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QUANTIFYING SHRINKING AND BOUNDEDLY COMPLETE BASES

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ABSTRACT. We investigate possible quantifications of R. C. James' classical work on bases and reflexivity of Banach spaces. By introducing new quantities measuring how far a basic sequence is from being shrinking and/or boundedly complete, we prove quantitative versions of James' famous characterisations of reflexivity in terms of bases. Furthermore, we establish quantitative versions of James' characterisations of reflexivity of Banach spaces with unconditional bases.

1. INTRODUCTION

James' classical paper [9] linking reflexivity and bases is deeply entrenched in modern Banach space theory; the now standard characterisation of reflexivity in terms of shrinkingness and bounded completeness of bases/basic sequences is proved therein. For a space with a basis a trade-off between various measures of (non-)weak compactness of the unit ball and closedness of the basis to be simultaneously shrinking and boundedly complete is naturally expected. In the present paper we investigate possible quantifications of the said notions with the aim of establishing quantitative analogues of James' criteria for reflexivity expressed in terms of bases/basic sequences. This line of research is particularly timely in the light of numerous recent results in this spirits (see, *e.g.*, [4, 12, 13, 14]).

In order to quantify James' criteria of reflexivity, it is thus necessary to introduce quantities measuring how far a basis is from being shrinking and/or boundedly complete. In Section 3, a quantity $\mathrm{sh}((x_n)_{n=1}^{\infty})$ measuring how far a basic sequence $(x_n)_{n=1}^{\infty}$ is from being shrinking is introduced and investigated. In Section 4, we introduce three equivalent quantities $\mathrm{bc}_1, \mathrm{bc}_2$, and bc_3 measuring (non-)bounded completeness of a basis. Besides the quantities sh and bc, we also need a quantity sep measuring non-separability of a set, a quantity $\alpha_Y(X)$ measuring how well a Banach space Y is from being isomorphically embedded into another Banach space X, and two important mutually equivalent quantities wk, wck measuring weak non-compactness of sets. In this paper, we quantify Theorems 2.1–2.4, respectively. In order to state the results let us introduce the following conventions.

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If X is a Banach space with a basis $(x_n)_{n=1}^\infty$, we denote by $(x_n^*)_{n=1}^\infty$ the sequence of coordinate functionals associated with the basis and by K the basis constant. When $(x_n)_{n=1}^\infty$ is unconditional, we denote by K_u the unconditional constant of $(x_n)_{n=1}^\infty$. Slightly abusing the notation, for a Banach space X with a fixed basis $(x_n)_{n=1}^\infty$, we set

$$V = [x_n^* \colon n \in \mathbb{N}].$$

Theorem A. *Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$. If K is the basis constant, then*

$$\operatorname{sh}((x_n)_{n=1}^\infty) \leqslant \widehat{\operatorname{d}}(B_{X^*}, V) \leqslant (K+1)\operatorname{sh}((x_n)_{n=1}^\infty),$$

and

$$\frac{1}{2K}\operatorname{bc}_2((x_n^*)_{n=1}^\infty) \leqslant \operatorname{sh}((x_n)_{n=1}^\infty) \leqslant K\operatorname{bc}_2((x_n^*)_{n=1}^\infty).$$

If, in addition, $(x_n)_{n=1}^\infty$ is unconditional, then

$$\frac{1}{K_u}\operatorname{sh}((x_n)_{n=1}^\infty) \leqslant \alpha_{\ell_1}(X) \leqslant \operatorname{sep}(B_{X^*}) \leqslant \widehat{\operatorname{d}}(B_{X^*}, V).$$

Theorem B. *Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$. If K is the basis constant, then*

$$\operatorname{sh}((x_n^*)_{n=1}^\infty) \leqslant \operatorname{bc}_2((x_n)_{n=1}^\infty) \leqslant 2K^2\operatorname{sh}((x_n^*)_{n=1}^\infty).$$

If, in addition, $(x_n)_{n=1}^\infty$ is unconditional, then

$$\frac{1}{K_u}\alpha_{c_0}(X) \leqslant \operatorname{bc}_1((x_n)_{n=1}^\infty) \leqslant K_u^3\alpha_{c_0}(X).$$

Theorem C. *Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$.*

(i) *If $(x_n)_{n=1}^\infty$ is boundedly complete, then*

$$\begin{cases} \operatorname{sh}((x_n)_{n=1}^\infty) & \leqslant & 4K^3\operatorname{wk}_X(B_X) \\ \operatorname{wck}_X(B_X) & \leqslant & (K+1)\widehat{\operatorname{d}}(B_{X^*}, V) \end{cases}$$

(ii) *If $(x_n)_{n=1}^\infty$ is shrinking, then*

$$\begin{cases} \operatorname{bc}_3((x_n)_{n=1}^\infty) & \leqslant & 2K^2\operatorname{wk}_X(B_X) \\ \operatorname{wck}_X(B_X) & \leqslant & (K+1)^2\operatorname{bc}_2((x_n)_{n=1}^\infty). \end{cases}$$

Theorem D. *Let X be a Banach space with an unconditional basis $(x_n)_{n=1}^\infty$.*

(i) *If X contains no isomorphic copies of ℓ_1, then*

$$\frac{1}{K_u^3K(K+1)^2}\operatorname{wck}_X(B_X) \leqslant \alpha_{c_0}(X) \leqslant \alpha_{\ell_1}(X^*) \leqslant \operatorname{wck}_X(B_X).$$

(ii) *If X contains no isomorphic copies of c_0, then*

$$\frac{1}{K_u(K+1)^2}\operatorname{wck}_X(B_X) \leqslant \alpha_{\ell_1}(X) \leqslant \operatorname{wck}_X(B_X).$$

(iii)

$$\frac{1}{K_u^3K(K+1)^2}\operatorname{wck}_X(B_X) \leqslant \operatorname{sep}(B_{X^{**}}) \leqslant \operatorname{wk}_X(B_X).$$

2. Preliminaries

Our notation and terminology are standard and mainly follow [16] and [15]. Throughout this paper, all Banach spaces are infinite-dimensional and real for the sake of convenience. By a *subspace* we understand as a closed, linear subspace and by an *operator* we mean a bounded, linear operator. An operator $T\colon X \to Y$ is *bounded below*, whenever there is $\gamma > 0$ such that $\|Tx\| \geqslant \gamma\|x\|$ $(x \in X)$; equivalently, when T is an isomorphism onto its range. If X is a Banach space, we denote by B_X its closed unit ball and by I_X the identity operator on X. For a subset A of X, $[A]$ stands for the closed linear span of A.

2.1. **Basics on Schauder bases.** A sequence $(x_n)_{n=1}^\infty$ in a Banach space X is called a *(Schauder) basis* for X whenever every $x \in X$ has a unique expansion $x = \sum_{n=1}^\infty a_n(x)x_n$ for some scalar sequence $(a_n(x))_{n=1}^\infty$. A sequence $(x_n)_{n=1}^\infty$ in a Banach space X is called *basic* if it is a basis for $[x_n\colon n \in \mathbb{N}]$. For every $n \in N$, the linear functional x_n^* on X given by $\langle x_n^*, x\rangle = a_n(x)$ $(x \in X)$ is well-defined and bounded. We call $(x_n^*)_{n=1}^\infty$ the *biorthogonal functionals* associated to the basis $(x_n)_{n=1}^\infty$.

The canonical basis projections $(P_n)_{n=1}^\infty$ associated to the basis $(x_n)_{n=1}^\infty$ are given by $P_n(x) = \sum_{i=1}^n \langle x_i^*, x\rangle x_i$ $(x \in X)$. Since each functional x_n^* is continuous, so is P_n $(n \in \mathbb{N})$. The Uniform Boundedness principle implies that $K := \sup_n \|P_n\| < \infty$. The number K is called the *basis constant* of $(x_n)_{n=1}^\infty$. We have $P_n^* x^* = \sum_{i=1}^n \langle x^*, x_i\rangle x_i^*$ $(x^* \in X^*)$. In particular, $(x_n^*)_{n=1}^\infty$ is a basic sequence with basis constant at most K. We denote by j be the canonical map from X to V^* defined by $\langle jx, x^*\rangle = \langle x^*, x\rangle$ for all $x \in X$ and $x^* \in V$. Then j is bounded below as

$$\frac{1}{K}\|x\| \leqslant \|jx\| \leqslant \|x\| \quad (x \in X). \tag{2.1}$$

and $(jx_n)_n$ is the sequence of biorthogonal functionals associated to the basic sequence $(x_n^*)_{n=1}^\infty$. We let $W = [jx_n\colon n \in \mathbb{N}]$.

Let X be a Banach space with an unconditional basis $(x_n)_{n=1}^\infty$. Then for every choice of unit scalars $\theta = (\theta_n)_{n=1}^\infty$ the map $M_\theta\colon X \to X$ defined by $M_\theta(x) = \sum_{n=1}^\infty \theta_n \langle x_n^*, x\rangle x_n$ $(x \in X)$ is continuous. The Uniform Boundedness principle implies that $\sup_\theta \|M_\theta\|$ is finite. The number $K_u := \sup_\theta \|M_\theta\|$ is called the *unconditional constant* of $(x_n)_{n=1}^\infty$. One observes readily that $K_u \geqslant K$.

Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$. Then $(x_n)_{n=1}^\infty$ is

- *shrinking* if the sequence of biorthogonal functionals $(x_n^*)_{n=1}^\infty$ is a basis for X^*, *i.e.*, when $X^* = V$.
- *boundedly complete* if for every scalar sequence $(a_n)_{n=1}^\infty$ with $\sup_n \|\sum_{i=1}^n a_i x_i\| < \infty$, the series $\sum_{n=1}^\infty a_n x_n$ converges.

We shall use the notation $\|x^*\|_n = \|x^*|_{[x_i\colon i>n]}\|$, $\quad (x^* \in X^*, n \in \mathbb{N})$, which renders to

$$\|x^*\|_n = \mathrm{d}(x^*, [x_i^*\colon i \leqslant n]). \tag{2.2}$$

(see [16, Proposition 4.1]).

It is known that a basis $(x_n)_{n=1}^\infty$ is shrinking if and only if $\|x^*\|_n \to 0$ as $n \to \infty$ for every $x^* \in X^*$. A basic sequence $(x_n)_{n=1}^\infty$ in a Banach space X is *shrinking* if it is a shrinking basis for $[x_n\colon n \in \mathbb{N}]$. Similarly, a basic sequence $(x_n)_{n=1}^\infty$ is called *boundedly complete* if it is a boundedly complete basis for $[x_n\colon n \in \mathbb{N}]$.

2.2. **The James space.** For illustratory purposes, we shall occasionally invoke the order-one quasi-reflexive *James space* $\mathcal{J}$, which is a sequence space comprising all scalar sequences $x = (a_n)_{n=1}^\infty$ for which $\lim_{n\to\infty} a_n = 0$ and

$$\|x\|_{\mathcal{J}} = \frac{1}{\sqrt{2}} \sup[|a_{p_1} - a_{p_2}|^2 + |a_{p_2} - a_{p_3}|^2 + \ldots + |a_{p_n} - a_{p_{n+1}}|^2 + |a_{p_{n+1}} - a_{p_1}|^2]^{\frac{1}{2}} < \infty,$$

where the supremum is taken over all n and all choices of integers $p_1 < p_2 < \ldots < p_{n+1}$.

The standard unit vectors $(e_n)_{n=1}^\infty$ form a monotone, shrinking normalised basis for $\mathcal{J}$, whereas $(\sum_{i=1}^n e_i)_{n=1}^\infty$ is a boundedly complete basis for $\mathcal{J}$.

2.3. **James' theorems.** Theorem A is directly motivated by the following theorem of James [9] that we shall now invoke.

Theorem 2.1. *Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$. Then the following assertions are equivalent:*

(i) *$(x_n)_{n=1}^\infty$ is shrinking;*
(ii) *$X^* = [x_n^*\colon n \in \mathbb{N}]$;*
(iii) *$(x_n^*)_{n=1}^\infty$ is a boundedly complete basic sequence.*

If in addition $(x_n)_{n=1}^\infty$ is unconditional, then (i)–(iii) *are equivalent to* (iv)–(v)*:*

(iv) *X^* is separable;*
(v) *X contains no isomorphic copies of ℓ_1.*

Similarly, Theorem B quantifies the following theorem of James [9].

Theorem 2.2. *Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$. Then the following assertions are equivalent:*

(i) *$(x_n)_{n=1}^\infty$ is boundedly complete;*
(ii) *$(x_n^*)_{n=1}^\infty$ is shrinking.*

If in addition $(x_n)_{n=1}^\infty$ is unconditional, then (i)-(ii) *are equivalent to the following:*

(iii) *X contains no subspaces isomorphic to c_0.*

James [9] proved that joint occurrence of shrinkingness and bounded completeness characterise reflexivity, which is out main motivation for Theorem C.

Theorem 2.3. *Let X be a Banach space with a basis $(x_n)_{n=1}^{\infty}$. Then X is reflexive if and only if $(x_n)_{n=1}^{\infty}$ is both boundedly complete and shrinking.*

The following theorem is again due to R. C. James [9] except that the last statement that had been proved earlier by S. Karlin [11] who employed different techniques.

Theorem 2.4. *Let X be a Banach space with an unconditional basis. The following assertions are equivalent:*

(i) *X is reflexive.*
(ii) *No subspace of X is isomorphic to either of ℓ_1 or c_0.*
(iii) *No subspace of either X or X^* is isomorphic to ℓ_1.*
(iv) *X^{**} is separable.*

2.4. **Measures of weak non-compactness and non-separability.** Let X be a Banach space and $A, B \subseteq X$ two non-empty sets. We set

- $\mathrm{d}(A,B) = \inf\{\|a-b\| : a \in A, b \in B\}$,
- $\widehat{\mathrm{d}}(A,B) = \sup\{\mathrm{d}(a,B) : a \in A\}$.

$\mathrm{d}(A,B)$ is the ordinary distance between A and B and $\widehat{\mathrm{d}}(A,B)$ is the (non-symmetrised) Hausdorff distance from A to B. The quantity $\widehat{\mathrm{d}}$ measures how much the set A sticks out from the set B and is used to define several measures of weak non-compactness. The present paper focuses on two standard equivalent measures. The first one is inspired by the Banach-Alaoglu theorem: for a bounded subset A of X we set:

$$\mathrm{wk}_X(A) = \widehat{\mathrm{d}}\big(\overline{A}^{\sigma(X^{**},X^*)}, X\big).$$

It is a direct consequence of the Banach–Alaoglu theorem that A is relatively weakly compact if and only if $\mathrm{wk}_X(A) = 0$. The second one is inspired by the Eberlein–Šmulyan theorem: for a bounded subset A of X we set:

$$\mathrm{wck}_X(A) = \sup\{\mathrm{d}(\mathrm{clust}_{X^{**}}((x_n)_{n=1}^{\infty}), X) : (x_n)_{n=1}^{\infty} \text{ is a sequence in } A\},$$

where $\mathrm{clust}_{X^{**}}((x_n)_{n=1}^{\infty})$ is the set of all weak*-cluster points of $(x_n)_{n=1}^{\infty}$ in X^{**}.

It follows easily from the Eberlein–Šmulyan theorem that $\mathrm{wck}_X(A) = 0$ whenever A is relatively weakly compact. The converse follows from the quantitative version of the Eberlein–Šmulyan theorem as proved in [2]. It follows from [3, Theorem 2.3] that

$$\mathrm{wck}_X(A) \leqslant \mathrm{wk}_X(A) \leqslant 2\,\mathrm{wck}_X(A). \tag{2.3}$$

It should be pointed out that $\mathrm{wck}_X(B_X) = 1$ for every non-reflexive Banach space X. This follows for example from [8, Theorem 1]. By (2.3), this also holds for $\mathrm{wk}_X(B_X)$. Hence

$$\mathrm{wck}_X(B_X) = \mathrm{wck}_{X^*}(B_{X^*}), \quad \mathrm{wk}_X(B_X) = \mathrm{wk}_{X^*}(B_{X^*}). \tag{2.4}$$

If Y is a subspace of a Banach space X and A is a bounded subset of Y, it is clear that A is relatively weakly compact in Y if and only if it is relatively weakly compact in X. This elementary fact has the following quantitative version:

$$\mathrm{wk}_X(A) \leqslant \mathrm{wk}_Y(A) \leqslant 2\,\mathrm{wk}_X(A), \quad \mathrm{wck}_X(A) \leqslant \mathrm{wck}_Y(A) \leqslant 2\,\mathrm{wck}_X(A). \tag{2.5}$$

In both cases the first inequality is trivial, whereas the second one follows from [7, Lemma 11]. It is worth mentioning that the factor 2 in the right-hand side of the inequalities (2.5) is optimal. Indeed, let $Y = c_0, X = \ell_\infty$, and A be the summing basis of c_0. A standard argument shows that $\mathrm{wck}_Y(A) = \mathrm{wk}_Y(A) = 1$ and $\mathrm{wck}_X(A) = \mathrm{wk}_X(A) = 1/2$.

Let A be a subset of a Banach space X, we set

$$\mathrm{sep}(A) = \inf\{\varepsilon > 0\colon A \subseteq C + \varepsilon B_X, C \subseteq X \text{ countable}\}.$$

Clearly, A is separable if and only if $\mathrm{sep}(A) = 0$.

Lemma 2.5. *Let X be a Banach space. Then* $\mathrm{sep}(B_X) \in \{0, 1\}$. *In particular,*

$$\mathrm{sep}(B_X) \leqslant \mathrm{sep}(B_{X^*}).$$

Proof. Suppose that $\mathrm{sep}(B_X) < 1$. Then there exist $\varepsilon < 1$ and a countable subset C_1 of X so that $B_X \subseteq C_1 + \varepsilon B_X$. This implies that

$$B_X \subseteq C_1 + \varepsilon(C_1 + \varepsilon B_X) = C_2 + \varepsilon^2 B_X,$$

where $C_2 = C_1 + \varepsilon C_1$ is countable.

Inductively, we get a sequence of countable sets $(C_n)_n$ so that $B_X \subseteq C_n + \varepsilon^n B_X$ for all n. Hence $\mathrm{sep}(B_X) \leqslant \varepsilon^n$ for all n and so $\mathrm{sep}(B_X) = 0$. □

2.5. **Measures of isomorphisms between Banach spaces.** The first-named author [5] introduced a quantity measuring how well a Banach space can be embedded into another Banach space. More precisely, let X, Y be Banach spaces. If X and Y are isomorphic, we set

$$\alpha_Y(X) = \sup\{\|T^{-1}\|^{-1}\colon T\colon Y \to X \text{ is an isomorphism with} \|T\| \leqslant 1\}.$$

If there is no isomorphism from Y into X, we set $\alpha_Y(X) = 0$. We have $\alpha_Y(X) = 1$ if and only if X contains almost isometric copies of Y. We also need a quantity [5] measuring how well a Banach space is from being isomorphic to a complemented subspace of another Banach space.

For a pair of Banach spaces X, Y we set

$$\beta_Y(X) = \sup\{(\|A\|\|B\|)^{-1}\colon A\colon X \to Y, B\colon Y \to X \text{ are operators such that } AB = I_Y\}.$$

If there are no such operators A, B, we set $\beta_Y(X) = 0$. Clearly, $\beta_Y(X) = 1$ if and only if for every $\varepsilon > 0$, there exists a subspace M of X so that M is $(1+\varepsilon)$-isomorphic to Y and M is $(1+\varepsilon)$-complemented in X. Let us collect some elementary observations that we shall later employ.

Lemma 2.6. *Let X, Y be Banach spaces. Then*

(i) $\beta_Y(X) \leqslant \beta_{Y^*}(X^*) \leqslant \alpha_{Y^*}(X^*)$.
(ii) $\alpha_{c_0}(X) = \beta_{c_0}(X)$ *if X is separable.*
(iii) $\alpha_{\ell_1}(X) \leqslant \mathrm{wck}_X(B_X)$.

Proof. (i) is straightforward, whereas (ii) follows from [6, Theorem 6]. (iii) follows from [13, Lemma 5]. □

3. Quantification of shrinking bases

Definition 3.1. Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$. We set

$$\mathrm{sh}_X((x_n)_{n=1}^\infty) = \sup_{x^* \in B_{X^*}} \limsup_n \|x^*\|_n.$$

According to the definition above, $(x_n)_{n=1}^\infty$ is shrinking if and only if $\mathrm{sh}((x_n)_{n=1}^\infty) = 0$.

If $(x_n)_n$ is a basic sequence, we are able to define $\mathrm{sh}_{[x_n : n \in \mathbb{N}]}((x_n)_n)$ naturally. For the sake of simplicity, we'll omit the subscripts X in $\mathrm{sh}_X((x_n)_n)$ for a basis $(x_n)_n$ and $[x_n : n \in \mathbb{N}]$ in $\mathrm{sh}_{[x_n : n \in \mathbb{N}]}((x_n)_n)$ for a basic sequence $(x_n)_n$.

Example 3.2.

(1) $\mathrm{sh}((e_n^*)_{n=1}^\infty) = 1$, where $(e_n^*)_{n=1}^\infty$ is the unit vector basis of ℓ_1.
(2) $\mathrm{sh}((s_n)_{n=1}^\infty) = 1$, where $(s_n)_{n=1}^\infty$ is the summing basis of c_0.
(3) $\mathrm{sh}((\sum_{i=1}^n e_i)_{n=1}^\infty) = 1$, where $(e_n)_{n=1}^\infty$ is the unit vector basis of the James space $\mathcal{J}$.

Proof. We only prove (3). Let $(f_n)_{n=1}^\infty$ be the biorthogonal functionals associated to $(e_n)_{n=1}^\infty$. Since $(e_n)_{n=1}^\infty$ is monotone, we get $\|f_1\| = 1$. Clearly, $\|\sum_{i=1}^n e_i\|_{\mathcal{J}} = 1$ for all n. Thus $\|f_1|_{[\sum_{i=1}^k e_i \,:\, k > n]}\| = 1$ for all n. □

Theorem 3.3. *Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$. Then*

$$\mathrm{sh}((x_n)_{n=1}^\infty) \leqslant \widehat{\mathrm{d}}(B_{X^*}, V) \leqslant (K+1)\,\mathrm{sh}((x_n)_{n=1}^\infty).$$

Proof. Let $0 < c < \mathrm{sh}((x_n)_{n=1}^\infty)$. By (2.2), there exist $x_0^* \in B_{X^*}$ and a strictly increasing sequence $(k_n)_n$ of positive integers so that $\mathrm{d}(x_0^*, [x_i^* \colon i \leqslant k_n]) > c$ for all n. We *claim* that $\mathrm{d}(x_0^*, V) \geqslant c$. Indeed, given $x^* \in V$ and $\varepsilon > 0$. We choose $y^* \in [x_i^* \colon i \leqslant N]$ for some N so that $\|x^* - y^*\| < \varepsilon$. Hence we get

$$\|x_0^* - x^*\| \geqslant \|x_0^* - y^*\| - \|y^* - x^*\| > c - \varepsilon.$$

Letting $\varepsilon \to 0$, we get the claim. Since c is arbitrary, we arrive at the first inequality. It remains to verify the second inequality.

For this, fix an arbitrary $0 < c < \widehat{\mathrm{d}}(B_{X^*}, V)$. We choose $x_0^* \in B_{X^*}$ with $\mathrm{d}(x_0^*, V) > c$. For each n, we get

$$\begin{aligned} c &\leqslant \|x_0^* - \sum_{i=1}^{n}\langle x_0^*, x_i\rangle x_i^*\| \\ &= \sup_{x\in B_X} |\langle x_0^*, x\rangle - \sum_{i=1}^{n}\langle x_0^*, x_i\rangle\langle x_i^*, x\rangle| \\ &= \sup_{x\in B_X} |\langle x_0^*, x - \sum_{i=1}^{n}\langle x_i^*, x\rangle x_i\rangle| \\ &= \sup_{x\in B_X} |\langle x_0^*, \sum_{i=n+1}^{\infty}\langle x_i^*, x\rangle x_i\rangle| \\ &\leqslant (K+1)\|x_0^*|_{[x_i\colon i>n]}\|. \end{aligned}$$

This implies that $c \leqslant (K+1)\limsup_n \|x_0^*\|_n$. As c is arbitrary, the proof is complete. □

Theorem 3.4. *Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$. Then*

$$\alpha_{\ell_1}(X) \leqslant \mathrm{sep}(B_{X^*}) \leqslant \widehat{\mathrm{d}}(B_{X^*}, V).$$

Proof. Let $0 < c < \alpha_{\ell_1}(X)$. Then there exists an operator $T\colon \ell_1 \to X$ such that

$$c\|z\| \leqslant \|Tz\| \leqslant \|z\| \quad (z \in \ell_1).$$

This implies that $T^*B_{X^*} \supseteq cB_{\ell_\infty} \supseteq cA$, where $A = \{(\theta_n)_{n=1}^\infty\colon |\theta_n| = 1 \ (n \in \mathbb{N})\}$. It is easy to see that $\mathrm{sep}(A) = 1$. Hence we get

$$c \leqslant \mathrm{sep}(T^*B_{X^*}) \leqslant \mathrm{sep}(B_{X^*}).$$

By the arbitrariness of c, we get the first inequality.

Let $c > \widehat{\mathrm{d}}(B_{X^*}, V)$ and $\varepsilon > 0$. Moreover, let $\mathcal{Q}$ be a countable, dense subset of $\mathbb{R}$ and let $C = \{\sum_{i=1}^n r_i x_i^*\colon n \in \mathbb{N}, r_1, r_2, \ldots, r_n \in \mathcal{Q}\}$. For $x^* \in B_{X^*}$ we choose $y^* \in V$ with $\|x^* - y^*\| < c$ and $z^* \in C$ with $\|y^* - z^*\| < \varepsilon$. Hence $\|x^* - z^*\| < c + \varepsilon$. This means that $\mathrm{sep}(B_{X^*}) \leqslant c + \varepsilon$. As c and ε were arbitrary, the proof is complete. □

Theorem 3.5. *Let X be a Banach space with an unconditional basis $(x_n)_{n=1}^\infty$. Then*

$$\mathrm{sh}((x_n)_{n=1}^\infty) \leqslant K_u\alpha_{\ell_1}(X).$$

Proof. Let $0 < c < \mathrm{sh}((x_n)_{n=1}^\infty)$. Then there exist $x_0^* \in B_{X^*}$ and a block basic sequence $(u_n)_n$ with respect to $(x_n)_{n=1}^\infty$ so that $\|u_n\| \leqslant 1$ and $\langle x_0^*, u_n\rangle > c$ for all n. Then, for each m and every choice of scalars $a_1, a_2, \ldots, a_m$, we get

$$K_u\|\sum_{n=1}^{m} a_n u_n\| \geqslant \|\sum_{n=1}^{m} |a_n| u_n\| \geqslant \sum_{n=1}^{m} |a_n|\langle x_0^*, u_n\rangle \geqslant c\sum_{n=1}^{m} |a_n|.$$

Let us define an operator $T\colon \ell_1 \to X$ by $Te_n = u_n$ $(n \in \mathbb{N})$. It is easy to see that $\|T\| \leqslant 1$ and $\|T^{-1}\| \leqslant \frac{K_u}{c}$. This implies that $\alpha_{\ell_1}(X) \geqslant \frac{c}{K_u}$. As c was arbitrary, the proof is complete. □

4. Quantifications of bounded completeness

Let $(x_n)_{n=1}^\infty$ be a bounded sequence in a Banach space X. We set

$$\mathrm{ca}((x_n)_{n=1}^\infty) = \inf_n \sup_{k,l\geqslant n} \|x_k - x_l\|.$$

Then $(x_n)_{n=1}^\infty$ is norm-Cauchy if and only if $\mathrm{ca}((x_n)_{n=1}^\infty) = 0$.

Definition 4.1. Let $(x_n)_{n=1}^\infty$ be a basis for a Banach space X. We set

$$\mathrm{bc}_1((x_n)_{n=1}^\infty) = \sup\Big\{\mathrm{ca}((\sum_{i=1}^n a_i x_i)_{n=1}^\infty)\colon (\sum_{i=1}^n a_i x_i)_{n=1}^\infty \subseteq B_X\Big\}.$$

Clearly, $(x_n)_{n=1}^\infty$ is boundedly complete if and only if $\mathrm{bc}_1((x_n)_{n=1}^\infty) = 0$.

Example 4.2.

(1) $\mathrm{bc}_1((e_n)_n) = 1$, where $(e_n)_n$ is the unit vector basis of c_0.
(2) $\mathrm{bc}_1((s_n)_n) = 1$, where $(s_n)_n$ is the summing basis of c_0.
(3) $\mathrm{bc}_1((e_n)_{n=0}^\infty) = 2$, where $(e_n)_{n=0}^\infty$ is the unit vector basis of c $(e_0 = (1, 1, 1, \ldots))$.
(4) $\mathrm{bc}_1((e_n)_n) = 1$, where $(e_n)_n$ is the unit vector basis of the James space $\mathcal{J}$.

Proof. Example (1) is straightforward. For (2), note that $\|\sum_{i=1}^n a_i s_i\| = \max_{1\leqslant k\leqslant n} |\sum_{i=k}^n a_i|$ for all n and all scalars $a_1, a_2, \ldots, a_n$. Given $(\sum_{i=1}^n a_i s_i)_n \subseteq B_{c_0}$. Then $|\sum_{i=k}^n a_i| \leqslant 1$ for all n and all $k \leqslant n$. This implies that

$$\|\sum_{i=n+1}^{n+m} a_i s_i\| = \max_{n+1\leqslant k\leqslant n+m} |\sum_{i=k}^{n+m} a_i| \leqslant 1 \quad (m, n \in \mathbb{N}).$$

Hence $\mathrm{ca}((\sum_{i=1}^n a_i s_i)_n) \leqslant 1$ and so $\mathrm{bc}_1((s_n)_n) \leqslant 1$.

On the other hand, observe that

$$\|\sum_{i=1}^n (-1)^i s_i\| = \max_{1\leqslant k\leqslant n} |\sum_{i=k}^n (-1)^i| \leqslant 1 \quad (n \in \mathbb{N}).$$

However, for each n, we have

$$\|\sum_{i=1}^{n+2n-1} (-1)^i s_i - \sum_{i=1}^n (-1)^i s_i\| = \max_{n+1\leqslant k\leqslant n+2n-1} |\sum_{i=k}^{n+2n-1} (-1)^i| \geqslant |\sum_{i=n+1}^{n+2n-1} (-1)^i| = 1.$$

This implies that $\mathrm{ca}((\sum_{i=1}^n (-1)^i s_i)_{n=1}^\infty) \geqslant 1$. Hence $\mathrm{bc}_1((s_n)_n) \geqslant 1$.
(3). Observe that $\|\sum_{i=0}^n a_i e_i\| = \max(|a_0|, |a_0 + a_1|, \ldots, |a_0 + a_n|)$ for all n and all scalars $a_0, a_1, \ldots, a_n$. We take $a_0 = -1, a_n = 2$ $(n = 1, 2, \ldots)$. Then $\|\sum_{i=0}^n a_i e_i\| = 1$ for all n. However, for all n, k, we get

$$\Big\|\sum_{i=0}^{n+k} a_i e_i - \sum_{i=0}^n a_i e_i\Big\| = \sup_{n+1\leqslant i\leqslant n+k} |a_i| = 2.$$

This means that $\mathrm{ca}((\sum_{i=0}^n a_i e_i)_{n=1}^\infty) = 2$. Hence $\mathrm{bc}_1((e_n)_{n=0}^\infty) = 2$.
(4). Note that $\|\sum_{i=n}^m e_i\|_{\mathcal{J}} = 1$ for all n, m. This implies that $\mathrm{bc}_1((e_n)_n) \geqslant 1$. For the second inequality, assume that $\|\sum_{i=1}^n a_i e_i\|_{\mathcal{J}} \leqslant 1$ for all n. Suppose $p_1 < p_2 < \ldots < p_{n+1}$. Since $\|\sum_{i=1}^{p_{n+1}} a_i e_i\|_{\mathcal{J}} \leqslant 1$, we get $\sum_{i=1}^n (a_{p_i} - a_{p_{i+1}})^2 + a_{p_{n+1}}^2 + a_{p_1}^2 \leqslant 2$. This implies that $\|\sum_{i=n}^m a_i e_i\|_{\mathcal{J}} \leqslant 1$ for all n, m and so $\mathrm{bc}_1((e_n)_n) \leqslant 1$. □

Definition 4.3. Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$. We set

- $\mathrm{bc}_2((x_n)_{n=1}^\infty) = \sup_{\varphi \in B_{V^*}} \mathrm{ca}((\sum_{i=1}^n \langle \varphi, x_i^* \rangle x_i)_{n=1}^\infty)$
- $\mathrm{bc}_3((x_n)_{n=1}^\infty) = \sup_{x^{**} \in B_{X^{**}}} \mathrm{ca}((\sum_{i=1}^n \langle x^{**}, x_i^* \rangle x_i)_{n=1}^\infty)$.

Theorem 4.4. *Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$. Then*

$$\mathrm{bc}_1((x_n)_{n=1}^\infty) \leqslant \mathrm{bc}_2((x_n)_{n=1}^\infty) \leqslant \mathrm{bc}_3((x_n)_{n=1}^\infty) \leqslant K\, \mathrm{bc}_1((x_n)_{n=1}^\infty).$$

Proof. Claim 1: $\mathrm{bc}_1((x_n)_{n=1}^\infty) \leqslant \mathrm{bc}_2((x_n)_{n=1}^\infty)$.

Given a sequence of scalars $a_1, \ldots, a_n$ so that $\|\sum_{i=1}^n a_i x_i\| \leqslant 1$ for all n, we have

$$|\sum_{i=1}^n b_i a_i| = |\langle \sum_{j=1}^n b_j x_j^*, \sum_{i=1}^n a_i x_i \rangle| \leqslant \|\sum_{j=1}^n b_j x_j^*\|$$

for all n and all scalars $b_1, b_2, \ldots, b_n$. By Helly's theorem, there exists $\varphi \in V^*$ so that $\|\varphi\| \leqslant 1$ and $\langle \varphi, x_n^* \rangle = a_n$ for all n. This proves Claim 1.

Claim 2: $\mathrm{bc}_2((x_n)_{n=1}^\infty) \leqslant \mathrm{bc}_3((x_n)_{n=1}^\infty)$.

Let $\varphi \in B_{V^*}$ and let $\varepsilon > 0$. We choose $x^{**} \in X^{**}$ so that $x^{**}|_V = \varphi$ and $\|x^{**}\| \leqslant 1+\varepsilon$. Hence

$$\mathrm{ca}((\sum_{i=1}^n \langle \varphi, x_i^* \rangle x_i)_n) = (1+\varepsilon)\, \mathrm{ca}((\sum_{i=1}^n \langle \frac{x^{**}}{1+\varepsilon}, x_i^* \rangle x_i)_{n=1}^\infty) \leqslant (1+\varepsilon)\, \mathrm{bc}_3((x_n)_{n=1}^\infty).$$

Thus

$$\mathrm{bc}_2((x_n)_{n=1}^\infty) \leqslant (1+\varepsilon)\, \mathrm{bc}_3((x_n)_{n=1}^\infty).$$

Letting $\varepsilon \to 0$, we arrive at the sought conclusion.

Claim 3: $\mathrm{bc}_3((x_n)_{n=1}^\infty) \leqslant K\, \mathrm{bc}_1((x_n)_{n=1}^\infty)$. For this, observe that $P_n^{**} x^{**} = \sum_{i=1}^n \langle x^{**}, x_i^* \rangle x_i$ for each $x^{**} \in X^{**}$. Hence for every $x^{**} \in B_{X^{**}}$, $\|\sum_{i=1}^n \langle x^{**}, x_i^* \rangle x_i\| \leqslant K$. This implies that

$$\mathrm{ca}((\sum_{i=1}^n \langle x^{**}, x_i^* \rangle x_i)_{n=1}^\infty) \leqslant K\, \mathrm{bc}_1((x_n)_{n=1}^\infty),$$

which completes the proof. □

Theorem 4.5. *Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$. Then*

$$\mathrm{sh}((x_n^*)_{n=1}^\infty) \leqslant \mathrm{bc}_2((x_n)_{n=1}^\infty) \leqslant 2K^2\, \mathrm{sh}((x_n^*)_{n=1}^\infty).$$

Proof. Claim 1: for every $\varphi \in V^*$ we have $\limsup_n \|\varphi|_{[x_i^* \colon i > n]}\| \leqslant \mathrm{ca}((\sum_{i=1}^n \langle \varphi, x_i^* \rangle x_i)_{n=1}^\infty)$.

Let $0 < c < \limsup_n \|\varphi|_{[x_i^* \colon i>n]}\|$. Then there exists a block basic sequence $(f_n)_{n=1}^\infty$ with respect to $(x_n^*)_{n=1}^\infty$ so that $\|f_n\| \leqslant 1$ and $|\langle \varphi, f_n\rangle| > c$ for all n. Let us write $f_n = \sum_{i=k_{n-1}+1}^{k_n} \langle f_n, x_i\rangle x_i^*$. Then we get

$$\Big\| \sum_{i=k_{n-1}+1}^{k_n} \langle \varphi, x_i^*\rangle x_i \Big\| \geqslant \Big|\Big\langle f_n, \sum_{i=k_{n-1}+1}^{k_n} \langle \varphi, x_i^*\rangle x_i \Big\rangle\Big| = |\langle \varphi, f_n\rangle| > c \quad (n \in \mathbb{N}).$$

This implies that $\mathrm{ca}((\sum_{i=1}^n \langle \varphi, x_i^*\rangle x_i)_n) > c$ and so Claim 1 is established.

Claim 2: $\mathrm{ca}((\sum_{i=1}^n \langle \varphi, x_i^*\rangle x_i)_{n=1}^\infty) \leqslant 2K^2 \limsup_n \|\varphi|_{[x_i^* \colon i>n]}\|$ for every $\varphi \in V^*$.

Let $0 < c < \mathrm{ca}((\sum_{i=1}^n \langle \varphi, x_i^*\rangle x_i)_{n=1}^\infty)$. Then there exists a strictly increasing sequence of positive integers $(k_n)_n$ so that $\| \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle \varphi, x_i^*\rangle x_i\| > c$ for all n. It follows from (2.1) that

$$\| \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle \varphi, x_i^*\rangle j x_i\| > \frac{c}{K} \quad (n \in \mathbb{N}).$$

For each n we choose $f_n \in B_V$ so that

$$\Big|\Big\langle \varphi, \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle jx_i, f_n\rangle x_i^* \Big\rangle\Big| = \Big| \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle \varphi, x_i^*\rangle\langle jx_i, f_n\rangle\Big| > \frac{c}{K}.$$

Since $\| \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle jx_i, f_n\rangle x_i^*\| \leqslant 2K$ for each n, we get $\|\varphi|_{[x_i^* \colon i>k_{2n-1}]}\| \geqslant \frac{c}{2K^2}$ and so $\limsup_n \|\varphi|_{[x_i^* \colon i>n]}\| \geqslant \frac{c}{2K^2}$. As c was arbitrary, the proof is complete. □

Theorem 4.6. *Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$. Then*

$$\frac{1}{2K}\, \mathrm{bc}_2((x_n^*)_{n=1}^\infty) \leqslant \mathrm{sh}((x_n)_{n=1}^\infty) \leqslant K\, \mathrm{bc}_2((x_n^*)_{n=1}^\infty).$$

Proof. Let $0 < c < \mathrm{bc}_2((x_n^*)_{n=1}^\infty)$. Take $f \in B_{W^*}$ so that $\mathrm{ca}((\sum_{i=1}^n \langle f, jx_i\rangle x_i^*)_{n=1}^\infty) > c$. We choose a strictly increasing sequence of integers $(k_n)_{n=1}^\infty$ so that $\| \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle f, jx_i\rangle x_i^*\| > c$ $(n \in \mathbb{N})$. For each n, we take $y_n \in B_X$ with $| \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle f, jx_i\rangle\langle x_i^*, y_n\rangle| > c$. Define $x^* \in X^*$ by $\langle x^*, x\rangle = \langle f, jx\rangle (x \in X)$. Then $x^* \in B_{X^*}$ and

$$\Big|\Big\langle x^*, \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle x_i^*, y_n\rangle x_i \Big\rangle\Big| = \Big| \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle x^*, x_i\rangle\langle x_i^*, y_n\rangle\Big| > c \quad (n \in \mathbb{N}).$$

Note that

$$\| \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle x_i^*, y_n\rangle x_i\| \leqslant 2K \quad (n \in \mathbb{N}).$$

Consequently,

$$\|x^*|_{[x_i \colon i>k_{2n-1}]}\| \geqslant \frac{c}{2K} \quad (n \in \mathbb{N}).$$

This implies

$$\limsup_{n\to\infty} \|x^*|_{[x_i \colon i>n]}\| \geqslant \frac{c}{2K}$$

so $\mathrm{sh}((x_n)_{n=1}^\infty) \geqslant \frac{c}{2K}$. As c was arbitrary, we arrive at the former inequality.

As to the latter one, let $0 < c < \mathrm{sh}((x_n)_{n=1}^\infty)$. Then there exists $x^* \in B_{X^*}$ so that $\limsup_n \|x^*|_{[x_i \colon i>n]}\| > c$. We choose a block basic sequence $(u_n)_{n=1}^\infty$ with respect to

$(x_n)_{n=1}^\infty$ so that $\|u_n\| \leqslant 1$ and $|\langle x^*, u_n\rangle| > c$ for all n. Write each $u_n = \sum_{i=k_{n-1}+1}^{k_n} \langle x_i^*, u_n\rangle x_i$. We define $f \in W^*$ by $\langle f, jx\rangle = \langle x^*, x\rangle$ $(x \in X)$. By (2.1), we get $\|f\| \leqslant K$. Moreover, we get

$$c < \Big| \sum_{i=k_{n-1}+1}^{k_n} \langle x_i^*, u_n\rangle\langle x^*, x_i\rangle\Big| = \Big|\Big\langle \sum_{i=k_{n-1}+1}^{k_n} \langle f, jx_i\rangle x_i^*, u_n\Big\rangle\Big| \leqslant \Big\| \sum_{i=k_{n-1}+1}^{k_n} \langle f, jx_i\rangle x_i^*\Big\| \quad (n \in \mathbb{N}).$$

Thus $\mathrm{ca}((\sum_{i=1}^n \langle f, jx_i\rangle x_i^*)_{n=1}^\infty) \geqslant c$ and then $\mathrm{bc}_2((x_n^*)_{n=1}^\infty) \geqslant \frac{c}{K}$. By the arbitrariness of c, we complete the proof. □

Theorem 4.7. *Let X be a Banach space with an unconditional basis $(x_n)_{n=1}^\infty$. Then*

$$\frac{1}{K_u}\alpha_{c_0}(X) \leqslant \mathrm{bc}_1((x_n)_{n=1}^\infty) \leqslant K_u^3\alpha_{c_0}(X).$$

Proof. Let $0 < c < \alpha_{c_0}(X)$. Then there exists a sequence $(y_n)_{n=1}^\infty$ in X so that

$$c \cdot \max_{1\leqslant i\leqslant n} |t_i| \leqslant \|\sum_{i=1}^n t_i y_i\| \leqslant \max_{1\leqslant i\leqslant n} |t_i| \quad (n \in \mathbb{N},\ t_1, t_2, \ldots, t_n \in \mathbb{K}). \tag{4.1}$$

Let $\varepsilon > 0$. It follows from a quantitative version of the Bessaga–Pełczyński Selection Principle ([5, Lemma 2.3]) that there exist a subsequence $(y_{k_n})_{n=1}^\infty$ of $(y_n)_{n=1}^\infty$ and a block basic sequence $(u_n)_{n=1}^\infty$ with respect to $(x_n)_{n=1}^\infty$ so that

$$(1-\varepsilon)\|\sum_{i=1}^n t_i u_i\| \leqslant \|\sum_{i=1}^n t_i y_{k_i}\| \leqslant (1+\varepsilon)\|\sum_{i=1}^n t_i u_i\| \quad (n \in \mathbb{N},\ t_1, t_2, \ldots, t_n \in \mathbb{K}) \tag{4.2}$$

Combining (4.1) and (4.2), we get

$$\frac{c}{1+\varepsilon} \max_{1\leqslant i\leqslant n} |t_i| \leqslant \|\sum_{i=1}^n t_i u_i\| \leqslant \frac{1}{1-\varepsilon} \max_{1\leqslant i\leqslant n} |t_i| \quad (n \in \mathbb{N},\ t_1, t_2, \ldots, t_n \in \mathbb{K}). \tag{4.3}$$

Write $u_n = \sum_{i=k_{n-1}+1}^{k_n} a_i x_i$. By [15, Proposition 1.c.7] and (4.3), we get

$$\|\sum_{j=1}^n a_j x_j\| \leqslant K_u \|\sum_{j=1}^n u_j\| \leqslant \frac{K_u}{1-\varepsilon} \quad (n \in \mathbb{N}) \tag{4.4}$$

and

$$\|\sum_{i=k_{n-1}+1}^{k_n} a_i x_i\| = \|u_n\| \geqslant \frac{c}{1+\varepsilon} \quad (n \in \mathbb{N}) \tag{4.5}$$

Using (4.4) and (4.5), we arrive at

$$\mathrm{bc}_1((x_n)_{n=1}^\infty) \geqslant \frac{c}{1+\varepsilon}\frac{1-\varepsilon}{K_u}.$$

Letting $\varepsilon \to 0$, we get $\mathrm{bc}_1((x_n)_{n=1}^\infty) \geqslant \frac{c}{K_u}$. Since c is arbitrary, we arrive at the former inequality.

It remains to verify the latter one. Let $0 < c < \mathrm{bc}_1((x_n)_{n=1}^\infty)$. Then there exists a scalar sequence $(a_n)_{n=1}^\infty$ so that $\|\sum_{i=1}^n a_i x_i\| \leqslant 1$ for all n and $\mathrm{ca}((\sum_{i=1}^n a_i x_i)_{n=1}^\infty) > c$. We choose $k_1 < k_2 < \ldots < k_n < \ldots$ so that $\|\sum_{i=k_{2n-1}+1}^{k_{2n}} a_i x_i\| > c$ for all n. Let

$u_n = \sum_{i=k_{2n-1}+1}^{k_{2n}} a_i x_i$. Then $\|u_n\| > c$ for every n. Given a finite choice of scalars $(t_n)_{n=1}^m$. Appealing again to [15, Proposition 1.c.7], we get

$$\begin{aligned}\|\sum_{n=1}^m t_n u_n\| &= \|\sum_{n=1}^m \sum_{i=k_{2n-1}+1}^{k_{2n}} t_n a_i x_i\| \\ &\leqslant K_u \max_{1\leqslant n\leqslant m} |t_n| \|\sum_{n=1}^m u_n\| \\ &\leqslant K_u \max_{1\leqslant n\leqslant m} |t_n| K_u \|\sum_{i=1}^{k_{2m}} a_i x_i\| \\ &\leqslant K_u^2 \max_{1\leqslant n\leqslant m} |t_n|.\end{aligned}$$

On the other hand, for each $1 \leqslant n \leqslant m$, we get

$$c|t_n| \leqslant \|t_n u_n\| \leqslant K_u \|\sum_{n=1}^m t_n u_n\|.$$

Hence

$$c \cdot \max_{1\leqslant n\leqslant m} |t_n| \leqslant K_u \|\sum_{n=1}^m t_n u_n\|.$$

In conclusion,

$$\frac{c}{K_u} \max_{1\leqslant n\leqslant m} |t_n| \leqslant \|\sum_{n=1}^m t_n u_n\| \leqslant K_u^2 \max_{1\leqslant n\leqslant m} |t_n| \tag{4.6}$$

We define an operator $T\colon c_0 \to X$ by $e_n \mapsto \frac{1}{K_u^2} u_n$. By (4.6), $\|T\| \leqslant 1$ and $\|T^{-1}\| \leqslant \frac{K_u^3}{c}$. Hence $\alpha_{c_0}(X) \geqslant \frac{c}{K_u^3}$. The arbitrariness of c completes the proof. □

5. Quantifications of reflexivity

Theorem 5.1. *Let X be Banach space with a basis $(x_n)_{n=1}^\infty$. Then*

(1) $\mathrm{bc}_3((x_n)_{n=1}^\infty) \leqslant 2K^2 \operatorname{wk}_X(B_X)$.
(2) $\mathrm{sh}((x_n)_{n=1}^\infty) \leqslant 4K^3 \operatorname{wk}_X(B_X)$.

Proof. (1). Let $0 < c < \mathrm{bc}_3((x_n)_{n=1}^\infty)$. Then there exist $x^{**} \in B_{X^{**}}$ and a strictly increasing sequence $(k_n)_{n=1}^\infty$ so that

$$\left\| \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle x^{**}, x_i^* \rangle x_i \right\| > c \quad (n \in \mathbb{N}).$$

By (2.1), for each n we choose $f_n \in B_V$ so that

$$\left| \left\langle f_n, \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle x^{**}, x_i^* \rangle x_i \right\rangle \right| > \frac{c}{K} \quad (n \in \mathbb{N}).$$

We *claim* that $\mathrm{d}(x^{**}, X) \geqslant \frac{c}{2K^2}$.

Indeed, for every $x \in X$, we get

$$\begin{aligned}2K\|x^{**} - x\| &\geqslant |\langle x^{**}, \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle f_n, x_i\rangle x_i^*\rangle - \langle \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle f_n, x_i\rangle x_i^*, x\rangle| \\ &= |\langle f_n, \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle x^{**}, x_i^*\rangle x_i\rangle - \langle \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle f_n, x_i\rangle x_i^*, x\rangle| \\ &\geqslant \frac{c}{K} - |\langle f_n, \sum_{i=k_{2n-1}+1}^{k_{2n}} \langle x_i^*, x\rangle x_i\rangle| \\ &\geqslant \frac{c}{K} - \|\sum_{i=k_{2n-1}+1}^{k_{2n}} \langle x_i^*, x\rangle x_i\|.\end{aligned}$$

Letting $n \to \infty$, we get $2K\|x^{**} - x\| \geqslant \frac{c}{K}$. This proves the claim. Consequently, we get $\mathrm{wk}_X(B_X) \geqslant \frac{c}{2K^2}$. The arbitrariness of c completes the proof.

(2). Combining Theorem 4.6, Theorem 4.4 and (1), we get

$$\mathrm{sh}((x_n)_{n=1}^\infty) \leqslant 2K^3 \,\mathrm{wk}_V(B_V). \tag{5.1}$$

By (2.5),

$$\mathrm{wk}_V(B_V) \leqslant 2\,\mathrm{wk}_{X^*}(B_{X^*}) \tag{5.2}$$

By (5.1), (5.2) and (2.4), we get

$$\mathrm{sh}((x_n)_{n=1}^\infty) \leqslant 4K^3 \,\mathrm{wk}_{X^*}(B_{X^*}) = 4K^3 \,\mathrm{wk}_X(B_X).$$

This completes the proof. □

Theorem 5.2. *Let X be a Banach space with a basis $(x_n)_{n=1}^\infty$.*

(1) *If $(x_n)_{n=1}^\infty$ is boundedly complete, then*

$$\mathrm{wck}_X(B_X) \leqslant (K+1)\widehat{\mathrm{d}}(B_{X^*}, V).$$

(2) *If $(x_n)_{n=1}^\infty$ is shrinking, then*

$$\mathrm{wck}_X(B_X) \leqslant (K+1)^2 \,\mathrm{bc}_2((x_n)_{n=1}^\infty)$$

Proof. (1). Let $0 < c < \mathrm{wck}_X(B_X)$. Then there exists a sequence $(y_n)_n$ in B_X so that $\mathrm{d}(\mathrm{clust}_{X^{**}}((y_n)_n), X) > c$. Take $x_0^{**} \in \mathrm{clust}_{X^{**}}((y_n)_n)$. We may choose a strictly increasing sequence $(k_n)_{n=1}^\infty$ so that $|\langle x_0^{**} - y_{k_n}, x_i^*\rangle| < \frac{1}{n}$ $(i = 1, 2, \ldots, n)$ This implies that $\lim_{n\to\infty}\langle x_i^*, y_{k_n}\rangle = \langle x_0^{**}, x_i^*\rangle$ for each i. Note that, for each m, we get

$$\|\sum_{i=1}^m \langle x_0^{**}, x_i^*\rangle x_i\| = \lim_{n\to\infty} \|\sum_{i=1}^m \langle x_i^*, y_{k_n}\rangle x_i\| \leqslant K.$$

By the hypothesis, $\sum_{i=1}^\infty \langle x_0^{**}, x_i^*\rangle x_i = x_0$ for some $x_0 \in X$. Moreover, $\|x_0\| \leqslant K$. Hence $\|x_0^{**} - x_0\| > c$. Take $x_0^* \in B_{X^*}$ so that $|\langle x_0^{**} - x_0, x_0^*\rangle| > c$. By the definition of x_0, we get $\langle x_n^*, x_0\rangle = \langle x_0^{**}, x_n^*\rangle$ for all n and so $\langle x_0^{**} - x_0, x^*\rangle = 0$ for all $x^* \in V$. Thus, for all $x^* \in V$, we get

$$(K+1)\|x_0^* - x^*\| \geqslant |\langle x_0^{**} - x_0, x_0^* - x^*\rangle| > c.$$

This implies that

$$(K+1)\widehat{\mathrm{d}}(B_{X^*}, V) \geqslant (K+1)\mathrm{d}(x_0^*, V) \geqslant c.$$

As c was arbitrarty, the proof of (1) is complete.

(2). Suppose that $(x_n)_{n=1}^\infty$ is shrinking. It follows from [9, Theorem 3] that $(x_n^*)_{n=1}^\infty$ is a boundedly complete basis for X^*. By (1) and Theorem 3.3, we get

$$\mathrm{wck}_{X^*}(B_{X^*}) \leqslant (K+1)^2 \,\mathrm{sh}((x_n^*)_{n=1}^\infty).$$

By Theorem 4.5,

$$\mathrm{wck}_{X^*}(B_{X^*}) \leqslant (K+1)^2 \,\mathrm{bc}_2((x_n)_{n=1}^\infty).$$

By (2.4), we arrive at the conclusion. □

Theorem 5.3. *Let X be a Banach space with an unconditional basis $(x_n)_{n=1}^\infty$.*

(1) *If X contains no isomorphic copies of ℓ_1, then*

$$\frac{1}{K_u^3 K(K+1)^2}\,\mathrm{wck}_X(B_X) \leqslant \alpha_{c_0}(X) \leqslant \alpha_{\ell_1}(X^*) \leqslant \mathrm{wck}_X(B_X).$$

(2) *If X contains no isomorphic copies of c_0, then*

$$\frac{1}{K_u(K+1)^2}\,\mathrm{wck}_X(B_X) \leqslant \alpha_{\ell_1}(X) \leqslant \mathrm{wck}_X(B_X).$$

Proof. (1). By Theorem 2.1, $(x_n)_{n=1}^\infty$ is shrinking. Combining Theorem 4.7, Theorem 4.4 and Theorem 5.2 (2), we get

$$\alpha_{c_0}(X) \geqslant \frac{1}{K_u^3}\,\mathrm{bc}_1((x_n)_{n=1}^\infty) \geqslant \frac{1}{K_u^3 K}\,\mathrm{bc}_2((x_n)_{n=1}^\infty) \geqslant \frac{1}{K_u^3 K(K+1)^2}\,\mathrm{wck}_X(B_X).$$

The second and third inequalities of (1) follow from Lemma 2.6 and (2.4).

(2). The right inequality of (2) follows from Lemma 2.6. By Theorem 2.2, $(x_n)_{n=1}^\infty$ is boundedly complete. By Theorem 5.2, Theorem 3.3 and Theorem 3.5, we get

$$\mathrm{wck}_X(B_X) \leqslant (K+1)\widehat{\mathrm{d}}(B_{X^*}, V) \leqslant (K+1)^2\,\mathrm{sh}((x_n)_{n=1}^\infty) \leqslant (K+1)^2 K_u \alpha_{\ell_1}(X).$$

The proof is complete. □

Theorem 5.4. *Let X be a Banach space with an unconditional basis. Then*

$$\frac{1}{K_u^3 K(K+1)^2}\,\mathrm{wck}_X(B_X) \leqslant \mathrm{sep}(B_{X^{**}}) \leqslant \mathrm{wk}_X(B_X).$$

Proof. Let $(x_n)_{n=1}^\infty$ be an unconditional basis for X. Let $c > \mathrm{wk}_X(B_X)$ be arbitrary. Let $\mathcal{Q}$ be a countable dense subset of $\mathbb{R}$ and $C = \{\sum_{i=1}^n r_i x_i \colon n \in \mathbb{N}, r_1, r_2, \ldots, r_n \in \mathcal{Q}\}$. It is easy to see that $B_{X^{**}} \subseteq C + cB_{X^{**}}$. Hence $\mathrm{sep}(B_{X^{**}}) \leqslant c$. As c was arbitrary, the proof of the second inequality is complete.

For the first inequality, we divide the proof into two cases. If X contains an isomorphic copy of ℓ_1, it follows from James' distortion theorem that $\alpha_{\ell_1}(X) = 1$. By Theorem 3.4, we get $\mathrm{sep}(B_{X^*}) = 1$. By Lemma 2.5, $\mathrm{sep}(B_{X^{**}}) = 1$. The first inequality clearly holds. If X contains no isomorphic copy of ℓ_1, we get, by Theorem 5.3 and Theorem 3.4,

$$\frac{1}{K_u^3 K(K+1)^2}\,\mathrm{wck}_X(B_X) \leqslant \alpha_{\ell_1}(X^*) \leqslant \mathrm{sep}(B_{X^{**}}).$$

This completes the proof. □

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