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Erratum and addendum to 'Recovering a compact Hausdorff space X from the compatibility ordering on $C(X)$'

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ERRATUM AND ADDENDUM TO 'RECOVERING A COMPACT HAUSDORFF SPACE X FROM THE COMPATIBILITY ORDERING ON $C(X)$'

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ABSTRACT. It was kindly pointed out by L. G. Cordeiro as well as independently by T. Bice and W. Kubiś that the proof of Theorem 1.1 from the paper 'Recovering a compact Hausdorff space X from the compatibility ordering on $C(X)$', Fund. Math. 242 (2018), 187–205 is flawed. We demonstrate that not only is the proof of the said statement erroneous but that there is indeed a counterexample to it; Theorems 1.2–1.3 remain unaffected though. We salvage the result in the class of totally disconnected compact spaces and we propose an amendment by a suitable modification of the compatibility ordering that yields the conclusion of Theorem 1.1 for arbitrary compact spaces.

It is most unfortunate that the statement recorded as [2, Theorem 1.1] is erroneous. In the present note we

- salvage the result for totally disconnected compact spaces by appealing to [1, Theorem 1.17] (Theorem 1.5),
- provide a counterexample to [2, Theorem 1.1] by constructing a compatibility isomorphism between the spaces of continuous functions on the unit disc and a closed annulus in the plane (Theorem 1.6),
- discuss a minor modification of the compatibility ordering which yields the conclusion of [2, Theorem 1.1] in full generality (Theorem 1.8), and
- present a cleaner argument for [2, Proposition 4.1] (Proposition 1.3) to ensure that [2, Theorems 1.2–1.3] are valid; this proposition is also required for the first clause presented above.

We refer to [2] for all unexplained notation and terminology. Let X be a topological space. For $f \in C(X)$ we set $\sigma(f) = \operatorname{int}\operatorname{supp} f = \operatorname{int}\overline{\{x \in X\colon f(x) \neq 0\}}$. Given a fixed compatibility isomorphism $T\colon C(X) \to C(Y)$, we define the mapping

$$\tau\colon \{\sigma(f)\colon f \in C(X)\} \to \{\sigma(g)\colon g \in C(Y)\} \quad \text{by} \quad \tau(\sigma(f)) = \sigma(Tf). \tag{1.1}$$

In [2, Proposition 3.8] it was proved that when X and Y are completely regular spaces, τ is a well-defined inclusion-preserving bijection. (A generalisation of this result may be found in [1, Theorem 1.16].)

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Moreover, the following lemma was proved ([2, Lemma 3.1])

Lemma 1.1. *Let X and Y be topological spaces. Suppose that $T\colon C(X) \to C(Y)$ is a compatibility isomorphism and $f, g \in C(X)$. Consider the following conditions:*

(i) *f and g are orthogonal,*
(ii) *Tf and Tg are orthogonal,*
(iii) *$T(f+g) = Tf + Tg$.*

Then (i) *and* (ii) *are equivalent and imply* (iii).

Lemma 1.1 has a natural converse, which was not included in [2]. We record it here as it will be required in the proof of Theorem 1.6.

Lemma 1.2. *Let X and Y be topological spaces. Suppose that $T\colon C(X) \to C(Y)$ is a bijection such that*

(i) *for $f, g \in C(X)$, $fg = 0$ if and only if $(Tf)(Tg) = 0$,*
(ii) *for $f, g \in C(X)$, if $fg = 0$, then $T(f+g) = Tf + Tg$,*
(iii) *for $f, g \in C(Y)$, if $fg = 0$, then $T^{-1}(f+g) = T^{-1}f + T^{-1}g$.*

Then T is a compatibility isomorphism.

Proof. Let $f, g \in C(X)$. Without loss of generality $f \neq 0$. Suppose that $f \preccurlyeq g$, that is $fg = f^2$. We have $f(g-f) = 0$, which (by (i)) implies $(Tf)(T(g-f)) = 0$, and (by (ii)) $Tf + T(g-f) = Tg$, so $(Tf)^2 = TfTf + (Tf)(T(g-f)) = (Tf)(Tg)$. Consequently $Tf \preccurlyeq Tg$. The proof for T^{-1} is completely analogous. □

1.1. **Clarification of the proof of Proposition 4.1.** [2, Proposition 4.1] is correct, yet its proof presented in [2] may leave a doubt due to the sentence 'By applying the closure and interior operations…'.) Below we present a complete proof of this key lemma.

Proposition 1.3 ([2, Proposition 4.1])**.** *Let X and Y be completely regular spaces such that there exists a compatibility isomorphism $T\colon C(X) \to C(Y)$. If $U \subseteq X$ is clopen, then $\tau(X\backslash U) = Y\backslash\tau(U)$. In particular, $\tau(U)$ is clopen.*

Proof. Let us first observe that for $h = T^{-1}(\mathbb{1}_Y) \in C(X)$ it is true that $\sigma(h) = X$. Indeed, assume $\sigma(h) \neq X$. This, of course, means, that $\operatorname{supp}(h) \neq X$, and (by complete regularity of X) there is a non-zero function $k \in C(X)$ orthogonal to h. By Lemma 1.1, $0 = T0 \neq Tk$ is orthogonal to $Th = \mathbb{1}_Y$, which is impossible.

Having established $\sigma(h) = X$, we now define $f = h \cdot \mathbb{1}_U$ and $g = h \cdot \mathbb{1}_{X\backslash U}$. Since h is continuous and U is clopen, the functions f, g are continuous, orthogonal and $h = f + g$. Thus, by Lemma 1.1, $\mathbb{1}_Y = Th = Tf + Tg$, with Tf, Tg continuous and orthogonal. Setting

$$A = \{y \in Y : Tf(y) \neq 0\} \quad \text{and} \quad B = \{y \in Y : Tg(y) \neq 0\},$$

it follows that $A \cup B = Y$ and the union is disjoint. Since, by continuity, A and B are open in Y, they are clopen. Thus

$$\sigma(Tf) = \operatorname{int}\overline{A} = A \quad \text{and} \quad \sigma(Tg) = \operatorname{int}\overline{B} = B.$$

But $\sigma(f) = U$ and $\sigma(g) = X\backslash U$ as U is clopen and $\sigma(h) = X$; we therefore have

$$\tau(U) = \tau(\sigma(f)) = \sigma(Tf) = A \quad \text{and} \quad \tau(X\backslash U) = \tau(\sigma(g)) = \sigma(Tg) = B,$$

whence $\tau(U)$ is clopen and $Y\backslash\tau(U) = Y\backslash A = B = \tau(X\backslash U)$. □

In order to state and prove [2, Theorem 1.1] restricted to the class of totally disconnected compact spaces, we require a piece of terminology. Let X be a compact space. Cordeiro calls a family $\mathcal{A} \subset C(X)$ *weakly regular* ([1, Definition 1.5(ii)]) when $\{\sigma(f) \colon f \in \mathcal{A}\}$ is a base for the topology on X.

Theorem 1.4 ([1, Theorem 1.17])**.** *Let X and Y be compact Hausdorff spaces. Suppose that $\mathcal{A}(X) \subset C(X), \mathcal{A}(Y) \subseteq C(Y)$ are weakly regular families. If $T \colon \mathcal{A}(X) \to \mathcal{A}(B)$ is a bijection such that*

$$\operatorname{supp} f \cap \operatorname{supp} g = \varnothing \iff \operatorname{supp} Tf \cap \operatorname{supp} Tg = \varnothing \quad (f, g \in \mathcal{A}(X)),$$

then there exists a unique homeomorphism $\psi \colon Y \to X$ such that $\psi(\sigma(Tf)) = \sigma(f)$ for every $f \in \mathcal{A}(X)$.

We are now ready to state and prove [2, Theorem 1.1] in the totally disconnected setting.

Theorem 1.5. *Let X and Y be totally disconnected compact Hausdorff spaces. If there exists a compatibility isomorphism $T \colon C(X) \to C(Y)$, then X and Y are homeomorphic.*

Proof. Since X and Y are compact and totally disconnected, the clopen subsets thereof form open bases for their topologies. Consequently, the families

$$\mathcal{A}(X) = \{f \in C(X) \colon \sigma(f) \text{ is clopen}\}, \quad \mathcal{A}(Y) = \{f \in C(Y) \colon \sigma(f) \text{ is clopen}\}$$

are weakly regular. We *claim* that $T(\mathcal{A}(X)) = \mathcal{A}(Y)$. For this, let us take arbitrary f in $\mathcal{A}(X)$, *i.e.*, $f \in C(X)$ such that $\sigma(f)$ is clopen. Then, by Proposition 1.3, $\sigma(Tf) = \tau(\sigma(f))$ is clopen, and so $Tf \in \mathcal{A}(Y)$. To prove the converse inclusion, note that [2, Proposition 3.8] states that τ given by (1.1) is a (well-defined) bijection; thus for any $f \in C(X)$ we have $\tau^{-1}(\sigma(Tf)) = \sigma(f)$, in particular, $\tau^{-1}(\sigma(g)) = \sigma(T^{-1}(g))$ for any $g \in \mathcal{A}(Y)$. Since T^{-1} is, by definition, also a compatibility isomorphism, another application of Proposition 1.3 yields that T^{-1} maps $\mathcal{A}(Y)$ into $\mathcal{A}(X)$, and we conclude that $T|_{\mathcal{A}(X)} \colon \mathcal{A}(X) \to \mathcal{A}(Y)$ is a bijection. We have $\sigma(f) = \operatorname{supp} f$ for $f \in \mathcal{A}(X) \cup \mathcal{A}(Y)$, since $\sigma(f)$ is clopen. Consequently, we notice that $T|_{\mathcal{A}(X)}$ meets the hypothesis of Theorem 1.4, hence X and Y are homeomorphic. □

1.2. **A counterexample to Theorem 1.1 in the connected case.** Even though in the totally disconnected case compatibility isomorphisms do recover the underlying spaces, this need not be so for compact, connected metric spaces.

Theorem 1.6. *There exists a compatibility isomorphism between the spaces of continuous functions on the closed unit disc and the annulus $\{z \in \mathbb{C} \colon 1/2 \leqslant |z| \leqslant 1\}$.*

We shall divide the proof of Theorem 1.6 into a sequence of independent, more digestible claims. As in [2], we denote by $\mathrm{RO}(X)$ the lattice of regularly open subsets of a topological space X with the operations $U \vee_{\mathrm{ro}} V = \operatorname{int} \overline{U \cup V}$ and $U \wedge_{\mathrm{ro}} V = U \cap V$.

Proof of Theorem 1.6. Let

$$A = \{z \in \mathbb{C} \colon 1/2 < |z| \leqslant 1\} \text{ and } P = \{z \in \mathbb{C} \colon 0 < |z| \leqslant 1\}.$$

The map $h \colon A \to P$ defined by

$$h(z) = (2|z| - 1)\frac{z}{|z|} \quad (z \in A)$$

is a homeomorphism from A onto P. Let

$$X = \{z \in \mathbb{C} \colon 1/2 \leqslant |z| \leqslant 1\} \text{ and } Y = \{z \in \mathbb{C} \colon |z| \leqslant 1\}.$$

Define $\varphi \colon \mathrm{RO}(X) \to \mathrm{RO}(Y)$ and $\psi \colon \mathrm{RO}(Y) \to \mathrm{RO}(X)$ by

$$\varphi(U) = \operatorname{int} \overline{h(U \cap A)} \text{ and } \psi(V) = \operatorname{int} \overline{h^{-1}(V \cap P)},$$

where the closures and interiors are taken in X and Y respectively.

Claim 1. φ is an order-preserving lattice isomorphism with inverse ψ.

Proof of Claim 1. It is clear that φ is order-preserving. By [2, Lemma 3.12], it is sufficient to show that φ is bijective.

Let $U \in \mathrm{RO}(X)$. We then have $h(U \cap A) \subseteq \varphi(U)$ since $h(U \cap A)$ is open in Y. Thus

$$\psi(\varphi(U)) = \operatorname{int} \overline{h^{-1}[\varphi(U) \cap P]} \supseteq \operatorname{int} \overline{h^{-1}[h(U \cap A)]} = \operatorname{int} \overline{U \cap A} = \operatorname{int} \overline{U} = U.$$

Conversely, let cl_A and cl_P denote the closure operations in A and P, respectively. Then

$$h^{-1}[\overline{h(U \cap A)} \cap P] = h^{-1}[\mathrm{cl}_P(h(U \cap A))] = \mathrm{cl}_A(U \cap A) \subseteq \overline{U}.$$

Hence

$$h^{-1}(\varphi(U) \cap P) \subseteq h^{-1}[\overline{h(U \cap A)} \cap P] \subseteq \overline{U}.$$

It follows that

$$\psi(\varphi(U)) = \operatorname{int} \overline{h^{-1}(\varphi(U) \cap P)} \subseteq \operatorname{int} \overline{U} = U.$$

This proves that $\psi(\varphi(U)) = U$. Similarly, $\varphi(\psi(V)) = V$ for every $V \in \mathrm{RO}(Y)$. Consequently, $\varphi \colon \mathrm{RO}(X) \to \mathrm{RO}(Y)$ is an order-preserving bijection. Hence φ is a lattice isomorphism. □

Claim 2. $U \in \mathrm{RO}(X)$ is non-empty and connected if and only if $\varphi(U) \in \mathrm{RO}(Y)$ is non-empty and connected.

Proof of Claim 2. Let $U \in \mathrm{RO}(X)$ be non-empty and connected. Obviously $\varphi(U)$ is non-empty. Since U is open in X, it is in fact path-connected. If $x_1, x_2 \in U \cap A$, there is a continuous path in U that runs from x_1 to x_2. A slight perturbation yields a path in $U \cap A$ that also runs from x_1 to x_2. Thus $U \cap A$ is path-connected. As a result $h(U \cap A)$ is path-connected. Thus it is open and connected in P and hence in Y too. Since

$$h(U \cap A) \subseteq \varphi(U) \subseteq \overline{h(U \cap A)},$$

$\varphi(U)$ is connected in Y. A similar argument shows that if $V = \varphi(U) \in \mathrm{RO}(Y)$ is non-empty and connected, then $U = \psi(V)$ is non-empty and connected. □

A function $f \in C(X)$ is said to be *decomposable* if it can be written as a sum of two orthogonal non-zero functions in $C(X)$. A function that is not decomposable is *indecomposable*.

For $f \in C(X)$, let $C(f) = \{x \in X : f(x) \neq 0\}$. Since X is locally connected, the connected components of $C(f)$ are open in X. We can then infer from the separability of $C(f)$ that it has at most countably many connected components. Denote the connected components of $C(f)$ by $(U_n)_{n\in J}$, where J is either a finite set or $\mathbb{N}$.

Claim 3. If $C(f)$ is connected, then f is indecomposable.

Proof of Claim 3. If we had $f = f_1 + f_2$ with non-zero, orthogonal, continuous functions f_1, f_2, then $C(f)$ would be the union of two non-empty, disjoint, open sets $C(f_1)$ and $C(f_2)$, which is impossible due to its connectedness. □

Claim 4. Let $f \in C(X)$ be a non-zero function and let $(U_n)_{n\in J}$ be the enumeration of the connected components of $C(f)$. Set $f_n = f \cdot \mathbb{1}_{U_n}$ for each $n \in J$. Then $f_n \in C(X)$. Each f_n is indecomposable. Furthermore, if $J = \mathbb{N}$, then $\|f_n\| \to 0$ as $n \to \infty$.

Proof of Claim 4. Let U be a connected component of $C(f)$. We first show that we have $f \cdot \mathbb{1}_U \in C(X)$ and that $f \cdot \mathbb{1}_U$ is indecomposable.

Denote $g := f \cdot \mathbb{1}_U$. Then clearly $f|_{\overline{U}} = g|_{\overline{U}}$, so g is continuous in $\overline{U}$ in the relative topology of $\overline{U}$. On the other hand, $g|_{X\setminus U} = 0$ in, so it is continuous in the relative topology of $X\setminus U$. In particular, g is continuous at all points of $\partial U = \overline{U} \cap (X\setminus U)$; at all other points g is continuous trivially.

It follows from Claim 3 that g is indecomposable.

It remains to show the final assertion. If $J = \mathbb{N}$ and $\|f_n\| \not\to 0$, we may choose an infinite subset J' of J and $\varepsilon > 0$ so that $\|f_n\| \geqslant \varepsilon$ for all $n \in J'$. For each $n \in J'$, there exists $x_n \in U_n$ so that $\varepsilon \leqslant \|f_n\| = |f_n(x_n)| = |f(x_n)|$. Replace J' by a further infinite subset if necessary to assume that $(x_n)_{n\in J'}$ converges to some $x_0 \in X$. If $x_0 \in C(f)$, then $x_0 \in U_{n_0}$ for some $n_0 \in J$. Since U_{n_0} is an open set, $x_n \in U_{n_0} \cap U_n$ for all sufficiently large $n \in J'$. But this means that $n = n_0$ for all sufficiently large $n \in J'$, which is absurd. Thus $x_0 \notin C(f)$. We now have

$$\lim_{n\in J'} \|f_n\| = \lim_{n\in J'} |f(x_n)| = |f(x_0)| = 0,$$

contrary to the choice of J'. This completes the proof. □

Let $f \in C(X)$ be a non-zero function. We say that $\sum_{n\in J} f_n$ is an *irreducible decomposition* of f if J is either finite or $\mathbb{N}$, $(f_n)_{n\in J}$ is a sequence of pairwise orthogonal functions in $C(X)$, each f_n is indecomposable and non-zero, and $f = \sum_{n\in J} f_n$, where the sum converges in $C(X)$ unformly if $J = \mathbb{N}$.

Claim 5. Every non-zero function $f \in C(X)$ (and in $C(Y)$) has an irreducible decomoposition. The decomposition is unique in the sense that if $f = \sum_{n\in J} f_n = \sum_{n\in J'} g_n$ are two irreducible decompositions, then there is a bijection $\pi\colon J' \to J$ so that $g_n = f_{\pi(n)}$ for all $n \in J'$.

Proof of Claim 5. Let $(U_n)_{n\in J}$ be an enumeration of the connected components of $C(f)$. It follows from Claim 4 that if we set $f_n = f\cdot \mathbb{1}_{U_n}$, $n\in J$, then each $f_n\in C(X)$ is indecomposable and non-zero. Furthermore, if $J=\mathbb{N}$, the sum $f=\sum_{n\in J} f_n$ converges in $C(X)$ since $(f_n)_{n=1}^\infty$ is a sequence pairwise orthogonal functions and $\|f_n\|\to 0$ as $n\to\infty$.

Suppose that f admits another irreducible decomposition $f=\sum_{n\in J'} g_n$. Let $n\in J'$. If $C(g_n)$ were not connected, we could find two non-empty disjoint open sets V_1, V_2 so that $C(g_n)=V_1\cup V_2$. As in the proof of Claim 4, one can verify that $g_n\mathbb{1}_{V_i}\in C(X)$, $i=1,2$. Then $g_n=g_n\mathbb{1}_{V_1}+g_n\mathbb{1}_{V_2}$ shows that g_n is decomposable, contrary to its choice. Hence, each $C(g_n)$ is connected. Therefore, $C(f)=\bigcup_{n\in J'} C(g_n)$ expresses $C(f)$ as a union of disjoint open connected non-empty sets. Thus, each set $C(g_n)$ $(n\in J')$ must be a connected component of $C(f)$. So there is a bijection $\pi\colon J'\to J$ so that $C(g_n)=U_{\pi(n)}=C(f_{\pi(n)})$. Finally,

$$g_n = f\cdot \mathbb{1}_{C(g_n)} = f\cdot \mathbb{1}_{U_\pi(n)} = f_{\pi(n)}. \qquad \square$$

Similarly, every non-zero function $g\in C(Y)$ has a unique (up to permutation) irreducible decomposition.

Denote the set of all indecomposable functions on X and Y, by $I(X)$ and $I(Y)$, respectively. For a connected non-empty set $U\in \mathrm{RO}(X)$ and $r\geqslant 0$, let $I(U,r)$ be the set of functions $f\in I(X)$ such that $\sigma(f)=U$ and $\|f\|=r$. Similarly define $I(V,r)$ for connected non-empty $V\in\mathrm{RO}(Y)$ and $r\geqslant 0$.

Claim 6. Let $U\in\mathrm{RO}(X)$ or $U\in\mathrm{RO}(Y)$ be non-empty and connected. If $r>0$, then the set $I(U,r)$ has cardinality $\mathfrak{c}$, of the continuum.

Proof of Claim 6. One can readily find $a\in U$, $\delta>0$, and $K>0$ such that the function $\alpha(x)=\min\{K\cdot d(x,U^c), r/2\}$ satisfies for each $x\in B(a,\delta)\subseteq U$, $\alpha(x)=r/2$. Clearly $[0,r/2]$ is the range of α and $C(\alpha)=U$ (as U is open). Next, for each $z\in B(a,\delta)$, we find a function $\beta_z\in C(X)$ with range $[0,r/2]$ such that $\operatorname{supp}\beta_z\subseteq B(a,\delta)$ and $(\beta_z)^{-1}(r/2)=\{z\}$; it is obvious from these conditions that $\beta_z\neq\beta_y$ whenever $z\neq y$, $z,y\in B(a,\delta)$. Setting, for each $z\in B(a,\delta)$, $\gamma_z=\alpha+\beta_z$, we see that $|\{\gamma_z\colon z\in B(a,\delta)\}|=|B(a,\delta)|=\mathfrak{c}$. But $\gamma_z\in I(U,r)$ for each $z\in B(a,\delta)$. Indeed, $\gamma_z(z)=\alpha(z)+\beta_z(z)=r/2+r/2=r$, so $[0,r]$ is the range of γ_z. Moreover, we have $C(\gamma_z)=U$ (by $C(\alpha)=U$ and the non-negativity of both α and β_z), so according to Claim 3 connectedness of U implies that γ_z is indecomposable.

Similarly, the set $I(V,r)$ has cardinality $\mathfrak{c}$ for every non-empty connected set $V\in\mathrm{RO}(Y)$ and $r>0$. $\square$

Construction of a norm-preserving bijection between the indecomposables. By Claim 2, if $U\in\mathrm{RO}(X)$ is non-empty and connected, so is $\varphi(U)\in\mathrm{RO}(Y)$. Thus, for each non-empty connected $U\in\mathrm{RO}(X)$, there is a bijection

$$S_U\colon \bigcup_{r>0} I(U,r)\to\bigcup_{r>0} I(\varphi(U),r)$$

such that S_U maps each $I(U,r)$ onto $I(\varphi(U),r)$ $(r>0)$. Indeed, by Claim 6, for each $r>0$ we may fix a bijection $S_U^r\colon I(U,r)\to I(\varphi(U),r)$ and define $S_U f = S_U^r f$ when $f\in I(U,r)$. S_U is well-defined as the sets $I(U,r)$ $(r>0)$ are pairwise disjoint.

Remark. Let $f\in C(X)$ be a non-zero function; we write $f=\sum_{n\in J} f_n$ in terms of its irreducible decomposition. By the construction of the irreducible decomposition in Claim 4 and the uniqueness proved in Claim 5, all sets $C(f_n)$ $(n\in J)$ must be connected components of $C(f)$. Since $C(f_n)\subseteq\sigma(f_n)\subseteq\overline{C(f_n)}$, $\sigma(f_n)$ is connected (and non-empty).

Construction of the sought compatibility isomorphism. We define $T\colon C(X)\to C(Y)$ in the following manner: $T0=0$ and

$$Tf=\sum_{n\in J} S_{\sigma(f_n)} f_n \quad (f\in C(X), f\neq 0).$$

Claim 7. $T\colon C(X)\to C(Y)$ is a well defined mapping. Moreover, if for $f,g\in C(X)$ we have $fg=0$, then $Tf\cdot Tg=0$ and $T(f+g)=Tf+Tg$.

Proof of Claim 7. For the well definedness, it is required to show that for $f\in C(X)$ we have $Tf\in C(Y)$. If J is finite, then assertion is trivial, so we may suppose that $J=\mathbb{N}$. Since $(f_n)_{n=1}^\infty$ is a sequence of pairwise orthogonal functions and $\sum f_n$ converges in $C(X)$, $\lim\|f_n\|=0$. Thus $f_n\in I(U_n,r_n)$, where $U_n=\sigma(f_n)$ and $r_n=\|f_n\|>0$.

By the very definition of S_{U_n}, $S_{U_n}f_n\in I(\varphi(U_n),r_n)$. Since $(U_n)_{n=1}^\infty$ is a sequence of pairwise disjoint sets in $\mathrm{RO}(X)$ and $\varphi\colon\mathrm{RO}(X)\to\mathrm{RO}(Y)$ is an order-preserving isomorphism, by Claim 1, $(\varphi(U_n))_{n=1}^\infty$ is a sequence of pairwise disjoint sets. Thus $(S_{U_n}f_n)_{n=1}^\infty$ is a sequence of pairwise orthogonal functions in $C(Y)$ with $\|S_{U_n}f_n\|=r_n$ for all n. Since $r_n\to 0$, it is now clear that $\sum S_{U_n}f_n$ converges in $C(Y)$.

Let $f,g\in C(X)$ be such that $fg=0$. If one of them is the zero function, then obviously $Tf\cdot Tg=0$ and $T(f+g)=Tf+Tg$. Suppose that both f,g are non-zero. Let $f=\sum_{n\in J_f} f_n$ and $g=\sum_{n\in J_g} g_n$ be their respective irreducible decompositions. For each $n\in J_f$,

$$\sigma(S_{\sigma(f_n)}f_n)=\varphi(\sigma(f_n))\subseteq\varphi(\sigma(f)).$$

Thus

$$\sigma(Tf)=\sigma\Big(\sum_{n\in J_f} S_{\sigma(f_n)}f_n\Big)=\operatorname{int}\overline{\bigcup_{n\in J_f} C(S_{\sigma(f_n)}f_n)}\subseteq\operatorname{int}\overline{\bigcup_{n\in J_f}\sigma(S_{\sigma(f_n)}f_n)}\subseteq\varphi(\sigma(f)),$$

where the last inclusion holds as $\bigcup_{n\in J_f}\sigma(S_{\sigma(f_n)}f_n)\subseteq\varphi(\sigma(f))$ and $\varphi(\sigma(f))\in\mathrm{RO}(Y)$. Similarly, $\sigma(Tg)\subseteq\varphi(\sigma(g))$. Since $fg=0$, $\sigma(f)\cap\sigma(g)=\varnothing$, and so $\varphi(\sigma(f))\cap\varphi(\sigma(g))=\varnothing$. Therefore, $Tf\cdot Tg=0$.

Finally, it is clear that

$$f+g=\sum_{n\in J_f} f_n+\sum_{n\in J_g} g_n$$

is the irreducible decomposition of $f+g$. By definition

$$T(f+g)=\sum_{n\in J_f} S_{\varphi(f_n)}f_n+\sum_{n\in J_g} S_{\varphi(g_n)}g_n=Tf+Tg.$$

This completes the proof of the statement. □

If $g \in C(Y)$ is a non-zero function, let $g = \sum_{n\in J} g_n$ be its irreducible decomposition. Then $C(g_n)$, $n \in J$, are the connected components of $C(g)$. Thus $\sigma(g_n)$ is connected (and non-empty). By Claim 2, $\varphi^{-1}(\sigma(g_n))$ is connected and non-empty.

Construction of the inverse to T. We define a map $\widetilde{T}\colon C(Y) \to C(X)$ as follows: $\widetilde{T}0 = 0$ and

$$\widetilde{T}g = \sum_{n\in J} S^{-1}_{\varphi^{-1}(\sigma(g_n))} g_n \quad (g \in C(Y), g \neq 0).$$

Claim 8. If $f, g \in C(Y)$ and $fg = 0$, then $\widetilde{T}f \cdot \widetilde{T}g = 0$ and $\widetilde{T}(f+g) = \widetilde{T}f + \widetilde{T}g$.

Proof of Claim 8. The proof is completely analogous to the proof of Claim 7. □

Claim 9. T and $\widetilde{T}$ are mutual inverses.

Proof of Claim 9. We have $T\widetilde{T}0 = 0$ and $\widetilde{T}T0 = 0$. Let $f \in C(X)$ be a non-zero function written in terms of its irreducible decomposition: $f = \sum_{n\in J} f_n$. Then $Tf = \sum_{n\in J} S_{\sigma(f_n)} f_n$. Set $g_n = S_{\sigma(f_n)} f_n$. By the very definition, $\sigma(g_n) = \varphi(\sigma(f_n))$. By Claim 2, the set $\sigma(g_n)$ is connected. Since the sequence $(\sigma(f_n))_{n\in J}$ comprises pairwise disjoint sets, by Claim 1, so does the sequence $(\varphi(\sigma(f_n)))_{n\in J} = (\sigma(g_n))_{n\in J}$

Thus the sequence $(S_{\sigma(f_n)} f_n)_{n\in J}$ comprises pairwise orthogonal functions, each of which is indecomposable by definition. Therefore, $\sum_{n\in J} S_{\sigma(f_n)} f_n$ is the irreducible decomposition of Tf. Let $g_n = S_{\sigma(f_n)} f_n$ $(n \in J)$. Since $S^{-1}_{\varphi^{-1}(\sigma(g_n))} = S^{-1}_{\sigma(f_n)}$,

$$\widetilde{T}Tf = \widetilde{T}\Big(\sum_{n\in J} g_n\Big) = \sum_{n\in J} S^{-1}_{\varphi^{-1}(\sigma(g_n))} g_n = \sum_{n\in J} S^{-1}_{\sigma(f_n)} g_n = \sum_{n\in J} f_n = f.$$

The proof for $T\widetilde{T}g = g$ for all $g \in C(Y)$ is similar. □

It follows from Claims 7–9 that T is bijective, both T and T^{-1} preserve orthogonality and both are orthogonality additive. By Lemma 1.2, T is a compatibility isomorphism. □

1.3. **A modification of the compatibility ordering.** Let X be a compact Hausdorff space and let f, g be scalar-valued continuous functions on X. We define the order relation $f \sqsubseteq g$ whenever there exists an open set $U \subseteq X$ such that

- $\operatorname{supp} f \subseteq U$,
- $g(x) = f(x)$ for all $x \in U$.

Let us call a (possibly non-linear) bijection $T\colon C(X) \to C(Y)$ a $\sqsubseteq$*-isomorphism* whenever

$$f \sqsubseteq g \iff Tf \sqsubseteq Tg \quad (f, g \in C(X)).$$

Lemma 1.7. *Let X and Y be compact Hausdorff spaces and let $T\colon C(X) \to C(Y)$ be a $\sqsubseteq$-isomorphism. If the functions $f_1, f_2 \in C(X)\backslash\{0\}$ have disjoint supports, so have Tf_1, Tf_2.*

Proof. Let us set $f = f_1 + f_2$, $g_1 = Tf_1$, $g_2 = Tf_2$, and $g = Tf$. By the assumption $\operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \varnothing$, and using normality of X, we easily see that $f_1 \sqsubseteq f$ and $f_2 \sqsubseteq f$; it follows that $g_1 \sqsubseteq g$ and $g_2 \sqsubseteq g$. Hence there exist open sets $U_1, U_2 \subseteq Y$ such that $\operatorname{supp} g_i \subseteq U_i$ and $g_i = g$ in U_i, $i = 1, 2$.

Assume, for a contradiction, that $\operatorname{supp} g_1 \cap \operatorname{supp} g_2 \neq \varnothing$. Then $U_1 \supseteq \operatorname{supp} g_1 \cap \operatorname{supp} g_2$, so $U_1 \cap \operatorname{supp} g_2 \neq \varnothing$. Since U_1 is open, $U_1 \cap C(g_2) \neq \varnothing$ (where $C(g_2) = \{y \in Y : g_2(y) \neq 0\}$). But $g = g_1$ in U_1 and $g = g_2$ in $U_2 \supseteq C(g_2)$. Therefore $g_1 = g = g_2$ in $U_1 \cap U_2 \supseteq U_1 \cap C(g_2)$, and since g_2 is non-zero in this non-empty set, all the functions are.

Thus we have obtained that $C := C(g_1) \cap C(g_2) \neq \varnothing$ and $g_1 = g_2 = g$ in $V := U_1 \cap U_2 \supseteq C$. Note also that $\varnothing \neq C \subseteq \operatorname{supp} g_1 \cap \operatorname{supp} g_2 \subseteq U_1 \cap U_2 = V$. Setting $h := g \cdot \mathbb{1}_V$, it is clear that h also equals $g_i \cdot \mathbb{1}_V$, $i = 1, 2$. More importantly, $h \in C(Y)$. Indeed, we have $g_1 = g_2 = g$ in V, so (also by openness of V) $\operatorname{supp} g \cap V = \operatorname{supp} g_1 \cap \operatorname{supp} g_2 \cap V$, but this equals $\operatorname{supp} g_1 \cap \operatorname{supp} g_2$ as the last set is contained in V. Thus $\operatorname{supp} g \cap V$ is compact, and it easily follows that h is indeed continuous (and non-zero as $C \neq \varnothing$). Observe that constant zero functions on X and Y are the least elements in $(C(X), \sqsubseteq)$ and $(C(Y), \sqsubseteq)$ respectively, so it is clear that $T(0) = 0$; in particular, $T^{-1}(h) \neq 0$.

We have $\operatorname{supp} h \subseteq V$, V is open and, for $i = 1, 2$, $h = g_i$ in V, *i.e.* $0 \neq h \sqsubseteq g_i$, whence $0 \neq T^{-1}(h) \sqsubseteq T^{-1}(g_i) = f_i$. This is a contradiction with our assumption $\operatorname{supp} f_1 \cap \operatorname{supp} f_2 = \varnothing$, and the proof is complete. □

Theorem 1.8. *Let X and Y be compact Hausdorff spaces. Suppose that there exists a $\sqsubseteq$-isomorphism $T\colon C(X) \to C(Y)$. Then X and Y are homeomorphic.*

Proof. Let $T\colon C(X) \to C(Y)$ be a $\sqsubseteq$-isomorphism. By Lemma 1.7, it has the property

$$\operatorname{supp} f \cap \operatorname{supp} g = \varnothing \iff \operatorname{supp} Tf \cap \operatorname{supp} Tg = \varnothing \quad (f, g \in C(X)).$$

That X and Y are homeomorphic now follows from Theorem 1.4. □

Remark 1.9. Theorem 1.6 demonstrates that even though compatibility isomorphisms have the property $f \sqsubseteq g \Rightarrow Tf \preccurlyeq Tg$ for f, g in the domain of a compatibility isomorphism T, they need not preserve the relation $\sqsubseteq$.

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