_i$, $i = 1, \ldots, d$, the set of all faces that are orthogonal to the basis vector $\mathbf{e}_i$ of the Cartesian coordinate system. By $\mathcal{E}(K)$ we denote the set of faces of an element K. We further denote by $\boldsymbol{n}$ the outer normal vector of a generic face $\sigma \in \mathcal{E}$.
- For any $\sigma \in \mathcal{E}$ we write $\sigma = K|L$ if $\sigma = \mathcal{E}(K) \cap \mathcal{E}(L)$. For any $\sigma = K|L \in \mathcal{E}_i,\ i \in 1, \ldots, d$, we also denote by $d_\sigma = h_i$ the distance between the barycenters of K and L.
- By $|K|$ and $|\sigma|$ we denote the (d– and $(d-1)$–dimensional) Lebesgue measure of an element K, and a face σ, respectively. Obviously, $|K| = h_i|\sigma|$ for any $\sigma \in \mathcal{E}(K) \cap \mathcal{E}_i$.
- A dual element D_σ is associated to a generic face $\sigma = K|L \in \mathcal{E}_{\text{int}}$, where $D_\sigma = D_{\sigma,K} \cup D_{\sigma,L}$, and $D_{\sigma,K}$ (resp. $D_{\sigma,L}$) is built by half of K (resp. L), see Figure 1 for an example of such cell. Note that $D_\sigma = D_{\sigma,K}$ if $\sigma = \mathcal{E}(K) \cap \mathcal{E}_{\text{ext}}$.

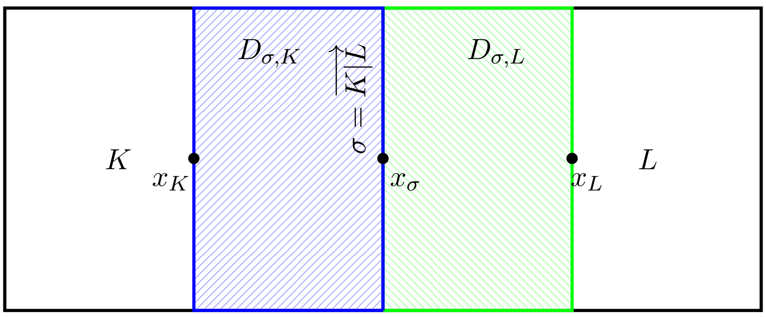

Figure 1: Dual grid

Function space. In order to introduce a finite volume approximation we define a space of piecewise constant functions Q_h on the primary grid Ω_h associated with the standard projection operator

$$\Pi_h^Q : L^1(\Omega_h) \to Q_h. \quad \Pi_h^Q \phi = \sum_{K \in \Omega_h} 1_K \frac{1}{|K|} \int_K \phi \mathrm{d}x.$$

Thanks to the above notations, we denote $(\varrho_h, n_h, \boldsymbol{u}_h) \in Q_h \times Q_h \times \mathbf{Q}_h$ as the piecewise constant approximation of the continuous functions $(\varrho, n, \boldsymbol{u})$, where $\boldsymbol{u}_h \in \mathbf{Q}_h := Q_h(\Omega_h; R^d)$ means $v_{i,h} \in Q_h$ for all $i = 1, \ldots, d$. The discrete pressure shall be denoted as $p_h = p(\varrho_h, n_h)$, where the function p is given in (1.2).

Remark 2.1. *Note that the physical domain Ω is smooth while the computational domain Ω_h is only Lipschitz (polygonal), thus $\Omega \neq \Omega_h$. To avoid problems from this mismatch, we assume*

$$\Omega_h \subset \Omega \text{ and } K \subset \Omega_h \text{ for any compact } K \subset \Omega \text{ for all } h \in (0, h_0).$$

Moreover, we extend the densities ϱ_h, n_h and $\boldsymbol{u}_h$ outside Ω_h to be $\underline{\varrho} > 0$, $\underline{n} > 0$, and zero, respectively. Consequently, we may replace $\int_{\Omega_h} \mathrm{d}x$ by $\int_\Omega \mathrm{d}x$ in the upcoming analysis, see [7] and [9, Chapter 13]. On the other hand, when shifting to periodic boundary conditions where the physical domain can be identified by a flat torus, we do not require the above assumption nor extension.

For a piecewise continuous function v_h we define

$$v_h^{\text{out}}(x) = \lim_{\delta \to 0+} v_h(x + \delta \boldsymbol{n}), \ x \in \sigma \in \mathcal{E}_{\text{int}}, \quad \text{and} \quad v_h^{\text{in}}(x) = \lim_{\delta \to 0+} v_h(x - \delta \boldsymbol{n}), \ x \in \sigma \in \mathcal{E},$$

Concerning $x \in \sigma \in \mathcal{E}_{\text{ext}}$ and $v_h \in \{\varrho_h, n_h, \boldsymbol{u}_h\}$, we implicitly define $v_h^{\text{out}}(x)$ according to the following boundary conditions

$$[\![\varrho_h]\!]_\sigma = 0 = [\![n_h]\!]_\sigma \text{ and } \{\!\{\boldsymbol{u}_h\}\!\}_\sigma = 0, \tag{2.1}$$

where we have denoted

$$\{\!\{v_h\}\!\} = \frac{v_h^{\text{in}} + v_h^{\text{out}}}{2}, \quad \text{and} \quad [\![v_h]\!] = v_h^{\text{out}} - v_h^{\text{in}}.$$

Diffusive upwind flux. Given the velocity field $\boldsymbol{u}_h \in \mathbf{Q}_h$, the upwind flux for any function $r_h \in Q_h$ is specified at each face $\sigma \in \mathcal{E}$ by

$$\mathrm{Up}[r_h, \boldsymbol{u}_h]|_\sigma = r_h^{\text{up}} u_\sigma = r_h^{\text{in}}[u_\sigma]^+ + r_h^{\text{out}}[u_\sigma]^- = \{\!\{r\}\!\} u_\sigma - \frac{1}{2}|u_\sigma| [\![r_h]\!],$$

where

$$u_\sigma = \{\!\{\boldsymbol{u}_h\}\!\}_\sigma \cdot \boldsymbol{n}, \quad [r_h]^\pm = \frac{r_h \pm |r_h|}{2} \quad \text{and} \quad r_h^{\text{up}} = \begin{cases} r_h^{\text{in}} & \text{if } u_\sigma \geq 0, \\ r_h^{\text{out}} & \text{if } u_\sigma < 0. \end{cases}$$

Furthermore, we consider a diffusive numerical flux function of the following form

$$\mathrm{F}_h^{\text{up}}(\varepsilon, r_h, \boldsymbol{u}_h) = \mathrm{Up}[r_h, \boldsymbol{u}_h] - h^\varepsilon [\![r_h]\!], \ \varepsilon > 0.$$

When r_h becomes a vector-valued function $\mathbf{r_h}$, e.g. $\mathbf{r_h} = \varrho_h \boldsymbol{u}_h$ in the momentum equation, we extend the above flux operator as

$$\mathbf{Up}(\varrho \boldsymbol{u}_h, \boldsymbol{v}_h) \equiv \big(\mathrm{Up}(\varrho u_{1,h}, \boldsymbol{v}_h), \ldots, \mathrm{Up}(\varrho u_{d,h}, \boldsymbol{v}_h)\big)^T \quad \text{and}$$
$$\mathbf{F}_h^{\text{up}}(\varepsilon, \varrho \boldsymbol{u}_h, \boldsymbol{v}_h) \equiv \big(\mathrm{F}_h^{\text{up}}(\varepsilon, \varrho u_{1,h}, \boldsymbol{v}_h), \ldots, \mathrm{F}_h^{\text{up}}(\varepsilon, \varrho u_{d,h}, \boldsymbol{v}_h)\big)^T.$$

It is easy to check the following relation for the convective fluxes for any $\varrho_h \in Q_h$ and $\boldsymbol{u}_h \in \mathbf{Q}_h$ that

$$\begin{aligned} &\sum_{\sigma\in\mathcal{E}_{\rm int}} \int_\sigma \left({\rm F}_h^{\rm up}(\varepsilon, \varrho_h, \boldsymbol{u}_h) \left[\!\!\left[\frac{1}{2}|\boldsymbol{u}_h|^2 \right]\!\!\right] - \mathbf{F}_h^{\rm up}(\varepsilon, \varrho_h\boldsymbol{u}_h, \boldsymbol{u}_h) \cdot [\![\boldsymbol{u}_h]\!] \right) {\rm d}S(x) \\ &= \frac{1}{2} \sum_{\sigma\in\mathcal{E}_{\rm int}} \int_\sigma \varrho_h^{\rm up} \, |[\![\boldsymbol{u}_h]\!]|^2 \, |u_\sigma| \, {\rm d}S(x) + h^\varepsilon \sum_{\sigma\in\mathcal{E}_{\rm int}} \int_\sigma \{\!\{\varrho_h\}\!\} \, |[\![\boldsymbol{u}_h]\!]|^2 \, {\rm d}S(x), \end{aligned} \tag{2.2}$$

see [9, Lemma 8.1].

Time discretization. For a given time step $\Delta t > 0$, we denote the approximation of a function v_h at time $t^k = k\Delta t$ by v_h^k for $k = 1, \ldots, N_T (= T/\Delta t)$. Then, we introduce the piecewise constant extension of discrete values,

$$v_h(t) = v_h^0 \text{ for } t \le 0, \;\; v_h(t, \cdot) = v_h^k \text{ for } t \in I^k := (t^{k-1}, t^k], \;\; k = 1, 2, \ldots, N_T. \tag{2.3}$$

Furthermore, we denote $v_h^{\triangleleft}(t) = v_h(t - \Delta t)$. The time derivative is approximated by the backward Euler method,

$$D_t v_h = \frac{v_h - v_h^{\triangleleft}}{\Delta t} = \frac{v_h(t) - v_h(t - \Delta t)}{\Delta t}.$$

Discrete differential operators. We define a divergence operator for any $\boldsymbol{u}_h \in Q_h$

$${\rm div}_h \boldsymbol{u}_h(x) = \sum_{K\in\Omega_h} 1_K ({\rm div}_h \boldsymbol{u}_h)_K, \quad ({\rm div}_h \boldsymbol{u}_h)_K = \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \, \{\!\{\boldsymbol{u}_h\}\!\} \cdot \boldsymbol{n} \tag{2.4}$$

and a gradient operator for any $r_h \in Q_h$

$$\nabla_{\mathcal{E}} r_h(x) := (\eth_{\mathcal{E}}^{(1)} r_h, \ldots, \eth_{\mathcal{E}}^{(d)} r_h)(x).$$

where

$$\eth_{\mathcal{E}}^{(i)} r_h(x) := \sum_{\sigma\in\mathcal{E}_i} 1_{D_\sigma} \left(\eth_{\mathcal{E}}^{(i)} r_h \right)_{D_\sigma}, \quad \left(\eth_{\mathcal{E}}^{(i)} r_h \right)_{D_\sigma} := \frac{[\![r_h]\!] \, \boldsymbol{n}_\sigma}{d_\sigma}, \quad \text{for all } \; \sigma \in \mathcal{E}_i,$$

Now we are ready to introduce a FV method for the approximation of two fluid system (1.1)–(1.5).

Definition 2.1 (FV method)**.** *We say* $(\varrho_h, n_h, \boldsymbol{u}_h) = \sum_{k=1}^{N_T} 1_{I^k} (\varrho_h^k, n_h^k, \boldsymbol{u}_h^k)$ *is a finite volume approximation of the two fluid system* (1.1)–(1.5) *if for all* $k = 1, \cdots, N_T$ *the triple* $(\varrho_h^k, n_h^k, \boldsymbol{u}_h^k) \in Q_h \times Q_h \times \mathbf{Q}_h$ *satisfies the following system of algebraic equations*

$$\begin{aligned} &D_t \varrho_h|_K + \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| {\rm F}_h^{\rm up}(\varepsilon_\gamma, \varrho_h, \boldsymbol{u}_h) = 0, \\ &D_t n_h|_K + \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| {\rm F}_h^{\rm up}(\varepsilon_\alpha, n_h, \boldsymbol{u}_h) = 0, \\ &D_t (\mathfrak{r}_h \boldsymbol{u}_h)_K + \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \big(\boldsymbol{F}_h^{\rm up}(\varepsilon_\gamma, \varrho_h\boldsymbol{u}_h, \boldsymbol{u}_h) + \boldsymbol{F}_h^{\rm up}(\varepsilon_\alpha, n_h\boldsymbol{u}_h, \boldsymbol{u}_h) \big), \\ &\quad + \frac{1}{|K|} \sum_{\sigma\in\mathcal{E}(K)} |\sigma| \left(\{\!\{p_h - \eta {\rm div}_h \boldsymbol{u}_h\}\!\} \, \boldsymbol{n} - \mu \frac{[\![\boldsymbol{u}_h]\!]}{d_\sigma} \right) = 0, \end{aligned} \tag{2.5}$$

equipped with the boundary conditions (2.1) *for all* $K \in \Omega_h$*, where* $p_h = p(\varrho_h, n_h)$*,* $\mathfrak{r}_h = \varrho_h + n_h$*,* $\eta = \frac{d-2}{d}\mu + \lambda$*, the parameters* ε_γ *and* ε_α *satisfy*

$$\begin{cases} \varepsilon_\gamma > 0 \text{ for } \gamma \geq 2, \text{ or } 0 < \varepsilon_\gamma < \min\{1, 2(\gamma - 1)\} \text{ for } \gamma \in (1,2), \\ \varepsilon_\alpha > 0 \text{ for } \alpha \geq 2, \text{ or } 0 < \varepsilon_\alpha < \min\{1, 2(\alpha - 1)\} \text{ for } \alpha \in (1,2). \end{cases} \tag{2.6}$$

Moreover, the initial data are give by

$$(\varrho_h^0, n_h^0, \boldsymbol{u}_h^0) = (\Pi_h^Q \varrho_0, \Pi_h^Q n_0, \Pi_h^Q \boldsymbol{u}_0).$$

Remark 2.2 (Periodic boundary condition)**.** *When considering periodic boundary conditions, we have* $\mathcal{E}_{\text{ext}} = \emptyset$ *and* $\mathcal{E}_{\text{int}} = \mathcal{E}$*. As a consequence, we do not require anymore the discrete boundary conditions* (2.1)*.*

Lemma 2.1 (Weak form of the FV method)**.** *Let* $(\varrho_h, n_h, \boldsymbol{u}_h)$ *be a FV approximation of the two-fluid system in the sense of Definition 2.1. Then we have:*

$$\int_\Omega D_t \varrho_h \phi_h \mathrm{d}x - \sum_{\sigma\in\mathcal{E}_{\text{int}}} \int_\sigma \mathrm{F}_h^{\text{up}}(\varepsilon_\gamma, \varrho_h, \boldsymbol{u}_h) [\![\phi_h]\!] \,\mathrm{d}S(x) = 0, \quad \text{for all } \phi_h \in Q_h; \tag{2.7a}$$

$$\int_\Omega D_t n_h \phi_h \mathrm{d}x - \sum_{\sigma\in\mathcal{E}_{\text{int}}} \int_\sigma \mathrm{F}_h^{\text{up}}(\varepsilon_\alpha, n_h, \boldsymbol{u}_h) [\![\phi_h]\!] \,\mathrm{d}S(x) = 0, \quad \text{for all } \phi_h \in Q_h; \tag{2.7b}$$

$$\begin{aligned} &\int_\Omega D_t(\mathfrak{r}_h \boldsymbol{u}_h) \cdot \boldsymbol{\phi}_h \mathrm{d}x - \sum_{\sigma\in\mathcal{E}_{\text{int}}} \int_\sigma \left(\boldsymbol{F}_h^{\text{up}}(\varepsilon_\gamma, \varrho_h \boldsymbol{u}_h, \boldsymbol{u}_h) + \boldsymbol{F}_h^{\text{up}}(\varepsilon_\alpha, n_h \boldsymbol{u}_h, \boldsymbol{u}_h)\right) \cdot [\![\boldsymbol{\phi}_h]\!] \,\mathrm{d}S(x) \\ &+ \mu \int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u}_h : \nabla_{\mathcal{E}} \boldsymbol{\phi}_h \mathrm{d}x + \eta \int_\Omega \mathrm{div}_h \boldsymbol{u}_h \,\mathrm{div}_h \boldsymbol{\phi}_h \mathrm{d}x = \int_\Omega p_h \mathrm{div}_h \boldsymbol{\phi}_h \mathrm{d}x, \quad \text{for all } \boldsymbol{\phi}_h \in \mathbf{Q}_h; \end{aligned} \tag{2.7c}$$

where ε_γ *and* ε_α *satisfy* (2.6)*.*

Note that the FV scheme (2.7) is an extension of Feireisl et al. [9, Chapter 11]. The solution to (2.7) enjoy similar properties as listed in the following remark, see [9, Lemma 11.2 and 11.3].

Remark 2.3 (Positivity of the density, internal energy balance, and existence of a numerical solution)**.** *Let* $(\varrho_h, n_h, \boldsymbol{u}_h)$ *be a solution to the FV scheme* (2.7)*. Then we have the following properties:*

1. **Positivity of the density.** *Let* $\varrho_0 > 0$ *and* $n_0 > 0$ *then* $\varrho_h(t) > 0$ *and* $n_h(t) > 0$*, respectively, for all* $t \in (0, T)$*.*

2. **Internal energy balance.**
There exist $\varrho_{h,\dagger} \in \mathrm{co}\{\varrho_h^{\text{in}}, \varrho_h^{\text{out}}\}$ *and* $n_{h,\dagger} \in \mathrm{co}\{n_h^{\text{in}}, n_h^{\text{out}}\}$ *for any* $\sigma \in \mathcal{E}_{\text{int}}$*,* $\varrho_h^\star \in \mathrm{co}\{\varrho_h^\triangleleft, \varrho_h\}$ *and* $n_h^\star \in \mathrm{co}\{n_h^\triangleleft, n_h\}$ *such that*

$$\begin{aligned} &\int_\Omega D_t H(\varrho_h) \mathrm{d}x + \int_\Omega (\varrho_h H'(\varrho_h) - H(\varrho_h)) \mathrm{div}_h \boldsymbol{u}_h \mathrm{d}x \\ &= -\frac{\Delta t}{2} \int_\Omega H''(\varrho_h^\star) |D_t \varrho_h|^2 \mathrm{d}x - \sum_{\sigma\in\mathcal{E}_{\text{int}}} \int_\sigma H''(\varrho_{h,\dagger}) [\![\varrho_h]\!]^2 \left(h^{\varepsilon_\gamma} + \frac{1}{2}|u_\sigma|\right) \mathrm{d}S(x) \leq 0, \end{aligned} \tag{2.8a}$$

$$\begin{aligned} &\int_\Omega D_t G(n_h) \mathrm{d}x + \int_\Omega (n_h G'(n_h) - G(n_h)) \mathrm{div}_h \boldsymbol{u}_h \mathrm{d}x \\ &= -\frac{\Delta t}{2} \int_\Omega G''(n_h^\star) |D_t n_h|^2 \mathrm{d}x - \sum_{\sigma\in\mathcal{E}_{\text{int}}} \int_\sigma G''(n_{h,\dagger}) [\![n_h]\!]^2 \left(h^{\varepsilon_\alpha} + \frac{1}{2}|u_\sigma|\right) \mathrm{d}S(x) \leq 0, \end{aligned} \tag{2.8b}$$

where we have denoted $\mathrm{co}\{A, B\} = [\min\{A, B\}, \max\{A, B\}]$*.*

3. **Existence of a solution.**
There exists a solution $(\varrho_h, n_h, \boldsymbol{u}_h)$ *to the FV scheme* (2.7) *for all* $t \in (0, T)$.

2.2 Stability

Now we are ready to show the energy stability.

Theorem 2.1 (Discrete energy balance). *Let* $(\varrho_h, n_h, \boldsymbol{u}_h)$ *be a solution of the FV method* (2.7). *Then, it holds*

$$D_t \int_\Omega \left(\frac{1}{2} \mathfrak{r}_h \left| \boldsymbol{u}_h \right|^2 + P(\varrho_h, n_h) \right) \, \mathrm{d}x + \mu \| \nabla_{\mathcal{E}} \boldsymbol{u}_h \|^2_{L^2(\Omega)} + \eta \| \mathrm{div}_h \boldsymbol{u}_h \|^2_{L^2(\Omega)} = -D_{\mathrm{num}}, \tag{2.9}$$

where $D_{\mathrm{num}} \geq 0$ *is the numerical dissipation given by*

$$\begin{aligned} D_{\mathrm{num}} = & \frac{\Delta t}{2} \int_\Omega \mathfrak{r}_h^{\triangleleft} |D_t \boldsymbol{u}_h|^2 \mathrm{d}x + \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma \mathfrak{r}_h^{\mathrm{up}} \left| u_\sigma \right| \left| [\![\boldsymbol{u}_h]\!] \right|^2 \mathrm{d}S(x) \\ & + h^{\varepsilon_\gamma} \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma \{\!\{ \varrho_h \}\!\} \left| [\![\boldsymbol{u}_h]\!] \right|^2 \mathrm{d}S(x) + h^{\varepsilon_\alpha} \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma \{\!\{ n_h \}\!\} \left| [\![\boldsymbol{u}_h]\!] \right|^2 \mathrm{d}S(x) \\ & + \frac{\Delta t}{2} \int_\Omega H''(\varrho_h^\star) |D_t \varrho_h|^2 \mathrm{d}x + \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma H''(\varrho_{h,\dagger}) \, [\![\varrho_h]\!]^2 \left(h^{\varepsilon_\gamma} + \frac{1}{2} |u_\sigma| \right) \mathrm{d}S(x), \\ & + \frac{\Delta t}{2} \int_\Omega G''(n_h^\star) |D_t n_h|^2 \mathrm{d}x + \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma G''(n_{h,\dagger}) \, [\![n_h]\!]^2 \left(h^{\varepsilon_\alpha} + \frac{1}{2} |u_\sigma| \right) \mathrm{d}S(x). \end{aligned} \tag{2.10}$$

Here, $\varrho_{h,\dagger} \in \mathrm{co}\{\varrho_h^{\mathrm{in}}, \varrho_h^{\mathrm{out}}\}$ *and* $n_{h,\dagger} \in \mathrm{co}\{n_h^{\mathrm{in}}, n_h^{\mathrm{out}}\}$ *for any* $\sigma \in \mathcal{E}_{\mathrm{int}}$*,* $\varrho_h^\star \in \mathrm{co}\{\varrho_h^{\triangleleft}, \varrho_h\}$ *and* $n_h^\star \in \mathrm{co}\{n_h^{\triangleleft}, n_h\}$.

Proof. First, setting $\boldsymbol{\phi}_h = \boldsymbol{u}_h \in \mathbf{Q}_h$ in (2.7c) we get

$$\begin{aligned} & \int_\Omega D_t(\mathfrak{r}_h \boldsymbol{u}_h) \cdot \boldsymbol{u}_h \mathrm{d}x + \mu \| \nabla_{\mathcal{E}} \boldsymbol{u}_h \|^2_{L^2} + \eta \| \mathrm{div}_h \boldsymbol{u}_h \|^2_{L^2} \\ & = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma \left(\mathbf{F}_h^{\mathrm{up}}(\varepsilon_\gamma, \varrho_h \boldsymbol{u}_h, \boldsymbol{u}_h) + \mathbf{F}_h^{\mathrm{up}}(\varepsilon_\alpha, n_h \boldsymbol{u}_h, \boldsymbol{u}_h) \right) \cdot [\![\boldsymbol{u}_h]\!] \, \mathrm{d}S(x) + \int_\Omega p_h \mathrm{div}_h \boldsymbol{u}_h \mathrm{d}x. \end{aligned} \tag{2.11}$$

Next, letting $\phi_h = \frac{1}{2} \left| \boldsymbol{u}_h \right|^2$ in (2.7a) and (2.7b) we find their sum

$$\int_\Omega D_t \mathfrak{r}_h \frac{1}{2} \left| \boldsymbol{u}_h \right|^2 \mathrm{d}x = \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma \left(\mathrm{F}_h^{\mathrm{up}}(\varepsilon_\gamma, \varrho_h, \boldsymbol{u}_h) + \mathrm{F}_h^{\mathrm{up}}(\varepsilon_\alpha, n_h, \boldsymbol{u}_h) \right) \left[\!\!\left[\frac{1}{2} \left| \boldsymbol{u}_h \right|^2 \right]\!\!\right] \mathrm{d}S(x). \tag{2.12}$$

Subtracting (2.12) from (2.11) and using (2.2) we derive

$$\begin{aligned} & D_t \int_\Omega \frac{1}{2} \mathfrak{r}_h \left| \boldsymbol{u}_h \right|^2 \mathrm{d}x + \mu \| \nabla_{\mathcal{E}} \boldsymbol{u}_h \|^2_{L^2(\Omega)} + \eta \| \mathrm{div}_h \boldsymbol{u}_h \|^2_{L^2(\Omega)} \\ & = \int_\Omega p_h \mathrm{div}_h \boldsymbol{u}_h \mathrm{d}x - \frac{\Delta t}{2} \int_\Omega \mathfrak{r}_h^{\triangleleft} \left| D_t \boldsymbol{u}_h \right|^2 \mathrm{d}x - \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma \mathfrak{r}_h^{\mathrm{up}} \left| [\![\boldsymbol{u}_h]\!] \right|^2 \left| u_\sigma \right| \mathrm{d}S(x) \\ & \quad - h^{\varepsilon_\gamma} \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma \{\!\{ \varrho_h \}\!\} \left| [\![\boldsymbol{u}_h]\!] \right|^2 \mathrm{d}S(x) - h^{\varepsilon_\alpha} \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} \int_\sigma \{\!\{ n_h \}\!\} \left| [\![\boldsymbol{u}_h]\!] \right|^2 \mathrm{d}S(x). \end{aligned} \tag{2.13}$$

where we have also used the following identity

$$\int_\Omega \left(D_t(\mathfrak{r}_h \boldsymbol{u}_h)\cdot \boldsymbol{u}_h - D_t \mathfrak{r}_h \frac{|\boldsymbol{u}_h|^2}{2} \right) \mathrm{d}x = \int_\Omega \left(D_t\Big(\frac{1}{2}\mathfrak{r}_h |\boldsymbol{u}_h|^2\Big) + \frac{\Delta t}{2}\mathfrak{r}_h^{\triangleleft} |D_t \boldsymbol{u}_h|^2 \right) \mathrm{d}x.$$

Finally, combining (2.13) and (2.8), we obtain

$$\begin{aligned}
& D_t \int_\Omega \left(\frac{1}{2}\mathfrak{r}_h |\boldsymbol{u}_h|^2 + G(n_h) + H(\varrho_h) \right) \mathrm{d}x + \mu \|\nabla_{\mathcal{E}} \boldsymbol{u}_h\|^2_{L^2(\Omega)} + \eta \|\mathrm{div}_h \boldsymbol{u}_h\|^2_{L^2(\Omega)} \\
&= -\frac{\Delta t}{2} \int_\Omega \mathfrak{r}_h^{\triangleleft} |D_t \boldsymbol{u}_h|^2 \mathrm{d}x - \frac{1}{2}\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma \mathfrak{r}_h^{\mathrm{up}} |u_\sigma| \, |[\![\boldsymbol{u}_h]\!]|^2 \,\mathrm{d}S(x) \\
&\quad - h^{\varepsilon_\gamma} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma \{\!\{\varrho_h\}\!\} \, |[\![\boldsymbol{u}_h]\!]|^2 \,\mathrm{d}S(x) - h^{\varepsilon_\alpha} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma \{\!\{n_h\}\!\} \, |[\![\boldsymbol{u}_h]\!]|^2 \,\mathrm{d}S(x) \\
&\quad - \frac{\Delta t}{2} \int_\Omega H''(\varrho_h^\star) |D_t \varrho_h|^2 \mathrm{d}x - \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma H''(\varrho_{h,\dagger}) \, [\![\varrho_h]\!]^2 \left(h^{\varepsilon_\gamma} + \frac{1}{2}|u_\sigma| \right) \mathrm{d}S(x) \\
&\quad - \frac{\Delta t}{2} \int_\Omega G''(n_h^\star) |D_t n_h|^2 \mathrm{d}x - \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} \int_\sigma G''(n_{h,\dagger}) \, [\![n_h]\!]^2 \left(h^{\varepsilon_\alpha} + \frac{1}{2}|u_\sigma| \right) \mathrm{d}S(x),
\end{aligned}$$

which completes the proof. □

2.3 Consistency

One more step towards the convergence analysis is the consistency of the numerical solutions. Analogously to [9, Theorem 11.2] we have the following consistency formulation.

Lemma 2.2. *Let* $(\varrho_h, n_h, \boldsymbol{u}_h)$ *be a solution of the approximate problem* (2.7) *on the time interval* $[0,T]$ *with* $\Delta t \approx h$, $\gamma > 1$ *and* $\alpha > 1$. *Then*

$$-\int_\Omega \varrho_h^0 \phi(0,\cdot)\mathrm{d}x = \int_0^T \int_\Omega \left[\varrho_h \partial_t \phi + \varrho_h \boldsymbol{u}_h \cdot \nabla\phi\right] \mathrm{d}x\mathrm{d}t + \int_0^T e_{1,h}(t,\phi)\mathrm{d}t, \tag{2.14a}$$

for any $\phi \in C_c^2([0,T)\times\Omega)$;

$$-\int_\Omega n_h^0 \phi(0,\cdot)\mathrm{d}x = \int_0^T \int_\Omega \left[n_h \partial_t \phi + n_h \boldsymbol{u}_h \cdot \nabla\phi\right] \mathrm{d}x\mathrm{d}t + \int_0^T e_{2,h}(t,\phi)\mathrm{d}t, \tag{2.14b}$$

for any $\phi \in C_c^2([0,T)\times\Omega)$;

$$\begin{aligned}
-\int_\Omega \mathfrak{r}_h^0 \boldsymbol{u}_h^0 \cdot \boldsymbol{\phi}(0,\cdot)\mathrm{d}x &= \int_0^T \int_\Omega \left[\mathfrak{r}_h \boldsymbol{u}_h \cdot \partial_t \boldsymbol{\phi} + \mathfrak{r}_h \boldsymbol{u}_h \otimes \boldsymbol{u}_h : \nabla \boldsymbol{\phi} + p_h \mathrm{div}\boldsymbol{\phi}\right] \mathrm{d}x\mathrm{d}t, \\
&- \mu \int_0^T \int_\Omega \nabla_{\mathcal{E}} \boldsymbol{u}_h : \nabla\boldsymbol{\phi} \mathrm{d}x\mathrm{d}t - \eta \int_0^T \int_\Omega \mathrm{div}_h \boldsymbol{u}_h \mathrm{div}\boldsymbol{\phi} \mathrm{d}x\mathrm{d}t + \int_0^T e_{3,h}(t,\boldsymbol{\phi})\mathrm{d}t
\end{aligned} \tag{2.14c}$$

for any $\boldsymbol{\phi} \in C_c^2([0,T)\times\Omega;\mathbb{R}^d)$;

$$\|e_{j,h}(\cdot,\phi)\|_{L^1(0,T)} \lesssim h^\beta \|\phi\|_{C^2},\ j=1,2,3 \ \textit{for some}\ \ \beta > 0.$$

2.4 Final proof of Theorem 1.1

With the stability and consistency results in hand, we are ready to show the final proof to Theorem 1.1, that is the convergence of numerical solutions resulting from the FV method (2.7).

Proof of Theorem 1.1. From the energy estimates (2.9) we deduce that at least for suitable subsequences,

$$\begin{aligned}
\varrho_h &\to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)),\ \varrho \geq 0,\\
n_h &\to n \text{ weakly-(*) in } L^\infty(0,T;L^\alpha(\Omega)),\ n \geq 0,\\
\boldsymbol{u}_h &\to \boldsymbol{u} \text{ weakly in } L^2(Q_T;\mathbb{R}^d),\\
\nabla_{\mathcal{E}} \boldsymbol{u}_h &\to \nabla_x \boldsymbol{u} \text{ weakly in } L^2(Q_T;\mathbb{R}^{d\times d}), \quad \text{where } \boldsymbol{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^d)),\\
\varrho_h \boldsymbol{u}_h &\to \varrho \boldsymbol{u} \ \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d)),\\
n_h \boldsymbol{u}_h &\to n \boldsymbol{u} \ \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{2\alpha}{\alpha+1}}(\Omega;\mathbb{R}^d)),
\end{aligned}$$

Furthermore, it holds

$$(\varrho_h + n_h)\boldsymbol{u}_h \otimes \boldsymbol{u}_h \to \overline{(\varrho + n)\boldsymbol{u} \otimes \boldsymbol{u}} \text{ weakly-(*) in } L^\infty(0,T;\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}_{sym})),$$

$$p(\varrho_h, n_h) \to \overline{p(\varrho, n)} \text{ weakly-(*) in } L^\infty(0,T;\mathcal{M}^+(\overline{\Omega})),$$

$$\frac{1}{2}(\varrho_h + n_h)|\boldsymbol{u}_h|^2 \to \overline{\frac{1}{2}(\varrho + n)|\boldsymbol{u}|^2} \text{ weakly-(*) in } L^\infty(0,T;\mathcal{M}^+(\overline{\Omega})),$$

$$P(\varrho_h, n_h) \to \overline{P(\varrho, n)} \text{ weakly-(*) in } L^\infty(0,T;\mathcal{M}^+(\overline{\Omega})).$$

Thus we may set

$$\mu_c := \left[\overline{(\varrho + n)\boldsymbol{u} \otimes \boldsymbol{u}} + \overline{p(\varrho, n)}\mathbb{I}\right] - \left[(\varrho + n)\boldsymbol{u} \otimes \boldsymbol{u} + p(\varrho, n)\mathbb{I}\right],$$

$$\mathcal{D} := \left[\overline{\frac{1}{2}(\varrho + n)|\boldsymbol{u}|^2} + \overline{P(\varrho, n)}\right] - \left[\frac{1}{2}(\varrho + n)|\boldsymbol{u}|^2 + P(\varrho, n)\right].$$

It then follows that

$$|\mu_c(\tau)| \lesssim \mathcal{D}(\tau) \text{ for a.e. } \tau \in (0,T),$$

see [20, Section 8], [1, Section 3.4] for similar details.

Consequently, passing to the limit for $h \to 0$ in the energy estimates (2.9) and the consistency formulation (2.14a)–(2.14c) we deduce that $(\varrho, n, \boldsymbol{u})$ is a dissipative weak solution in the sense of Definition 1.1, which prove Item 1 of Theorem 1.1. Further, employing Proposition 1.1, we obtain Item 2 of Theorem 1.1, which completes the proof of Theorem 1.1. □

3 Conclusion

We studied in multi-dimensions a two-fluid model describing the motion of a mixture of two compressible barotropic fluids with the *full range* of adiabatic exponents $\gamma > 1$ and $\alpha > 1$. We attack the *global-in-time* solutions with *large initial data* for such a system by the concept of *dissipative weak solutions*:

- We proved the existence of dissipative weak solutions via a finite volume approximation method, see Theorem 1.1

- As a by-product, we observed the convergence of the finite volume approximation towards a strong solution on the lifespan of the latter, see also Theorem 1.1.

As far as we know, this is the first numerical attempt in the mathematical analysis of the two-fluid model, which also provides a general framework for the convergence analysis of numerical solutions for the two-fluid model (1.1)–(1.5). The current convergence analysis can be easily adapted to other numerical methods. For example, we may keep the finite volume approximation of the density equations and replace the approximation of the momentum equation by either a finite element method or a finite difference method studied in Chapter 13 and Chapter 14 of Feireisl et al. [9].

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Statements and Declarations

Conflict of interests: On behalf of all authors, the corresponding author states that there is no conflict of interest.
Data availability: Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

A Appendix

This appendix is dedicated to the proof of Proposition 1.1. To this end, we establish the relative energy inequality to (1.1)–(1.5) satisfied for any dissipative weak solutions and any suitable test functions. Then the proof is finished with the help of Gronwall-type argument. Note that we shall present the proof in case of Dirichlet boundary conditions. The case of periodic boundary conditions can be carried out analogously and the details are therefore omitted.

A.1 Relative energy inequality

Let $(\varrho, n, \boldsymbol{u})$ be a dissipative weak solution to (1.1)–(1.5) and $(r, b, \mathbf{U})$ belongs to

$$\begin{cases} r, b \in C^1(\overline{Q_T}),\ \ r, b > 0 \text{ in } \overline{Q_T}, \\ \mathbf{U} \in C^1(\overline{Q_T}; \mathbb{R}^d),\ \ \mathbf{U}|_{\partial\Omega} = \mathbf{0}. \end{cases} \tag{A.1}$$

Similar to [19], we introduce the relative energy as

$$\mathcal{E}\Big((\varrho, n, \boldsymbol{u}) \,\Big|\, (r, b, \mathbf{U})\Big)(\tau) := \int_\Omega \Big[\frac{1}{2}\mathfrak{r}|\boldsymbol{u} - \mathbf{U}|^2$$
$$+ H(\varrho) - H(r) - H'(r)(\varrho - r) + G(n) - G(b) - G'(b)(n - b)\Big]\mathrm{d}x. \tag{A.2}$$

Notice that we may rewrite the relative energy in an equivalent form as follows

$$\mathcal{E}\Big((\varrho,n,\boldsymbol{u})\,\Big|\,(r,b,\mathbf{U})\Big)(\tau)=\int_\Omega\left(\frac{1}{2}\mathfrak{r}|\boldsymbol{u}|^2+H(\varrho)+G(n)\right)\mathrm{d}x$$
$$+\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{U}|^2-H'(r)\varrho\right)\mathrm{d}x+\int_\Omega\left(\frac{1}{2}n|\mathbf{U}|^2-G'(b)n\right)\mathrm{d}x$$
$$-\int_\Omega\mathfrak{r}\boldsymbol{u}\cdot\mathbf{U}\mathrm{d}x+\int_\Omega\left(r^\gamma+b^\alpha\right)\mathrm{d}x. \tag{A.3}$$

The crucial observation is that the integrals on the right-hand side of (A.3) can be expressed through the weak formulations (1.7)–(1.10) with suitable choices of test functions. To handle the density-dependent terms, testing the continuity equation (1.7) by $\frac{1}{2}|\mathbf{U}|^2$ and $H'(r)$ gives

$$\int_0^\tau\int_\Omega\Big(\varrho\mathbf{U}\cdot\partial_t\mathbf{U}+\varrho\boldsymbol{u}\cdot\nabla_x\mathbf{U}\cdot\mathbf{U}\Big)\mathrm{d}x\mathrm{d}t=\left[\int_\Omega\frac{1}{2}\varrho|\mathbf{U}|^2\mathrm{d}x\right]_{t=0}^{t=\tau}; \tag{A.4}$$

$$\int_0^\tau\int_\Omega\Big(\varrho\partial_t H'(r)+\varrho\boldsymbol{u}\cdot\nabla_x H'(r)\Big)\mathrm{d}x\mathrm{d}t=\left[\int_\Omega\varrho H'(r)\mathrm{d}x\right]_{t=0}^{t=\tau}; \tag{A.5}$$

In the same manner, we test the continuity equation (1.8) by $\frac{1}{2}|\mathbf{U}|^2$ and $G'(b)$ to obtain

$$\int_0^\tau\int_\Omega\Big(n\mathbf{U}\cdot\partial_t\mathbf{U}+n\boldsymbol{u}\cdot\nabla_x\mathbf{U}\cdot\mathbf{U}\Big)\mathrm{d}x\mathrm{d}t=\left[\int_\Omega\frac{1}{2}n|\mathbf{U}|^2\mathrm{d}x\right]_{t=0}^{t=\tau}; \tag{A.6}$$

$$\int_0^\tau\int_\Omega\Big(n\partial_t G'(b)+n\boldsymbol{u}\cdot\nabla_x G'(b)\Big)\mathrm{d}x\mathrm{d}t=\left[\int_\Omega nG'(b)\mathrm{d}x\right]_{t=0}^{t=\tau}. \tag{A.7}$$

Upon choosing $\mathbf{U}$ as a test function in the momentum equation (1.9),

$$\int_0^\tau\int_\Omega\Big(\mathfrak{r}\boldsymbol{u}\cdot\partial_t\mathbf{U}+\mathfrak{r}\boldsymbol{u}\otimes\boldsymbol{u}:\nabla_x\mathbf{U}+p(\varrho,n)\mathrm{div}_x\mathbf{U}-\mathbb{S}(\nabla_x\boldsymbol{u}):\nabla_x\mathbf{U}\Big)\mathrm{d}x\mathrm{d}t$$
$$+\int_0^\tau\int_{\overline{\Omega}}\nabla_x\mathbf{U}:\mathrm{d}\mu_c(t)\mathrm{d}t=\left[\int_\Omega\mathfrak{r}\boldsymbol{u}\cdot\mathbf{U}\mathrm{d}x\right]_{t=0}^{t=\tau}. \tag{A.8}$$

Taking (A.4)–(A.8) and the balance of total energy (1.10) into account, we may estimate (A.3), in agreement with the compressible Navier-Stokes system [11], to arrive at

$$\left[\mathcal{E}\Big((\varrho,n,\boldsymbol{u})\,\Big|\,(r,b,\mathbf{U})\Big)\right]_{t=0}^{t=\tau}+\int_0^\tau\int_\Omega\Big(\mathbb{S}(\nabla_x\boldsymbol{u}-\nabla_x\mathbf{U}):(\nabla_x\boldsymbol{u}-\nabla_x\mathbf{U})\Big)\mathrm{d}x\mathrm{d}t+\int_{\overline{\Omega}}\mathrm{d}\mathcal{D}(\tau)$$
$$\le\int_0^\tau\int_\Omega\mathfrak{r}(\mathbf{U}-\boldsymbol{u})\cdot\Big(\partial_t\mathbf{U}+\boldsymbol{u}\cdot\nabla_x\mathbf{U}\Big)\mathrm{d}x\mathrm{d}t$$
$$+\int_0^\tau\int_\Omega\mathbb{S}(\nabla_x\mathbf{U}):(\nabla_x\mathbf{U}-\nabla_x\boldsymbol{u})\,\mathrm{d}x\mathrm{d}t-\int_0^\tau\int_\Omega p(\varrho,n)\mathrm{div}_x\mathbf{U}\mathrm{d}x\mathrm{d}t$$
$$+\int_0^\tau\int_\Omega\Big[\left(1-\frac{\varrho}{r}\right)\gamma r^{\gamma-1}\partial_t r-\varrho\boldsymbol{u}\cdot\gamma r^{\gamma-2}\nabla_x r\Big]\mathrm{d}x\mathrm{d}t$$
$$+\int_0^\tau\int_\Omega\Big[\left(1-\frac{n}{b}\right)\alpha b^{\alpha-1}\partial_t b-n\boldsymbol{u}\cdot\alpha b^{\alpha-2}\nabla_x b\Big]\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&\leq \int_0^\tau \int_\Omega \mathfrak{r}(\mathbf{U}-\boldsymbol{u}) \cdot \Big(\partial_t \mathbf{U} + \boldsymbol{u}\cdot\nabla_x \mathbf{U}\Big) \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \boldsymbol{u}) \,\mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega (r-\varrho)\partial_t \left(\frac{\gamma}{\gamma-1} r^{\gamma-1}\right) \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega (b-n)\partial_t \left(\frac{\alpha}{\alpha-1} b^{\alpha-1}\right) \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega (r\mathbf{U} - \varrho \boldsymbol{u}) \cdot \nabla_x \left(\frac{\gamma}{\gamma-1} r^{\gamma-1}\right) \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega (b\mathbf{U} - n \boldsymbol{u}) \cdot \nabla_x \left(\frac{\alpha}{\alpha-1} b^{\alpha-1}\right) \mathrm{d}x\mathrm{d}t. \\
&- \int_0^\tau \int_\Omega \Big[p(\varrho,n) - p(r,b)\Big] \mathrm{div}_x \mathbf{U} \mathrm{d}x\mathrm{d}t - \int_0^\tau \int_{\overline{\Omega}} \nabla_x \mathbf{U} : \mathrm{d}\mu_c(t) \mathrm{d}t. \qquad \text{(A.9)}
\end{aligned}
$$

Remark A.1. *Compared with [19], our relative energy inequality (A.9) holds for any test functions belonging to the class (A.1) and incorporates the phenomena of oscillations and concentrations.*

A.2 Weak–strong uniqueness principle

Basically, the proof of weak–strong uniqueness principle consists of:

- choosing the classical solution $(\tilde{\varrho}, \tilde{n}, \tilde{\boldsymbol{u}})$ as the test function $(r, b, \mathbf{U})$ in the relative energy inequality (A.9);
- estimating each term on the right-hand side of the relative energy inequality in a suitable manner;
- application of Gronwall-type argument.

To do this, assume that $(\tilde{\varrho}, \tilde{n}, \tilde{\boldsymbol{u}})$ is a strong solution to (1.1)–(1.5) starting from the smooth initial data $(\tilde{\varrho}_0, \tilde{n}_0, \tilde{\boldsymbol{u}}_0)$ with strictly positive $\tilde{\varrho}_0$ and $\tilde{n}_0$. Let $(\varrho, n, \boldsymbol{u})$ be a dissipative weak solution to (1.1)–(1.5) emanating from the same initial data. It follows from (A.9) that

$$
\begin{aligned}
&\mathcal{E}\Big((\varrho, n, \boldsymbol{u}) \Big| (\tilde{\varrho}, \tilde{n}, \tilde{\boldsymbol{u}})\Big)(\tau) + \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \boldsymbol{u} - \nabla_x \tilde{\boldsymbol{u}}) : (\nabla_x \boldsymbol{u} - \nabla_x \tilde{\boldsymbol{u}})\Big) \mathrm{d}x\mathrm{d}t + \int_{\overline{\Omega}} \mathrm{d}\mathcal{D}(\tau) \\
&\leq \int_0^\tau \int_\Omega (\tilde{\boldsymbol{u}} - \boldsymbol{u}) \cdot \Big[\mathfrak{r}\Big(\partial_t \tilde{\boldsymbol{u}} + \boldsymbol{u}\cdot\nabla_x \tilde{\boldsymbol{u}}\Big) - \mathrm{div}_x \mathbb{S}(\nabla_x \tilde{\boldsymbol{u}})\Big] \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega (\tilde{\varrho} - \varrho)\partial_t \left(\frac{\gamma}{\gamma-1} \tilde{\varrho}^{\gamma-1}\right) \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega (\tilde{n} - n)\partial_t \left(\frac{\alpha}{\alpha-1} \tilde{n}^{\alpha-1}\right) \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega (\tilde{\varrho}\tilde{\boldsymbol{u}} - \varrho \boldsymbol{u}) \cdot \nabla_x \left(\frac{\gamma}{\gamma-1} \tilde{\varrho}^{\gamma-1}\right) \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_\Omega (\tilde{n}\tilde{\boldsymbol{u}} - n \boldsymbol{u}) \cdot \nabla_x \left(\frac{\alpha}{\alpha-1} \tilde{n}^{\alpha-1}\right) \mathrm{d}x\mathrm{d}t. \\
&- \int_0^\tau \int_\Omega \Big[p(\varrho,n) - p(\tilde{\varrho},\tilde{n})\Big] \mathrm{div}_x \tilde{\boldsymbol{u}} \mathrm{d}x\mathrm{d}t - \int_0^\tau \int_{\overline{\Omega}} \nabla_x \tilde{\boldsymbol{u}} : \mathrm{d}\mu_c(t) \mathrm{d}t. \qquad \text{(A.10)}
\end{aligned}
$$

In light of the fact that $(\tilde{\varrho}, \tilde{n}, \tilde{\boldsymbol{u}})$ solves (1.1) in the classical sense, we furthermore rewrite the relative energy inequality as (see for instance [16] for similar calculations)

$$
\begin{aligned}
&\mathcal{E}\Big((\varrho, n, \boldsymbol{u}) \Big| (\tilde{\varrho}, \tilde{n}, \tilde{\boldsymbol{u}})\Big)(\tau) + \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \boldsymbol{u} - \nabla_x \tilde{\boldsymbol{u}}) : (\nabla_x \boldsymbol{u} - \nabla_x \tilde{\boldsymbol{u}})\Big) \mathrm{d}x\mathrm{d}t + \int_{\overline{\Omega}} \mathrm{d}\mathcal{D}(\tau) \\
&\leq \int_0^\tau \int_\Omega (\tilde{\boldsymbol{u}} - \boldsymbol{u}) \cdot \Big[(\mathfrak{r} - \tilde{\mathfrak{r}})\partial_t \tilde{\boldsymbol{u}} + (\mathfrak{r}\boldsymbol{u} - \tilde{\mathfrak{r}}\tilde{\boldsymbol{u}}) \cdot \nabla_x \tilde{\boldsymbol{u}}\Big] \mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_\Omega (\varrho - \tilde{\varrho})(\tilde{\boldsymbol{u}} - \boldsymbol{u}) \cdot \nabla_x \left(\frac{\gamma}{\gamma-1} \tilde{\varrho}^{\gamma-1}\right) \mathrm{d}x\mathrm{d}t
\end{aligned}
$$

$$+\int_0^\tau \int_\Omega (n-\tilde{n})(\tilde{\boldsymbol{u}}-\boldsymbol{u})\cdot\nabla_x\left(\frac{\alpha}{\alpha-1}\tilde{n}^{\alpha-1}\right)\mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau \int_\Omega \Big[\Big(\varrho^\gamma-\tilde{\varrho}^\gamma-\gamma\tilde{\varrho}^{\gamma-1}(\varrho-\tilde{\varrho})\Big)+\Big(n^\alpha-\tilde{n}^\alpha-\alpha\tilde{n}^{\alpha-1}(n-\tilde{n})\Big)\Big]\mathrm{div}_x\tilde{\boldsymbol{u}}\mathrm{d}x\mathrm{d}t$$

$$-\int_0^\tau \int_{\overline{\Omega}} \nabla_x\tilde{\boldsymbol{u}}:\mathrm{d}\mu_c(t)\mathrm{d}t=:\sum_{j=1}^5\mathcal{R}^{(j)}. \tag{A.11}$$

Notice that we may rewrite $\mathcal{R}^{(1)}$ as

$$\mathcal{R}^{(1)}=\int_0^\tau \int_\Omega (\tilde{\boldsymbol{u}}-\boldsymbol{u})\cdot(\mathfrak{r}-\tilde{\mathfrak{r}})\Big[\partial_t\tilde{\boldsymbol{u}}+\tilde{\boldsymbol{u}}\cdot\nabla_x\tilde{\boldsymbol{u}}\Big]\mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau \int_\Omega \mathfrak{r}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\cdot\nabla_x\tilde{\boldsymbol{u}}\cdot(\tilde{\boldsymbol{u}}-\boldsymbol{u})\mathrm{d}x\mathrm{d}t=:\mathcal{R}_1^{(1)}+\mathcal{R}_2^{(1)}.$$

Obviously, it holds

$$|\mathcal{R}_2^{(1)}|\lesssim\int_0^\tau \mathcal{E}\Big((\varrho,n,\boldsymbol{u})\,\Big|\,(\tilde{\varrho},\tilde{n},\tilde{\boldsymbol{u}})\Big)(t)\mathrm{d}t. \tag{A.12}$$

Next, to show the estimate of $\mathcal{R}_1^{(1)}$, we first recall [9, Lemma 14.3] for the following estimates.

$$\Big[\Big(\varrho^\gamma-\tilde{\varrho}^\gamma-\gamma\tilde{\varrho}^{\gamma-1}(\varrho-\tilde{\varrho})\Big)+\Big(n^\alpha-\tilde{n}^\alpha-\alpha\tilde{n}^{\alpha-1}(n-\tilde{n})\Big)\Big]$$

$$\gtrsim\begin{cases}(\varrho-\tilde{\varrho})^2+(n-\tilde{n})^2, & \text{if } \varrho\in[1/2\min\limits_{Q_T}\tilde{\varrho},2\max\limits_{Q_T}\tilde{\varrho}],\, n\in[1/2\min\limits_{Q_T}\tilde{n},2\max\limits_{Q_T}\tilde{n}],\\ 1+\varrho^\gamma+n^\alpha, & \text{otherwise.}\end{cases} \tag{A.13}$$

Moreover, following [12], we may decompose any measurable function $f(t,x)$ as the sum of "essential part" and "residual part":

$$[f]_{\mathrm{ess}}(t,x)=\begin{cases}f(t,x), & \text{if } \varrho\in[1/2\min\limits_{Q_T}\tilde{\varrho},2\max\limits_{Q_T}\tilde{\varrho}],\, n\in[1/2\min\limits_{Q_T}\tilde{n},2\max\limits_{Q_T}\tilde{n}],\\ 0, & \text{otherwise;}\end{cases}$$

$$[f]_{\mathrm{res}}(t,x)=f(t,x)-[f]_{\mathrm{ess}}(t,x).$$

Thus,

$$\mathcal{R}_1^{(1)}=\int_0^\tau \int_\Omega (\tilde{\boldsymbol{u}}-\boldsymbol{u})\cdot[\mathfrak{r}-\tilde{\mathfrak{r}}]_{\mathrm{ess}}\Big[\partial_t\tilde{\boldsymbol{u}}+\tilde{\boldsymbol{u}}\cdot\nabla_x\tilde{\boldsymbol{u}}\Big]\mathrm{d}x\mathrm{d}t$$

$$+\int_0^\tau \int_\Omega (\tilde{\boldsymbol{u}}-\boldsymbol{u})\cdot[\mathfrak{r}-\tilde{\mathfrak{r}}]_{\mathrm{res}}\Big[\partial_t\tilde{\boldsymbol{u}}+\tilde{\boldsymbol{u}}\cdot\nabla_x\tilde{\boldsymbol{u}}\Big]\mathrm{d}x\mathrm{d}t. \tag{A.14}$$

Observe first that

$$\left|\int_0^\tau \int_\Omega (\tilde{\boldsymbol{u}}-\boldsymbol{u})\cdot[\mathfrak{r}-\tilde{\mathfrak{r}}]_{\mathrm{ess}}\Big[\partial_t\tilde{\boldsymbol{u}}+\tilde{\boldsymbol{u}}\cdot\nabla_x\tilde{\boldsymbol{u}}\Big]\mathrm{d}x\mathrm{d}t\right|$$

$$\le\int_0^\tau \|\partial_t\tilde{\boldsymbol{u}}+\tilde{\boldsymbol{u}}\cdot\nabla_x\tilde{\boldsymbol{u}}\|_{L^\infty(\Omega)}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{L^2(\Omega)}\|[\mathfrak{r}-\tilde{\mathfrak{r}}]_{\mathrm{ess}}\|_{L^2(\Omega)}\mathrm{d}t$$

$$\lesssim\varepsilon\int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x\boldsymbol{u}-\nabla_x\tilde{\boldsymbol{u}}):(\nabla_x\boldsymbol{u}-\nabla_x\tilde{\boldsymbol{u}})\Big)\mathrm{d}x\mathrm{d}t+C(\varepsilon)\left(\|[\varrho-\tilde{\varrho}]_{\mathrm{ess}}\|^2_{L^2(\Omega)}+\|[n-\tilde{n}]_{\mathrm{ess}}\|^2_{L^2(\Omega)}\right)$$

$$\lesssim \varepsilon \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \boldsymbol{u} - \nabla_x \tilde{\boldsymbol{u}}) : (\nabla_x \boldsymbol{u} - \nabla_x \tilde{\boldsymbol{u}})\Big) \mathrm{d}x\mathrm{d}t + C(\varepsilon) \int_0^\tau \mathcal{E}\Big((\varrho, n, \boldsymbol{u}) \Big| (\tilde{\varrho}, \tilde{n}, \tilde{\boldsymbol{u}})\Big)(t)\mathrm{d}t, \tag{A.15}$$

where we have employed the generalized Korn's inequality in the second step and the first item of (A.17) in the third step. Next, it follows that

$$\begin{aligned} &\left| \int_0^\tau \int_\Omega (\tilde{\boldsymbol{u}} - \boldsymbol{u}) \cdot [\mathfrak{r} - \tilde{\mathfrak{r}}]_{\mathrm{res}} \Big[\partial_t \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{u}} \cdot \nabla_x \tilde{\boldsymbol{u}}\Big] \mathrm{d}x\mathrm{d}t \right| \\ &\lesssim \int_0^\tau \int_\Omega |\boldsymbol{u} - \tilde{\boldsymbol{u}}| \Big(|[\varrho - \tilde{\varrho}]_{\mathrm{res}}| + |[n - \tilde{n}]_{\mathrm{res}}| \Big) \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{A.16}$$

We make a decomposition as follows.

$$\begin{aligned} &\int_0^\tau \int_\Omega |\boldsymbol{u} - \tilde{\boldsymbol{u}}| |[\varrho - \tilde{\varrho}]_{\mathrm{res}}| \mathrm{d}x\mathrm{d}t \\ &= \int_0^\tau \int_{\varrho \leq 1/2 \min_{\overline{Q_T}} \tilde{\varrho}} |\boldsymbol{u} - \tilde{\boldsymbol{u}}| |[\varrho - \tilde{\varrho}]_{\mathrm{res}}| \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\varrho \geq 2 \max_{\overline{Q_T}} \tilde{\varrho}} |\boldsymbol{u} - \tilde{\boldsymbol{u}}| |[\varrho - \tilde{\varrho}]_{\mathrm{res}}| \mathrm{d}x\mathrm{d}t; \end{aligned}$$

Making use of the generalized Korn's inequality and the second item of (A.17),

$$\begin{aligned} &\int_0^\tau \int_{\varrho \leq 1/2 \min_{\overline{Q_T}} \tilde{\varrho}} |\boldsymbol{u} - \tilde{\boldsymbol{u}}| |[\varrho - \tilde{\varrho}]_{\mathrm{res}}| \mathrm{d}x\mathrm{d}t \lesssim \int_0^\tau \|[1]_{\mathrm{res}}\|_{L^2(\Omega)} \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^2(\Omega)} \mathrm{d}t \\ &\lesssim \varepsilon \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \boldsymbol{u} - \nabla_x \tilde{\boldsymbol{u}}) : (\nabla_x \boldsymbol{u} - \nabla_x \tilde{\boldsymbol{u}})\Big) \mathrm{d}x\mathrm{d}t + C(\varepsilon) \int_0^\tau \mathcal{E}\Big((\varrho, n, \boldsymbol{u}) \Big| (\tilde{\varrho}, \tilde{n}, \tilde{\boldsymbol{u}})\Big)(t)\mathrm{d}t. \end{aligned} \tag{A.17}$$

In the same spirit,

$$\begin{aligned} &\int_0^\tau \int_{\varrho \geq 2 \max_{\overline{Q_T}} \tilde{\varrho}} |\boldsymbol{u} - \tilde{\boldsymbol{u}}| |[\varrho - \tilde{\varrho}]_{\mathrm{res}}| \mathrm{d}x\mathrm{d}t \lesssim \int_0^\tau \int_\Omega |[1]_{\mathrm{res}}| \sqrt{\varrho} |\boldsymbol{u} - \tilde{\boldsymbol{u}}| \sqrt{\varrho} \mathrm{d}x\mathrm{d}t \\ &\lesssim \int_0^\tau \int_\Omega |[1]_{\mathrm{res}}| \sqrt{\mathfrak{r}} |\boldsymbol{u} - \tilde{\boldsymbol{u}}| \sqrt{\varrho} \mathrm{d}x\mathrm{d}t \lesssim \int_0^\tau \mathcal{E}\Big((\varrho, n, \boldsymbol{u}) \Big| (\tilde{\varrho}, \tilde{n}, \tilde{\boldsymbol{u}})\Big)(t)\mathrm{d}t. \end{aligned} \tag{A.18}$$

The estimate of the second integral in (A.16) can be carried in exactly the same manner. Therefore, combining (A.12), (A.14)–(A.18),

$$|\mathcal{R}^{(1)}| \lesssim \varepsilon \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \boldsymbol{u} - \nabla_x \tilde{\boldsymbol{u}}) : (\nabla_x \boldsymbol{u} - \nabla_x \tilde{\boldsymbol{u}})\Big) \mathrm{d}x\mathrm{d}t + C(\varepsilon) \int_0^\tau \mathcal{E}\Big((\varrho, n, \boldsymbol{u}) \Big| (\tilde{\varrho}, \tilde{n}, \tilde{\boldsymbol{u}})\Big)(t)\mathrm{d}t. \tag{A.19}$$

It is easy to see that $\mathcal{R}^{(2)}$ and $\mathcal{R}^{(3)}$ can be estimated analogously as above. Next, we observe that

$$\begin{aligned} &\left| \Big(\varrho^\gamma - \tilde{\varrho}^\gamma - \gamma \tilde{\varrho}^{\gamma-1}(\varrho - \tilde{\varrho})\Big) + \Big(n^\alpha - \tilde{n}^\alpha - \alpha \tilde{n}^{\alpha-1}(n - \tilde{n})\Big) \right| \\ &\lesssim H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) + G(n) - G(\tilde{n}) - G'(\tilde{n})(n - \tilde{n}), \end{aligned}$$

whence

$$|\mathcal{R}^{(4)}| \lesssim \int_0^\tau \mathcal{E}\Big((\varrho, n, \boldsymbol{u}) \Big| (\tilde{\varrho}, \tilde{n}, \tilde{\boldsymbol{u}})\Big)(t)\mathrm{d}t. \tag{A.20}$$

Finally,

$$|\mathcal{R}^{(5)}| = \left| \int_0^\tau \int_{\overline{\Omega}} \nabla_x \tilde{\boldsymbol{u}} : \mathrm{d}\mu_c(t)\mathrm{d}t \right| \leq \|\nabla_x \tilde{\boldsymbol{u}}\|_{L^\infty(\overline{Q_T})} \int_0^\tau \int_{\overline{\Omega}} \mathrm{d}\mathcal{D}(t)\mathrm{d}t. \tag{A.21}$$

Consequently, collecting the estimates above, we conclude from (A.11) that, upon choosing $\varepsilon > 0$ sufficiently small,

$$\mathcal{E}\Big((\varrho, n, \boldsymbol{u})\Big|(\tilde{\varrho}, \tilde{n}, \tilde{\boldsymbol{u}})\Big)(\tau) + \int_0^\tau \int_\Omega \Big(\mathbb{S}(\nabla_x \boldsymbol{u} - \nabla_x \tilde{\boldsymbol{u}}) : (\nabla_x \boldsymbol{u} - \nabla_x \tilde{\boldsymbol{u}})\Big) \mathrm{d}x\mathrm{d}t + \int_{\overline{\Omega}} \mathrm{d}\mathcal{D}(\tau)$$

$$\lesssim \int_0^\tau \mathcal{E}\Big((\varrho, n, \boldsymbol{u})\Big|(\tilde{\varrho}, \tilde{n}, \tilde{\boldsymbol{u}})\Big)(t)\mathrm{d}t + \int_0^\tau \int_{\overline{\Omega}} \mathrm{d}\mathcal{D}(t)\mathrm{d}t,$$

which immediately finishes the proof of Proposition 1.1 by Gronwall's inequality. □

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