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On interplay between operators, bases, and matrices

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ON INTERPLAY BETWEEN OPERATORS, BASES, AND MATRICES

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Abstract. Given a bounded linear operator T on separable Hilbert space, we develop an approach allowing one to construct a matrix representation for T having certain specified algebraic or asymptotic structure. We obtain matrix representations for T with preassigned bands of the main diagonals, with an upper bound for all of the matrix elements, and with entrywise polynomial lower and upper bounds for these elements. In particular, we substantially generalize and complement our results on diagonals of operators from [40]. Moreover, we extend a theorem by Stout (1981), and (partially) answer his open question. Several of our results have no analogues in the literature.

1. Introduction: a glimpse at matrix representations

Following conventional approach to describing operators on finite-dimensional spaces as matrices, one may represent a bounded linear operator T on infinite-dimensional separable Hilbert space H as the matrix

$$A_T := (\langle Tu_n, u_m\rangle)_{n,m=1}^{\infty}$$

with respect to an orthonormal bases $(u_n)_{n=1}^{\infty} \subset H$ and to try to relate the properties of T to the properties of A_T. This very natural idea looks naive to some extent, and the study of operators on infinite-dimensional spaces through their matrix representations goes back to the birth of operator theory in the beginning of 20-th century, and most notably, to Schur's multiplication and Weyl-von Neumann's perturbation theorem for selfadjoint operators.

While such a coordinatisation approach was neglected in favor of more revealing and standard by now textbook techniques, there was still a number of interesting applications of matrix representations scattered around the literature. Most of them are related to the studies in this paper and serve as motivations to what follows. So to put the paper into a proper context,

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we review several significant directions related to matrix representations of bounded operators.

1.1. **Diagonals.** The study of diagonals of operators on infinite-dimensional Hilbert spaces and also of the related issues go back to $70-80$'s with most essential results due to Fan, Fong, and Herrero. See e.g. [17],[26] and [18] as samples of the research from that period. The studies got a new impetus with foundational works of Kadison and Arveson [29],[30] and [3], who discovered a subtle structure in the set of possible diagonals of selfadjoint projections and, more generally, normal operators with finite spectrum, thus providing infinite-dimensional counterparts of the famous Schur-Horn theorem. The papers by Arveson and Kadison gave rise to a number of generalisations in various directions, including similar results for elements of von Neumann algebras, diagonals of operator tuples, applications to frame theory, etc. See, in particular, [8], [31], [32], [36] and [40]. As an illustration we mention the next theorem proved recently in [28].

Theorem 1.1. *A complex-valued sequence $(d_k)_{k=1}^\infty$ is a diagonal of a unitary operator on H if and only if $\sup_{k\ge1}|d_k|\le 1$ and*

$$2(1-\inf_{k\ge1}|d_k|)\le\sum_{k=1}^\infty(1-|d_k|).$$

A nice survey of recent developments in the theory of operator diagonals can be found in [34], see also [40] and the references therein.

In [40], we have changed a perspective by describing the diagonals for a given operator T rather than the set of possible diagonals for operator classes. Among other things, it was proved in [40] that if the essential numerical range of a bounded operator T on H has a non-empty interior, then a sequence from the interior is the diagonal of T if it approaches the boundary not too fast, satisfying so-called Blaschke-type condition. Such a condition is often optimal. For a more detailed discussion of some of the results from [40], see Section 2 below.

It seems that the methods of [40] opens a much wider venue than the one sketched in [40], and we hope the results of this paper justify this claim.

1.2. **Banded matrices.** One of the basic advantages in dealing with operator matrices is that for several important classes of operators their matrices have so-called banded structure. Recall that for $n\in\mathbb{N}$ the operator is said to be n-diagonal operator if it is is unitarily equivalent to a (finite or infinite) direct sum of (finite or infinite) n-diagonal matrices. The n-diagonal operators are often called band-diagonal when particular value of n is not crucial.

It is well known (and easy to prove) that selfadjont operators are 3-diagonal. This fact constitutes a basis for the classical approach to the study of selfadjoint (mostly unbounded) operators via associated 3-diagonal Jacobi matrices. However, any 3-diagonal unitary operator is diagonal. In fact, any unitary operator is 5-diagonal, and this number of diagonals is optimal. The relevance of this fact for mathematical physics was recognized comparatively recently, mainly due to so-called CMV-representations developed in [11] and [12], see also [44] for an additional insight. Of course, not only the mere fact of three or five diagonality of the matrix, but also the availability of concrete and convenient basis is important here.

However, band-diagonality is quite a rare phenomena. In particular, as proved in [51], see also [22], every normal operator with spectral measure not supported on a set of planar measure zero is not band-diagonal, and so, in particular, the multiplication operator $(Mf)(z) = zf(z)$ on $L^2(\mathbb{D})$ on the unit disc $\mathbb{D}$ is not band-diagonal. Moreover, there are non-band-diagonal operators in the intersection of all Schatten p-classes for $p > 2$, and the set of all non-band-diagonal operators is dense in the space $B(H)$ of bounded linear operators on H. At the same time the set of band-diagonal operators is not norm-dense in $B(H)$ being quite a small subset of $B(H)$ in various senses. For a discussion of restrictions posed by band-diagonality from the point of view of C^*-algebras, see e.g. [10]. An illuminating discussion of band-diagonality can be found in [22].

Another closely related topic concerns universal matrix representations with sparsified structure, that is representations possessing many zeros. It seems such representations go back to [52], where they were called staircase representations. One may prove that any $T \in B(H)$ admits a universal block three-diagonal form with the exponential control on block sizes. The representations are useful in commutator theory, e.g. in the study of Pearcy-Topping problem on compact commutators. Their modern and pertinent discussion can be found in [41]. As examples of other applications of the three diagonal block representaions we mention [37], where the Olsen lifting problem was treated, and [21], addressing representations of operators in $B(H)$ as linear combinations of operators of simple form (e.g. diagonal).

The same issues of sparsifying and arranging certain arrays of zero (and not only) elements in finite matrices still form a vast area of research. Being unable to give any reasonable account we refer to [27] and [16] as an illustration of problems and approaches considered there.

1.3. **Big matrices.** One of the basic results in the classical harmonic analysis, due to de Leeuw, Kahane, and Katznelson, says that for any $(a_j)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ there exists a periodic $f \in C([0, 2\pi])$ such that the Fourier coefficients $(\hat{f}_j)_{j\in\mathbb{Z}}$ of f satisfy $|\hat{f}_j| \geq |a_j|, j \in \mathbb{Z}$, so that $L^2([0, 2\pi])$-functions are indistinguishable from $C([0, 2\pi])$-functions by the size of their Fourier coefficients. Later on, numerous extensions of this result were found for other classes of f including also some spaces of functions analytic in $\mathbb{D}$, see e.g [4] for the survey of statements of that flavor, called plank type in view of similarity of the employed methods to the ones in Bang's theorem on covering convex body by planks.

On the way to obtaining noncommutative counterparts of the above domination result, Lust-Piquard proved in [35] the next elegant theorem on a possible size of matrices of bounded Hilbert space operators (which can be formulated for matrices in both $\mathbb{Z}$- and $\mathbb{N}$- settings, and we prefer the latter convention).

Theorem 1.2. *For every matrix of complex numbers $A = (a_{ij})_{i,j\geq 1}$ such that*

$$\|A\|_{\ell_\infty(\ell_2)} := \sup_i (\sum_j |a_{ij}|^2)^{1/2} < \infty \qquad \text{and} \qquad \|A^*\|_{\ell_\infty(\ell_2)} < \infty, \tag{1.1}$$

and every basis $(u_j)_{j=1}^\infty \subset H$ there exists $T \in B(H)$ satisfying

$$\|T\| \leq K \max\{\|A\|_{\ell_\infty(\ell_2)}, \|A^*\|_{\ell_\infty(\ell_2)}\} \qquad \text{and} \qquad |\langle Tu_j, u_i\rangle| \geq |a_{ij}| \tag{1.2}$$

for all $i, j \geq 1$.

Clearly, the assumption (1.1) is necessary for any estimates as (1.2) to hold, since (1.1) is satisfied by $A_T = (\langle Tu_j, u_i\rangle_{i,j\geq 1})$ in view of $A_T \in B(\ell^2(\mathbb{N}))$. Apart from being instructive as such, Theorem 1.2 appeared to be crucial in the characterization of wide classes of Schur multipliers on $B(\ell^2)$, see e.g. [13].

1.4. **Small matrices.** The Schur algebras, their structure and also the related notion of Schur multipliers is a natural scheme where matrix representations appear as a part of the very first definitions and surrounding basic results. Without going into details, of this separate and involved area, we emphasize a result not referring to specific notions and important for our further considerations.

In his study of the Schur multiplication on $B(H)$, Stout discovered in [49] that if $\langle Te_n, e_n\rangle \in c_0(\mathbb{N})$ for some orthonormal set $(e_n)_{n=1}^\infty$, i.e. 0 belongs to the essential numerical range $W_e(T)$ of T (in fact, Stout used a different definition of $W_e(T)$), then the size of matrix elements $\langle Tu_n, u_m\rangle$ can also be made small for an appropriate basis $(u_n)_{n=1}^\infty$. More precisely, the next theorem holds, see [49, Theorem 2.3].

Theorem 1.3. *Let $T \in B(H)$. Then the following properties are equivalent.*

(i) $0 \in W_e(T)$;

(ii) *For every $\epsilon > 0$ there exists a basis $(u_n)_{n=1}^\infty$ such that $|\langle Tu_m, u_n\rangle| < \epsilon$ for all $m, n \in \mathbb{N}$;*

(iii) *For every $(\alpha_n)_{n=1}^\infty \notin \ell_1(\mathbb{N})$ there exists a basis $(u_n)_{n=1}^\infty$ such that*

$$|\langle Tu_n, u_n\rangle| \leq \alpha_n \tag{1.3}$$

for all $n \in \mathbb{N}$.

For motivation of Theorem 1.3, and its relation to the structure of Schur algebars, see [49] and [50]. The statements similar but slightly weaker than Theorem 1.3 appeared to be crucial for the study of various matrix norms on $B(H)$, in particular for comparing them to each other and also to the operator norm on $B(H)$. See e.g. [20] for this direction of research.

1.5. **Order properties.** Quite often, the order properties of matrix elements prove to be useful. In particular, Radjavi and Rosenthal showed in [42] that for any $T \in B(H)$ there exists a selfadjoint $S \in B(H)$ such that T and S have no common invariant subspaces, and as a consequence the operators T and S generate $B(H)$. A key step in their approach is to note that if $T \in B(H)$ is not a multiple of the identity, then there exists an orthonormal basis $\{u_n\}_{n=1}^\infty$ such that $\langle Tu_n, u_m\rangle \neq 0$ for all n and m. In fact, the statement is true even for a sequence of bounded operators $(T_j)_{j=1}^\infty$ on H rather than a single operator T. This matrix statement was used used frequently in similar contexts.

In the study of cyclicity and multi-cyclicity phenomena, the next observation due to Deddens played an important role: if for $T \in B(H)$ there exists a basis $(u_n)_{n=1}^\infty$ in H such that $\langle Tu_n, u_m\rangle$ are real for all n and $m \in \mathbb{N}$, then $T \oplus T^*$, where T^* is adjoint of T, is not cyclic. In particular, if S is the unilateral shift on $H^2(\mathbb{D})$, then $S \oplus S^*$ is not cyclic. See [25] for more on that, though rather unfortunately bases leading to "real" representations of T have not been studied subsequently in the literature.

1.6. **Halmos problem.** The next problem appeared while developing the theory of integral operators, but it is quite natural as such, e.g. in the study of Schur multiplication and associated Schur operator algebras on Hilbert spaces, see e.g. [49]. Let us say that $T \in B(H)$ is absolutely bounded if the matrix $A_T := (|\langle Tu_j, u_i\rangle|)_{i,j\geq 1}$ defines a bounded operator on $\ell^2(\mathbb{N})$ for some orthonormal basis $(u_j)_{j=1}^\infty$ in H, and totally absolutely bounded if $A_T \in B(\ell^2(\mathbb{N}))$ for any orthonormal basis $(u_j)_{j=1}^\infty$ in H. Clearly, not every $T \in B(H)$ is absolutely bounded. The simplest example is probably T on $\ell^2(\mathbb{N})$ given by the so-called Hilbert matrix $(a_{ij})_{i,j\geq 1}$, where $a_{ij} = (i-j)^{-1}$ for $i \neq j$ and $a_{ij} = 0$, otherwise, see e.g. [24, Chapter]. In [23] Halmos asked for a characterization of absolutely bounded and totally absolutely bounded T (see [24] for more on theses notions and their motivations.) Independently and almost simultaneously, it was proved in [47] and [45] that T is totally absolutely bounded if and only if $T = \lambda + S$, where S is a Hilbert-Schmidt operator. However, the description of absolutely bounded T is still out of reach, though apart from Halmos, the problem was posed explicitly in in [47], [45], [46] and [49]. A discussion of this and related matters can be found in [24, Chapter 10 and Theorem 16.3]. Obviously, the problem reflects the current lack of understanding on how the entries of A_T may change when $(u_j)_{j=1}^\infty$ is varying. Another illustration of this problem is the study of operator diagonals discussed above, where the explicit description of the set of diagonals is known for very particular choices of T, even if T is selfadjoint, let along the description of such a set of diagonals for fixed T.

2. Tools, results and strategy

2.1. **Numerical ranges.** In our studies of matrix representations for a bounded operator T on a separable (complex) Hilbert space H we will rely on elementary properties of its numerical range $W(T) := \{\langle Tx, x\rangle : \|x\| = 1\}$ and its essential numerical range $W_e(T)$. Recall that for $T \in B(H)$ its essential numerical range $W_e(T)$ can be defined as

$$W_e(T) := \{\lambda \in \mathbb{C} : \langle Te_n, e_n\rangle \to \lambda,\ n \to \infty\}. \tag{2.1}$$

for some orthonormal sequence $(e_n)_{n=1}^\infty \subset H$. In fact, the orthonormal sequences in (2.1) can be replaced by orthonormal bases, see e.g. Theorem 1.3, (ii). Alternatively, $\lambda \in W_e(T)$ if and only if for every $\epsilon > 0$ and every subspace M of finite codimension there exists a unit vector $x \in M$ such that $|\langle Tx, x\rangle - \lambda| < \epsilon$. Clearly, $W_e(T) \subset \overline{W(T)}$. For any $T \in B(H)$, the set $W_e(T)$ is non-empty, compact and convex, and moreover $W_e(T)$ contains the essential spectrum of T. Thus, in view of convexity of $W_e(T)$,

$$\operatorname{Int} \operatorname{conv}\sigma_e(T) \subset \operatorname{Int} W_e(T). \tag{2.2}$$

So, for any contraction T with $\sigma(T) \supset \mathbb{T}$ (e.g. unilateral or bilateral shift) one has $\mathbb{D} \subset \operatorname{Int}\, W_e(T)$, hence the latter set is as large as possible in this case. The convexity of $W(T)$ also implies that $\operatorname{Int} W_e(T) \subset W(T)$. Since $W_e(T) = W_e((I-P)T(I-P))$ for every finite rank projection P, one has

$$\operatorname{Int} W_e(T) \subset W((I-P)T|_{(I-P)}). \tag{2.3}$$

Thus, to be able to find a fixed $\lambda \in W(T)$ in the numerical range of any restriction of T to a finite-codimensional subspace of H it is natural to assume that $\lambda \in \operatorname{Int} W_e(T)$. [IN FACT, MUST ONE ASSUME the latter property ? NO: ZERO OPERATOR IS A COUNTEREXAMPLE] The property (2.3) is vital in various inductive arguments given below. Note that, moreover, $\lambda \in \operatorname{Int} W_e(T)$ implies that there exists an infinite rank (orthogonal) projection P such that $PTP = \lambda P$. The basic properties of $W_e(T)$ can be found e.g. in [19] or [1]. Some of their analogons for tuples of bounded operators are proved in [33]. A unified approach to the essential numerical range for tuples and its relatives, including the properties mentioned above, has been provided in [39, Section].

Finally, for auxiliary estimates, we will need the next plank type result from [4, Theorem 2].

Theorem 2.1. *Let $(u_j)_{j=1}^n \subset H$ be a tuple of unit vectors, and let $(a_j)_{j=1}^n \subset \mathbb{R}_+$ be such that $\sum_{j=1}^n a_j^2 = 1$. Then there exists a unit vector $v \in H$ such that $|\langle v, u_j\rangle| \geq a_j$ for all j.*

2.2. **Results.** Extending and complementing various results on operator diagonals from the literature, it was proved in [40, Theorem] that for every $T \in B(H)$ and every $(\lambda)_{n=1}^\infty \subset \operatorname{Int} W_e(T)$ satisfying

$$\sum_{n=1}^\infty \operatorname{dist}\{\lambda_n, \partial W_e(T)\} = \infty, \tag{2.4}$$

there exists an orthonormal basis $(u_n)_{n=1}^\infty$ in H such that $\langle Tu_n, u_n\rangle = \lambda_n, n \in \mathbb{N}$. The assumption (2.4), introduced in [40] and called there *non-Blaschke* *type*, is close to be optimal and allows one to construct diagonals for general $T \in B(H)$. Moreover, it has operator-valued counterparts leading to construction of operator-value diagonals and generalisations of the main results from [7].

It is natural to ask whether (2.4) has further implications and can be used to preassign a part of the matrix of T larger than the main diagonal. We prove that under (2.4), apart from producing the main diagonal $(\langle Tu_n, u_n\rangle)_{n=1}^{\infty}$ by (2.4), the matrix of T can be sparsified by arranging zero matrix elements in any band outside of $(\langle Tu_n, u_n\rangle)_{n=1}^{\infty}$. The result is opposite in a sense to the series of results concerning matrices with banded structure discussed in the introduction. The obtained sparsification is rather mild, and in this sense the result is certainly weaker than the results on banded structure. On the other hand, it concerns general $T \in B(H)$ rather than very specific classes of $B(H)$ (normal, unitary, selfadjoint), and it complements the results on banded matrix representation.

Theorem 2.2. *Let $T \in B(H)$, and let $(\lambda)_{n=1}^{\infty} \subset \operatorname{Int} W_e(T)$ be such that $\sum_{n=1}^{\infty} \operatorname{dist}\{\lambda_n, \partial W_e(T)\} = \infty$. Then for every $K \in \mathbb{N}$ there exists an orthonormal basis $(u_n)_{n=1}^{\infty}$ in H such that*

$$\langle Tu_n, u_n\rangle = \lambda_n, \qquad n \in \mathbb{N}, \tag{2.5}$$

and

$$\langle Tu_j, u_k\rangle = 0, \qquad 1 \le |j-k| \le K. \tag{2.6}$$

Th next corollary of Theorem 2.2 is straightforward.

Corollary 2.3. *Let $T \in B(H)$ be such that $0 \in \operatorname{Int} W_e(T)$. For every $K \in \mathbb{N}$ there exists an orthonormal basis $(u_n)_{n=1}^{\infty}$ in H such that*

$$\langle Tu_j, u_k\rangle = 0 \tag{2.7}$$

for all $j, k \in \mathbb{N}$ with $|j-k| \le K$.

To distinguish the representations satisfying (2.7) from the representations with band structure, one may call the matrix of $T \in B(H)$ with respect to an orthonormal basis $(u_n)_{n=1}^{\infty}$ *windowed* with window of width K if $\langle Tu_j, u_k\rangle = 0$ for all j and k such that $|j-k| \le K$. In this terminology, any $T \in B(H)$ with $0 \in \operatorname{Int} W_e(T)$ allows a window matrix representation of any (finite) width.

If $(\lambda_n)_{n=1}^{\infty} \subset \operatorname{Int} W_e(T)$ is well-separated from the boundary of $W_e(T)$, then by e.g. Theorem 2.2 the sequence $(\lambda_n)_{n=1}^{\infty}$ is realizable as the main diagonal $(\langle Tu_n, u_n\rangle)_{n=1}^{\infty}$. It appears that in this case there are only size restrictions on the other two main diagonals $(\langle Tu_n, u_{n+1}\rangle)_{n=1}^{\infty}$ and $(\langle Tu_{n+1}, u_n\rangle)_{n=1}^{\infty}$ of T. The next statement supports this claim. For simplicity we formulate it for contractions.

Theorem 2.4. *Let $T \in B(H)$, $\|T\| \leq 1$ and let $\varepsilon > 0$ be fixed. Let $(\lambda_n)_{n=1}^{\infty} \subset \operatorname{Int} W_e(T)$, and let $(\mu_n)_{n=1}^{\infty}, (\nu_n)_{n=1}^{\infty} \subset \mathbb{C}$ satisfy*

$$\operatorname{dist}\{\lambda_n, \partial W_e(T)\} > 2\varepsilon, \qquad \sup_{n\geq 1}(|\mu_n|, |\nu_n|) < \frac{\varepsilon\sqrt{\varepsilon}}{16}$$

for all $n \in \mathbb{N}$. Then there exists an orthonormal basis $(u_n)_{n=1}^{\infty}$ in H such that for all $n \in \mathbb{N}$:

(1) $\langle Tu_n, u_n\rangle = \lambda_n$;

(2) $\langle Tu_n, u_{n+1}\rangle = \mu_n$;

(3) $\langle Tu_{n+1}, u_n\rangle = \nu_n$.

Next, instead of preassigning a part of the A_T we are interested in making the entries A_T vanishing fast enough at infinity. The method used in the proof of Theorems 2.2 and 2.4 works here as well, and a weaker assumption that $0 \in W_e(T)$ will suffice for this purpose. In the following result one spreads out the estimate in (1.3) over the whole of matrix A_T of T with respect to an appropriate orthonormal basis $(u_n)_{n=1}^{\infty}$.

Theorem 2.5. *Let $T \in B(H)$ satisfy $0 \in W_e(T)$, and let $(a_j)_{j=1}^{\infty}$ be a sequence of positive numbers such that $(a_j)_{j=1}^{\infty} \notin \ell^1$. Then there exists an orthonormal basis $(u_n)_{n=1}^{\infty}$ in H such that*

$$|\langle Tu_n, u_j\rangle| \leq \sqrt{a_n a_j} \tag{2.8}$$

for all $n, j \in \mathbb{N}$.

Note that for the diagonal elements $\langle Tu_n, u_n\rangle$, $n \geq 1$, the estimate (2.8) yields $|\langle Tu_n, u_n\rangle| \leq a_n$ for all $n \in \mathbb{N}$, which is precisely Stout's condition (1.3). Thus Theorem 2.5 is a generalization of Stout's result to the matrix context. See also an open question on [49, p. 45] containing a version of Theorem 2.5. One may also consult [40, Theorem] for generalizations of (1.3) in the framework of diagonals for operator tuples. Note that (2.8) is essentially the same as in (1.3) for the elements of A_T near the main diagonal, but weakens away from the main diagonal, e.g. one looses $\sqrt{a_n}$ in the first row or column.

By considering $T - \lambda$ for any $\lambda \in W_e(T)$ and applying Theorem 2.5 to $T - \lambda$ one gets the following corollary concerning arbitrary $T \in B(H)$.

Corollary 2.6. *For every sequence of positive numbers $(a_j)_{j=1}^{\infty}$ satisfying $(a_j)_{j=1}^{\infty} \notin \ell^1$ and every $T \in B(H)$ there exists an orthonormal basis $(u_n)_{n=1}^{\infty}$ in H such that*

$$|\langle Tu_n, u_j\rangle| \leq \sqrt{a_n a_j}$$

for all $n, j \in \mathbb{N}$, $n \neq j$.

Choosing $(a_j)_{j=1}^{\infty} \in \ell^2(\mathbb{N})\backslash\ell^1(\mathbb{N})$ in Theorem 2.5, we get the next corollary complementing Corollary 2.3 in the case when e.g. $\operatorname{Int} W_e(T) = \emptyset$.

Corollary 2.7. *For all* $T \in B(H)$, $\lambda \in W_e(T)$, *and* $K \in \mathbb{N}$ *there exists an orthonormal basis* $(u_n)_{n=1}^{\infty}$ *in* H *such that*

$$T = \lambda + W + S,$$

where $W \in B(H)$ *satisfy* $\langle Wu_j, u_k \rangle = 0, |j - k| \leq K$, *and* S *is a Hilbert-Schmidt operator on* H.

Next we consider a problem opposite in a sense to the one addressed in Theorem 2.5. For fixed operator $T \in B(H)$ we would like to find a basis $(u_n)_{n=1}^{\infty}$ giving rise to a matrix A_T of T with large entries $\langle Tu_n, u_j \rangle$. Since the task of getting the lower bounds for $\langle Tu_n, u_j \rangle$ seems to be more demanding than the one concerning the upper bounds, we restrict ourselves to the polynomial scale of bounds. However, to illustrate the sharpness of our lower estimates we provide the upper estimates as well. In terms of the size of constructed A_T, the result given below is of course weaker than Theorem 1.2. However, while in Theorem 1.2 one looks just for any operator satisfying (1.2), our theorem produces an operator having large matrix entries and belonging to the unitary orbit of a fixed $T \in B(H)$ if $W_e(T)$ is large enough.

Theorem 2.8. *Let* $T \in B(H)$ *be an operator which is not of the form* $T = \lambda + K$ *for some* $\lambda \in \mathbb{C}$ *and a compact operator* $K \in B(H)$. *Then there exist an orthonormal basis* $(u_n)_{n=1}^{\infty} \subset H$ *and strictly positive constants* c_1, c_2, *and* d *such that*

$$|\langle Tu_n, u_n \rangle| \geq d, \qquad n \in \mathbb{N},$$

and

$$\frac{c_1 \min\{n, j\}^{1/2}}{\max\{n, j\}^{3/2}} \leq |\langle Tu_n, u_j \rangle| \leq \frac{c_2}{\max\{n, j\}^{1/2}}$$

for all $n, j \in \mathbb{N}, n \neq j$.

Accidentally, the assumption of Theorem 2.8 is equivalent to the fact that T is a commutator (or that T is similar to an operator having an infinite-dimensional zero compression), see e.g. [1, p. 440] and also [9]. Some methods and observations from the early days of commutator theory are similar in spirit to our proof of Theorem 2.8. However, we are not aware of any deeper relations between our results and the commutator properties. On the other hand, estimates of the matrix elements were, in particular, important in the study of commutators in e.g. [9, Section 3], [14] and [15].

2.3. **Strategy.** It would be instructive and helpful to underline a general idea behind our arguments, since it may possibly be modified and used in similar contexts as well. While fine details of the proofs of above theorems differ from each other and require the corresponding adjustments, the general approach can be roughly described as follows.

Usually it is rather simple to find an orthonormal sequence (u_n) satisfying the required property. The main problem is to ensure that this sequence is a basis. This is achieved in the following way. First fix a countable sequence (y_m) of unit vectors which generates all the space. Write $\mathbb{N} = \bigcup_{m=1}^\infty A_m$ (disjoint union), where the sets A_m are properly chosen.

The orthonormal basis (u_n) is constructed inductively. Let $n \geq 1$ and suppose that the vectors $u_1, \dots, u_{n-1}$ have already been constructed. Choose a unit vector v_n orthogonal to all (finitely many) vectors that appeared in the construction up till now and satisfying the required property. Then u_n will be a small perturbation of v_n,

$$u_n - \sqrt{1-c_n}v_n + c_n(I - P_{n-1})y_{m(n)},$$

the where P_{n-1} is the orthogonal projection onto the subspace $M_{n-1} = \bigvee_{j=1}^{n-1} u_j$, $n \in A_{m(n)}$ and c_n is "small".

If we do it properly then u_n will be a unit vector still satisfying the required property, $u_n \perp u_1, \dots, u_{n-1}$ and

$$\operatorname{dist}^2\{y_{m(n)}, M_n\} \leq (1-c_n^2)\operatorname{dist}\{y_{m(n)}, M_{n-1}\}.$$

So if the numbers c_n are chosen properly, then we will have

$$\lim_{n\to\infty} \operatorname{dist}\{y_m, M_n\} = 0$$

for all $m \in \mathbb{N}$. So $y_m \in \bigvee_{n=1}^\infty u_n$ for all m, and so the constructed sequence (u_n) will be an orthonormal basis satisfying the required property.

3. Matrices with several given diagonals:proofs

We start with a proof of Theorem 2.2 being a generalization of [40, Corollary 4.2] in case of a single operator. (Concerning the setting of tuples see Section 6.) The argument employed there is a good illustration of the strategy described above, and its variants will be used several times in the sequel.

Proof of Theorem 2.2 Let $(y_m)_{m=1}^\infty$ be a sequence of unit vectors in H such that $\bigvee_{m=1}^\infty y_m = H$.

For $r = 0, 1, \dots, K$ let $B_r = \{n \in \mathbb{N} : n = r \mod (K+1)\}$, and note that there exists $r_0 \in \{0, \dots, K\}$ such that $\sum_{n\in B_{r_0}} \operatorname{dist}\{\lambda_n, \partial W_e(T)\} = \infty$.

Represent B_{r_0} as $B_{r_0} = \bigcup_{m=1}^\infty A_m$, where $A_m \cap A_n = \emptyset, m \neq n$, and

$$\sum_{n\in A_m} \operatorname{dist}\{\lambda_n, \partial W_e(T)\} = \infty$$

for all $m \in \mathbb{N}$.

For $n \in B_{r_0}$ let $m(n)$ be the unique integer satisfying $n \in A_{m(n)}$.

We construct the vectors $u_n, n \geq 1$, inductively. Let $n \geq 1$ and suppose that the vectors $u_1, \dots, u_{n-1}$ have already been constructed.

Suppose first that $n \notin B_{r_0}$. Let $\hat{n} = \min\{n' \in B_{r_0} : n' > n\}$. Find

$$u_n \in \{u_j, Tu_j, T^*u_j \quad j = 1, \dots, n-1, y_{\hat{n}}, Ty_{\hat{n}}, T^*y_{\hat{n}}\}^\perp$$

such that $\langle Tu_n, u_n\rangle = \lambda_n$. Clearly $u_n \perp u_1, \dots, u_{n-1}$,

$$\langle Tu_j, u_n\rangle = 0, \qquad j = \max\{1, n-K\}, \dots, n-1,$$

and

$$\langle Tu_n, u_j\rangle = \langle u_n, T^* u_j\rangle = 0, \qquad j = \max\{1, n-K\}, \dots, n-1.$$

So u_n satisfies (2.5) and (2.6).

Suppose now that $n \in B_{r_0}$. Let P_{n-1} be the orthogonal projection onto the subspace $M_{n-1} := \bigvee_{j=1}^{n-1} u_j$.

If $y_{m(n)} \in M_{n-1}$ then find a unit vector

$$u_n \in \{u_j, Tu_j, T^* u_j : j = 1, \dots, n-1\}^\perp$$

such that $\langle Tu_n, u_n\rangle = \lambda_n$. Then u_n satisfies (2.5) and (2.6) again.

Suppose that $y_{m(n)} \notin M_{n-1}$. Set

$$b_n = \frac{(I - P_{n-1}) y_{m(n)}}{\|(I - P_{n-1}) y_{m(n)}\|}.$$

Let

$$\rho_n := |\langle Tb_n, b_n\rangle - \lambda_n| \qquad \text{and} \qquad \delta_n := \frac{1}{2}\text{dist}\,\{\lambda_n, \partial W_e(T)\}.$$

If $\langle Tb_n, b_n\rangle = \lambda_n$ then set $\mu_n = \lambda_n$. If $\langle Tb_n, b_n\rangle \neq \lambda_n$ then let $\mu_n \in \mathbb{C}$ be the unique number satisfying

$$\frac{\langle Tb_n, b_n\rangle - \lambda_n}{\rho_n} = \frac{\lambda_n - \mu_n}{\delta_n}.$$

Clearly $\mu_n \in \text{Int}\, W_e(T)$. Using (2.3), find a unit vector

$$v_n \in \{u_j, Tu_j, T^* u_j, j = 1, \dots, n-1; b_n, Tb_n, T^* b_n\}^\perp$$

satisfying $\langle Tv_n, v_n\rangle = \mu_n$. Set

$$u_n = \sqrt{\frac{\rho}{\rho_n + \delta_n}}\, v_n + \sqrt{\frac{\delta_n}{\rho_n + \delta_n}}\, b_n.$$

Since $v_n \perp b_n$, we have $\|u_n\| = 1$. We have $u_n \perp u_1, \dots, u_{n-1}$ and

$$\begin{aligned}
\langle Tu_n, u_n\rangle =& \frac{\rho_n}{\rho_n + \varepsilon_n}\langle Tv_n, v_n\rangle + \frac{\delta_n}{\rho_n + \delta_n}\langle Tb_n, b_n\rangle \\
=& \frac{\rho_n \mu_n + \delta_n \langle Tb_n, b_n\rangle}{\rho_n + \delta_n} \\
=& \frac{\rho_n}{\rho_n + \delta_n}(\mu_n - \lambda_n) + \frac{\delta_n}{\rho_n + \delta_n}(\langle Tb_n, b_n\rangle - \lambda_n) + \lambda_n \\
=& \lambda_n.
\end{aligned}$$

Choose integer j such that $\max\{1, n-K\} \leq j \leq n-1$. Then $j, j+1, \dots, n-1 \notin B_{r_0}$. By construction, $u_j, \dots, u_{n-1} \perp y_{m(n)}$. So

$$b_n = \frac{(I - P_{n-1}) y_{m(n)}}{\|(I - P_{n-1}) y_{m(n)}\|} = \frac{(I - P_{j-1}) y_{m(n)}}{\|(I - P_{j-1}) y_{m(n)}\|},$$

hence b_n is a linear combination of $u_1, \dots, u_{j-1}, y_{m(n)}$, and so $Tu_j \perp b_n$.

Thus we have

$$\langle Tu_j, u_n\rangle = \sqrt{\frac{\rho_n}{\rho_n+\delta_n}}\langle Tu_j, v_n\rangle + \sqrt{\frac{\delta_n}{\rho_n+\delta_n}}\langle Tu_j, b_n\rangle = 0.$$

Similarly one can prove

$$\langle Tu_n, u_j\rangle = \langle u_n, T^*u_j\rangle = \sqrt{\frac{\rho_n}{\rho_n+\delta_n}}\langle v_n, T^*u_j\rangle + \sqrt{\frac{\delta_n}{\rho_n+\delta_n}}\langle b_n, T^*u_j\rangle = 0.$$

If we continue the construction inductively, we construct an orthonormal system $(u_n)_{n=1}^\infty$ satisfying (2.5) and (2.6).

It remains to show that it is a basis. Note that

$$\begin{aligned}\|(I-P_n)y_{m(n)}\|^2 =& \|(I-P_{n-1})y_{m(n)}\|^2 - |\langle (I-P_{n-1})y_{m(n)}, u_n\rangle|^2\\ =& \|(I-P_{n-1})y_{m(n)}\|^2 - \|(I-P_{n-1})y_{m(n)}\|^2\cdot|\langle b_n, u_n\rangle|^2\\ =& \|(I-P_{n-1})y_{m(n)}\|^2\Big(1-\frac{\delta_n}{\rho_n+\delta_n}\Big)\end{aligned}$$

and

$$\begin{aligned}\ln\|(I-P_n)y_{m(n)}\|^2 \le& \ln\|(I-P_{n-1})y_{m(n)}\|^2 + \ln\Big(1-\frac{\delta_n}{\rho_n+\delta_n}\Big)\\ \le& \ln\|(I-P_{n-1})y_{m(n)}\|^2 - \frac{\delta_n}{\rho_n+\delta_n}\\ \le& \ln\|(I-P_{n-1})y_{m(n)}\|^2 - \frac{\operatorname{dist}\{\lambda_n, \partial W_e(T)\}}{6\|T\|}.\end{aligned}$$

Now for fixed $m\in\mathbb{N}$, we have

$$\lim_{n\to\infty}\ln\|(I-P_n)y_m\|^2 \le -\sum_{n\in A_m}\frac{\operatorname{dist}\{\lambda_n, \partial W_e(T)\}}{6\|T\|} = -\infty.$$

So $y_m\in\bigvee_{n=1}^\infty u_n$ for all m. Hence $(u_n)_{n=1}^\infty$ is an orthonormal basis, and the proof is finished. □

Under assumptions somewhat stronger than those in Theorem 2.2, our techniques allows one to construct three diagonals of T with upper and lower sub-diagonals depending only on the sup-norm of its main diagonal. To this aim, we first prove the next auxiliary lemma.

Lemma 3.1. *Let $T\in B(H)$, and let $a\in \operatorname{Int} W_e(T)$ be such that*

$$\operatorname{dist}\{a, \partial W_e(T)\} > \varepsilon > 0.$$

Let $M\subset H$ be a subspace of finite codimension. Then there exists a unit vector $u\in M$ satisfying the following conditions:

(i) *$\langle Tu, u\rangle = a$;*

(ii) *if* $w = Tu - \langle Tu, u\rangle u$, $w' = T^*u - \langle T^*u, u\rangle u$ *and* $\alpha, \beta \in \mathbb{C}$, *then there exists* $z \in \bigvee\{w, w'\}$ *such that*

$$\langle w, z\rangle = \alpha, \qquad \langle w', z\rangle = \beta \quad \text{and} \quad \|z\| \leq \frac{2(|\alpha| + |\beta|)}{\varepsilon}.$$

Proof. Since $a \pm \varepsilon$ and $a \pm i\varepsilon \in \operatorname{Int} W_e(T)$, the property (2.3) implies that there exists a unit vector $x_1 \in M$ such that $\langle Tx_1, x_1\rangle = a + \varepsilon$.

Similarly, there exists a unit vector $x_2 \in M \cap \{x_1, Tx_1, T^*x_1\}^\perp$ such that $\langle Tx_2, x_2\rangle = a + i\varepsilon$, and unit vectors $x_3 \in M \cap \{x_j, Tx_j, T^*x_j : j = 1, 2\}^\perp$ and $x_4 \in M \cap \{x_j, Tx_j, T^*x_j : j = 1, 2, 3\}^\perp$ with $\langle Tx_3, x_3\rangle = a - \varepsilon$ and $\langle Tx_4, x_4\rangle = a - i\varepsilon$.

Let

$$u = \frac{1}{2}(x_1 + x_2 + x_3 + x_4).$$

Then $u \in M$, $\|u\| = 1$ and

$$\langle Tu, u\rangle = \frac{1}{4}\Big(\langle Tx_1, x_1\rangle + \langle Tx_2, x_2\rangle + \langle Tx_3, x_3\rangle + \langle Tx_4, x_4\rangle +\Big) = a.$$

Let

$$w = Tu - \langle Tu, u\rangle u \qquad \text{and} \qquad w' = T^*u - \langle T^*u, u\rangle u.$$

Let $\eta \in \mathbb{C}$. Then

$$\begin{aligned}
\langle w + \eta w', x_1\rangle =& \langle Tu, x_1\rangle - \langle Tu, u\rangle \cdot \langle u, x_1\rangle + \eta\langle T^*u, x_1\rangle - \eta\langle T^*u, u\rangle \cdot \langle u, x_1\rangle \\
=& \frac{1}{2}\langle Tx_1, x_1\rangle - \frac{a}{2} + \frac{\eta\langle T^*x_1, x_1\rangle}{2} - \frac{\eta\bar{a}}{2} \\
=& \frac{a + \varepsilon}{2} - \frac{a}{2} + \frac{\eta(\bar{a} + \varepsilon)}{2} - \frac{\eta\bar{a}}{2} \\
=& \frac{\varepsilon(1 + \eta)}{2}
\end{aligned}$$

and similarly,

$$\begin{aligned}
\langle w + \eta w', x_2\rangle =& \langle Tu, x_2\rangle - \langle Tu, u\rangle \cdot \langle u, x_2\rangle + \eta\langle T^*u, x_2\rangle - \eta\langle T^*u, u\rangle \cdot \langle u, x_2\rangle \\
=& \frac{1}{2}\langle Tx_2, x_2\rangle - \frac{a}{2} + \frac{\eta\langle T^*x_2, x_2\rangle}{2} - \frac{\eta\bar{a}}{2} \\
=& \frac{a + i\varepsilon}{2} - \frac{a}{2} + \frac{\eta(\bar{a} - i\varepsilon)}{2} - \frac{\eta\bar{a}}{2} \\
=& \frac{i\varepsilon(1 - \eta)}{2}.
\end{aligned}$$

So

$$\begin{aligned}
\|w + \eta w'\| \geq& \max\big\{|\langle w + \eta w', x_1\rangle|, |\langle w + \eta w', x_2\rangle|\big\} \\
=& \frac{\varepsilon}{2}\max\{|1 + \eta|, |1 - \eta|\} \geq \frac{\varepsilon}{2}.
\end{aligned}$$

In the same way, one can show

$$\|\eta w + w'\| \geq \frac{\varepsilon}{2}$$

for all $\eta \in \mathbb{C}$.

Let L be the two-dimensional subspace generated by w and w'. Let P and P' be the orthogonal projections from L onto the one-dimensional subspace generated by w and w', respectively. We then have

$$\|(I-P')w\| \ge \frac{\varepsilon}{2} \qquad \text{and} \qquad \|(I-P)w'\| \ge \frac{\varepsilon}{2}.$$

Let $\alpha, \beta \in \mathbb{C}$, and let

$$z = \frac{\alpha(I-P')w}{\|(I-P')w\|^2} + \frac{\beta(I-P)w'}{\|(I-P)w\|^2}.$$

Then

$$\langle w, z\rangle = \frac{\alpha\langle w, (I-P')w\rangle}{\|(I-P')w\|^2} = \alpha \qquad \text{and} \qquad \langle w', z\rangle = \beta.$$

Finally

$$\|z\| \le \frac{|\alpha|}{\|(I-P')w\|} + \frac{|\beta|}{\|(I-P)w'\|} \le \frac{2(|\alpha|+|\beta|)}{\varepsilon}.$$

□

The proof of Theorem 2.4 is similar to the proof of Theorem 2.2, but it is technically more demanding.

Proof of Theorem 2.4 Note that the assumption $\|T\| \le 1$ implies that $|\varepsilon| \le 1$.

Fix an orthonormal basis $(y_m)_{m=1}^{\infty}$ in H.

We construct the basis $(u_n)_{n=1}^{\infty}$ inductively.

For $n = 1$, find a unit vector $u_1 \in H$ with $\langle Tu_1, u_1\rangle = \lambda_1$ such that the vectors

$$w_1 := Tu_1 - \langle Tu_1, u_1\rangle u_1 \qquad \text{and} \qquad w_1' := T^*u_1 - \langle T^*u_1, u_1\rangle u_1$$

satisfy condition (ii) of Lemma (3.1), i.e., for all $\alpha, \beta \in \mathbb{C}$ there exists $z \in \bigvee\{w_1, w_1'\}$ such that

$$\langle w_1, z\rangle = \alpha, \qquad \langle w_1', z\rangle = \beta, \quad \text{and} \quad \|z\| \le \frac{2(|\alpha|+|\beta|)}{2}.$$

Set formally $v_1 = u_1$, $z_1 = b_1 = 0$.

Let $n \ge 2$ and suppose that the orthonormal vectors $u_1, \dots, u_{n-1}$ satisfying (1), (2) and (3) have already been constructed. To run the induction, we also assume that $u_1, \dots, u_{n-1}$ satisfy

$$u_{n-1} = v_{n-1} + z_{n-1} + b_{n-1},$$

where $\|z_{n-1}\| \le \frac{\sqrt{\varepsilon}}{2}$, $\|b_{n-1}\| \le \frac{\varepsilon\sqrt{\varepsilon}}{32}$, and

$$v_{n-1} \perp \{u_1, \dots, u_{n-2}, z_{n-1}, b_{n-1}, Tz_{n-1}, Tb_{n-1}, T^*z_{n-1}, T^*b_{n-1}\}.$$

Moreover, if

$$w_{n-1} := Tv_{n-1} - \langle Tv_{n-1}, v_{n-1}\rangle v_{n-1} \qquad \text{and} \qquad w_{n-1}' := T^*v_{n-1} - \langle T^*v_{n-1}, v_{n-1}\rangle v_{n-1},$$

then supposing, in addition, that w_{n-1} and w'_{n-1} satisfy condition (ii) of Lemma 3.1, i.e., for all $\alpha, \beta \in \mathbb{C}$ there exists $z \in \bigvee\{w_{n-1}, w'_{n-1}\}$ with

$$\langle w_{n-1}, z\rangle = \alpha, \qquad \langle w'_{n-1}, z\rangle = \beta \quad \text{and} \quad \|z\| \le \frac{2(|\alpha| + |\beta|)}{2}.$$

Denote by P_{n-1} the orthogonal projection onto the subspace $M_{n-1} := \bigvee_{j=1}^{n-1} u_j$.

Denote by A_n the set of all positive integers m such that $y_m \notin M_{n-1}$ and

$$\sup\big\{|\langle (I - P_{n-1})y_m, z\rangle| : z \in \bigvee\{u_{n-1}, Tu_{n-1}, T^*u_{n-1}\}, \|z\| = 1\big\}$$

does not exceed

$$\|(I - P_{n-1})y_m\| \cdot \frac{\varepsilon}{32}.$$

Since $y_m \to 0, m \to \infty$, weakly, the set A_n contains all but a finite number of $m \in \mathbb{N}$, so in particular $A_n \neq \emptyset$.

Let $m(n)$ be any number $m \in A_n$ minimizing the quantity $m + \text{card}\big\{k : 2 \le k \le n-1, m(k) = m\big\}$. (If there are more than one such numbers, then fix any of them).

Let

$$b_n = \frac{(I - P_{n-1})y_{m(n)}}{\|(I - P_{n-1})y_{m(n)}\|} \cdot \frac{\varepsilon\sqrt{\varepsilon}}{32}.$$

Observe that

$$\|z_{n-1} + b_{n-1}\|^2 \le \Big(\frac{\sqrt{\varepsilon}}{2} + \frac{\varepsilon\sqrt{\varepsilon}}{32}\Big)^2 \le \varepsilon \cdot \Big(\frac{1}{2} + \frac{1}{32}\Big)^2 \le \frac{8\varepsilon}{25} \le \frac{8}{25},$$

$$1 - \|z_{n-1} + b_{n-1}\|^2 \ge \frac{17}{25},$$

and

$$\sqrt{1 - \|z_{n-1} + b_{n-1}\|^2} \ge \frac{4}{5}.$$

By Lemma 3.1, there exists $z_n \in \bigvee\{w_{n-1}, w'_{n-1}\}$ such that

$$\langle w_{n-1}, z_n\rangle = \frac{\mu_{n-1} - \langle Tu_{n-1}, b_n\rangle}{\sqrt{1 - \|z_{n-1} + b_{n-1}\|^2}},$$

$$\langle w'_{n-1}, z_n\rangle = \frac{\bar{\nu}_{n-1} - \langle T^*u_{n-1}, b_n\rangle}{\sqrt{1 - \|z_{n-1} + b_{n-1}\|^2}},$$

and

$$\begin{aligned}\|z_n\| \le& \frac{2}{\varepsilon}\Big(\Big|\frac{\mu_{n-1} - \langle Tu_{n-1}, b_n\rangle}{\sqrt{1 - \|z_{n-1} + b_{n-1}\|^2}}\Big| + \Big|\frac{\bar{\nu}_{n-1} - \langle T^*u_{n-1}, b_n\rangle}{\sqrt{1 - \|z_{n-1} + b_{n-1}\|^2}}\Big|\Big)\\ \le& \frac{2}{\varepsilon} \cdot \frac{5}{4}\Big(\frac{2\varepsilon\sqrt{\varepsilon}}{16} + 2\|b_n\|\Big) \le 5\sqrt{\varepsilon}\Big(\frac{1}{16} + \frac{1}{32}\Big)\\ \le& \frac{\sqrt{\varepsilon}}{2}.\end{aligned}$$

Note that as above,

$$\|z_n+b_n\|^2 \le \frac{8\varepsilon}{25} \le \frac{8}{25}, \qquad 1-\|z_n+b_n\|^2 \ge \frac{17}{25} \quad \text{and} \quad \sqrt{1-\|z_n+b_n\|^2} \ge \frac{4}{5}.$$

Set

$$\lambda_n' = \frac{\lambda_n - \langle T(z_n+b_n), z_n+b_n\rangle}{1-\|z_n+b_n\|^2}.$$

We have

$$\begin{aligned}|\lambda_n'-\lambda_n| \le & \left|1-\frac{1}{1-\|z_n+b_n\|^2}\right| + \frac{\|z_n+b_n\|^2}{1-\|z_n+b_n\|^2}\\ =&\frac{2\|z_n+b_n\|^2}{1-\|z_n+b_n\|^2}\\ \le&\frac{2\varepsilon\frac{8}{25}}{\frac{17}{25}} \le \varepsilon.\end{aligned}$$

So $\operatorname{dist}\{\lambda_n', \partial W_e(T)\} > \varepsilon$. By Lemma 3.1 there exists a unit vector v_n such that

$$v_n \perp \bigvee_{j=1}^{n-1}\{u_j, Tu_j, T^*u_j\},$$

$$\begin{aligned}v_n \perp \{y_{m(n)}, Ty_{m(n)}, T^*y_{m(n)}\},\\ \{v_n, Tv_n, T^*v_n\} \perp&\{z_n, Tz_n, T^*z_n, b_n, Tb_n, T^*b_n\},\\ \langle Tv_n, v_n\rangle =&\lambda_n',\end{aligned}$$

and the vectors

$$w_n = Tv_n - \langle Tv_n, v_n\rangle v_n \qquad \text{and} \qquad w_n' = T^*v_n - \langle T^*v_n, v_n\rangle v_n$$

satisfy condition (ii) of Lemma 3.1.

Note that w_n and w_n' are orthogonal to $\bigvee\{z_n, b_n\}$ and similarly Tw_n, T^*w_n, Tw_n', and T^*w_n' are orthogonal to $\bigvee\{z_n, b_n\}$.

Let

$$u_n = \sqrt{1-\|z_n+b_n\|^2}v_n + z_n + b_n.$$

Since $v_n \perp z_n, b_n$, we have $\|u_n\| = 1$.

By definition, $v_n \perp M_{n-1}$ and $b_n \perp M_{n-1}$. Moreover,

$$z_n \in \bigvee\{w_{n-1}, w_{n-1}'\} \subset \{u_1, \dots, u_{n-2}\}^\perp.$$

Furthermore, $w_{n-1}, w_{n-1}' \perp v_{n-1}$ and

$$\{w_{n-1}, w_{n-1}'\} \subset \bigvee\{Tv_{n-1}, T^*v_{n-1}, v_{n-1}\} \subset \{z_{n-1}, b_{n-1}\}^\perp.$$

So $z_n \perp u_{n-1}$ and $u_n \perp u_1, \dots, u_{n-1}$.

Moreover,

$$Tz_n \in \bigvee\{Tw_{n-1}, Tw_{n-1}' \subset \bigvee\{Tv_{n-1}, T^2v_{n-1}, TT^*v_{n-1}\} \subset \{z_n, b_n\}^\perp.$$

We have

$$\langle Tu_n, u_n\rangle = (1-\|z_n+b_n\|^2)\langle Tv_n, v_n\rangle + \big\langle T(z_n+b_n), z_n+b_n\big\rangle$$
$$= (1-\|z_n+b_n\|^2)\lambda'_n + \big\langle T(z_n+b_n), z_n+b_n\big\rangle = \lambda_n.$$

Moreover,

$$\begin{aligned}\langle Tu_{n-1}, u_n\rangle =&\langle Tu_{n-1}, z_n\rangle + \langle Tu_{n-1}, b_n\rangle\\ =&\sqrt{1-\|z_{n-1}+b_{n-1}\|^2}\langle Tv_{n-1}, z_n\rangle + \langle Tu_{n-1}, b_n\rangle\\ =&\sqrt{1-\|z_{n-1}+b_{n-1}\|^2}\langle w_{n-1}, z_n\rangle + \langle Tu_{n-1}, b_n\rangle\\ =&\mu_{n-1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle Tu_n, u_{n-1}\rangle =&\langle Tz_n, u_{n-1}\rangle + \langle Tb_n, u_{n-1}\rangle\\ =&\sqrt{1-\|z_{n-1}+b_{n-1}\|^2}\langle Tz_n, v_{n-1}\rangle + \langle Tb_n, u_{n-1}\rangle\\ =&\sqrt{1-\|z_{n-1}+b_{n-1}\|^2}\langle z_n, w'_{n-1}\rangle + \langle Tb_n, u_{n-1}\rangle\\ =&\nu_{n-1}.\end{aligned}$$

We have

$$\begin{aligned}\big|\langle (I-P_{n-1})y_{m(n)}, u_n\rangle\big| =&\big|\langle (I-P_{n-1})y_{m(n)}, z_n+b_n\rangle\big|\\ \ge&\big|\langle (I-P_{n-1})y_{m(n)}, b_n\rangle\big| - \big|\langle (I-P_{n-1})y_{m(n)}, z_n\rangle\big|\\ \ge&\|(I-P_{n-1})y_{m(n)}\|\cdot\frac{\varepsilon\sqrt{\varepsilon}}{32}\\ &-\|(I-P_{n-1})y_{m(n)}\|\cdot\|z_n\|\cdot\frac{\varepsilon}{32}\\ \ge&\|(I-P_{n-1})y_{m(n)}\|\cdot\frac{\varepsilon\sqrt{\varepsilon}}{64}.\end{aligned}$$

So

$$\begin{aligned}\|(I-P_n)y_{m(n)}\|^2 =&\|(I-P_{n-1})y_{m(n)}\|^2 - |\langle (I-P_{n-1})y_{m(n)}, u_n\rangle|^2\\ \le&\|(I-P_{n-1})y_{m(n)}\|^2\Big(1-\frac{\varepsilon^3}{2^{12}}\Big).\end{aligned}$$

Clearly

$$\|(I-P_n)y_m\| \le \|(I-P_{n-1})y_m\|$$

for all $m\in\mathbb{N}$, $m\ne m(n)$.

Constructing the vectors u_n inductively, we obtain an orthonormal system $(u_n)_{n=1}^\infty$ satisfying (1), (2) and (3). It remain to show that it is a basis, i.e., that $\bigvee_{j=1}^\infty u_j$ contains all vectors $y_m, m\in\mathbb{N}$. This is clearly true if y_m belongs to the space M_n for some n. If $y_m\notin\bigcup_{n=1}^\infty M_n$, then $m(n)=m$ for infinitely values of n. Hence

$$\lim_{n\to\infty}\|(I-P_n)y_m\|^2 \le \lim_{r\to\infty}\Big(1-\frac{\varepsilon^3}{2^{12}}\Big)^r = 0,$$

so that $y_m\in\bigvee_{n=1}^\infty u_n$.

Thus $(u_n)_{n=1}^{\infty}$ is an orthonormal basis satisfying conditions (1),(2) and (3).

4. Matrices with small entries: proofs

In this section, we extend Stout's bound from Theorem 1.3, (iii) by providing a similar bound for all matrix elements for $T \in B(H)$ rather than merely the diagonal of T for an appropriate basis in H. To this aim, we need a simple lemma.

Lemma 4.1. *Let $(a_j)_{j=1}^{\infty}$ be a sequence of positive numbers which does not belong to ℓ^1. Then there exists a sequence $(a'_j)_{j=1}^{\infty}$ which does not belong to $\ell^1(\mathbb{N})$ such that $0 < a'_j \le \max\{1, a_j\}$ for all $j \in \mathbb{N}$ and $\lim_{j\to\infty} \frac{a'_j}{a_j} = 0$.*

Proof. Set $n_0 = 0$. We construct numbers $n_k, k \in \mathbb{N}$ inductively.

If $k \in \mathbb{N}$ and the numbers $n_0, n_1, \dots, n_{k-1}$ have already been constructed, then find $n_k > n_{k-1}$ such that

$$\sum_{j=n_{k-1}+1}^{n_k} a_j \ge k.$$

For $n_{k-1} + 1 \le j \le n_k$ then set $a'_j = \min\{1, k^{-1} a_j\}$. So $\sum_{j=n_{k-1}+1}^{n_k} a'_j \ge 1$.

If the numbers a'_j are constructed in this way, then clearly $\sum_{j=1}^{\infty} a'_j = \infty$ and $\lim_{j\to\infty} \frac{a'_j}{a_j} = 0$. □

Proof of Theorem 2.5 Without loss of generality we may assume that $\|T\| \le 1$.

Using Lemma 4.1, find a sequence $(a'_j)_{n=1}^{\infty}$ such that $0 < a'_n \le \min\{1, a_j\}$ for all n, $(a'_j)_{n=1}^{\infty} \notin \ell^1$ and $\lim_{n\to\infty} \frac{a'_j}{a_j} = 0$.

Let $(y_m)_{m\in\mathbb{N}}$ be a sequence of unit vectors in H such that $\bigvee_{m\in\mathbb{N}} y_m = H$.

Find mutually disjoint sets $A_m \subset \mathbb{N}$ such that $\bigcup_{m\in\mathbb{N}} A_m = \mathbb{N}$ and

$$\sum_{j\in A_m} a'_j = \infty$$

for all $m \in \mathbb{N}$. We may assume that $m \in \bigcup_{k=1}^{m} A_k$ for all m.

For $n \in \mathbb{N}$ denote by $m(n)$ the uniquely determined integer satisfying $n \in A_{m(n)}$.

Define also

$$d(n) := \min\left\{ r \in \mathbb{N}, r \ge n : \frac{a'_k}{a_k} < a_j \text{ for all } k \ge r \right\}.$$

So $m(n) \le n \le d(n)$ for all $n \in \mathbb{N}$.

We construct the orthonormal basis $(u_n)_{n=1}^{\infty}$ inductively. Let $n \ge 1$ and suppose that the vectors $u_1, u_2, \dots, u_{n-1} \in H$ have already been constructed.

Since $0 \in W_e(T)$, there exists a unit vector $v_n \in H$ orthogonal to the union of the sets

$$\{u_j, Tu_j, T^*u_j : j = 1, \dots, n-1\}$$

and

$$\{y_j, Ty_j, T^*y_j : j = 1, 2, \dots, d(n)\},$$

and such that

$$|\langle Tv_n, v_n\rangle| < \frac{a'_n}{2}.$$

Let P_{n-1} be the orthogonal projection onto the subspace $M_{n-1} := \bigvee_{j=1}^{n-1} u_j$. If $y_{m(n)} \in M_{n-1}$ then set $u_n = v_n$. Suppose that $y_{m(n)} \notin M_{n-1}$. Let

$$w_n = \frac{(I - P_{n-1})y_{m(n)}}{\|(I - P_{n-1})y_{m(n)}\|} \quad \text{and} \quad c_n = \frac{\sqrt{a'_n}}{2}.$$

Set

$$u_n = \sqrt{1 - c_n^2}\, v_n + c_n w_n.$$

Note that $w_n \in \bigvee\{u_j : j \leq n-1\} \vee \{y_{m(n)}\}$. So $w_n \perp v_n$, and similarly, $Tv_n \perp w_n$ and $T^*v_n \perp w_n$. In particular, $\|u_n\| = 1$ and $u_n \perp u_1, \dots, u_{n-1}$. Moreover,

$$\begin{aligned} \|(I - P_n)y_{m(n)}\|^2 =& \|(I - P_{n-1})y_{m(n)}\|^2 - |\langle (I - P_{n-1})y_{m(n)}, u_n\rangle|^2 \\ =& \|(I - P_{n-1})y_{m(n)}\|^2 - |\langle (I - P_{n-1})y_{m(n)}, c_n w_n\rangle|^2 \\ =& \|(I - P_{n-1})y_{m(n)}\|^2 (1 - c_n)^2 \\ =& \|(I - P_{n-1})y_{m(n)}\|^2 \left(1 - \frac{a'_n}{4}\right). \end{aligned} \tag{4.1}$$

We have

$$\begin{aligned} |\langle Tu_n, u_n\rangle| =& \left|(1 - c_n^2)\langle Tv_n, v_n\rangle + c_n^2 \langle Tw_n, w_n\rangle\right| \\ \leq& |\langle Tv_n, v_n\rangle| + c_n^2 \\ \leq& \frac{a'_n}{2} + \frac{a'_n}{4} \\ <& a_n. \end{aligned}$$

To estimate the non-diagonal terms, let $j \leq n-1$. We distinguish two cases:

1) If $n \geq d(j)$, then

$$|\langle Tu_j, u_n\rangle = |\langle Tu_j, c_n w_n\rangle| \leq c_n = \frac{\sqrt{a'_n}}{2} \leq \frac{\sqrt{a_n a_j}}{2} < \sqrt{a_n a_j}.$$

Similarly,

$$|\langle Tu_n, u_j\rangle| = |\langle T^*u_j, u_n\rangle| < \sqrt{a_n a_j},$$

so that the bound (2.8) holds in this case.

2) If the opposite case $n < d(j)$ holds. Then

$$\begin{aligned}|\langle Tu_j, u_n\rangle| =& c_n|\langle Tu_j, w_n\rangle| \\ \le& c_n|\langle Tv_j, w_n\rangle| + c_n c_j|\langle Tw_j, w_n\rangle| \\ \le& c_n|\langle Tv_j, w_n\rangle| + c_n c_j.\end{aligned}$$

Let L be the subspace generated by the vectors $\{v_n : 1 \le n \le n-1\}$ and $y_{m(1)}, \dots, y_{m(n)}$. We have

$$w_n \in \bigvee\{u_1, \dots, u_{n-1}, y_{m(n)}\} \subset \{v_1, \dots, v_{n-1}, w_1, \dots, w_{n-1}, y_{m(n)}\}.$$

By induction, $w_n \in L$.

Note that the vectors $v_1, \dots, v_{n-1}$ are mutually orthogonal. Indeed, if $k < k' \le j-1$, then by construction, we have $v_{k'} \perp u_1, \dots, u_k$ and $v_{k'} \perp y_{m(k)}$, since $m(k) \le k < k' \le d(k')$. Moreover, v_k is a linear combination of $u_1, \dots, u_k, y_{m(k)}$. So $v_k \perp v_{k'}$.

Similarly, $Tv_k \perp v_{k'}$ and $T^* v_k \perp v_{k'}$.

Thus the subspace $L \cap \{v_j\}^\perp$ contains the vectors $v_k, 1 \le k \le n-1, k \ne j$, and the vectors $y_{m(1)}, \dots, y_{m(n)}$ since $m(k) \le k \le n < d(j)$ for all $k \le n$.

Similarly, $Tv_j \perp y_{m(k)}$ for any k such that $1 \le k \le n$.

Hence $Tv_j \perp (L \cap \{v_j\}^\perp)$.

Therefore,

$$|\langle Tv_j, w_n\rangle| = |\langle Tv_j, v_j\rangle| \cdot |\langle w_n, v_j\rangle| \le |\langle Tv_j, v_j\rangle| \le \frac{a'_j}{2},$$

and then

$$|\langle Tu_j, u_n\rangle| \le \frac{c_n a'_j}{2} + c_n c_j \le \frac{\sqrt{a'_n} \cdot a'_j}{4} + \frac{\sqrt{a'_n a'_j}}{4} < \sqrt{a'_n a'_j} \le \sqrt{a_n a_j}.$$

Similarly,

$$|\langle Tu_n, u_j\rangle| = |\langle T^* u_j, u_n\rangle| \le \sqrt{a_n a_j}.$$

Thus, the two estimates above imply that (2.8) holds in the second case too.

Constructing the vectors $u_n, n \in \mathbb{N}$, in this way, we get the orthonormal system $(u_n)_{n=1}^\infty$ such that

$$|\langle Tu_n, u_j\rangle| \le \sqrt{a_n a_j}$$

for all $n, j \in \mathbb{N}$.

We claim that, moreover, $(u_n)_{n=1}^\infty$ is an (orthonormal) basis of H. Let $m \in \mathbb{N}$ be fixed.

We have

$$\|(I - P_k)y_m\| \le \|(I - P_{k-1})y_m\|$$

for all $k \in \mathbb{N}$, and taking into account (4.1),

$$\|(I - P_k)y_m\|^2 \le \|(I - P_{k-1})y_m\|^2 \Big(1 - \frac{a'_k}{4}\Big)$$

for all $k \in A_m$. So

$$\lim_{k\to\infty} \|(I - P_k)y_m\|^2 \le \prod_{k\in A_m} \Big(1 - \frac{a'_k}{4}\Big)$$

and

$$\lim_{k\to\infty} \ln \|(I - P_k)y_m\|^2 \le \sum_{k\in A_m} \ln\Big(1 - \frac{a'_k}{4}\Big) \le -\sum_{k\in A_m} \frac{a'_k}{4} = -\infty.$$

Hence $y_m \in \bigvee\{u_n : n \in \mathbb{N}\}$. Since $\bigvee_{m\in\mathbb{N}} y_m = H$, the claim follows. This finishes the proof. □

5. Matrices with large entries: proofs

This section is devoted to the proof of Theorem 2.8.

We will need the next Lemma similar in spirit to considerations in [9, Section 2], see also [43]. It is probably of independent interest.

Recall that $T \in B(H)$ is compact if and only if $W_e(T) = \{0\}$. So T is of the form $T = \lambda I + K$ for some $\lambda \in \mathbb{C}$ and a compact operator $K \in B(H)$ if and only if $W_e(T)$ is a singleton, i.e., the diameter

$$\operatorname{diam} W_e(T) = \max\{|\lambda - \mu| : \lambda, \mu \in W_e(T)\} = 0.$$

Lemma 5.1. *Let $T \in B(H)$ be an operator which is not of the form $T = \lambda I + K$ for some $\lambda \in \mathbb{C}$ and a compact operator $K \in B(H)$. Then $\operatorname{diam} W_e(T) > 0$. Let $0 < C < \frac{\operatorname{diam} W_e(T)}{4}$ and $0 < D < \frac{\operatorname{diam} W_e(T)}{2\sqrt{2}}$. Then for any subspace $M \subset H$ of finite codimension there exists a unit vector $u \in M$ such that*

$$|\langle Tu, u\rangle| \ge D,$$
$$\|Tu - \langle Tu, u\rangle u\| \ge C$$

and

$$\|T^*u - \langle T^*u, u\rangle u\| \ge C.$$

Proof. Since $T \ne \lambda I + K$, the set $W_e(T)$ contains at least two points. Let $\lambda, \nu \in W_e(T)$ satisfy $|\lambda - \nu| = \operatorname{diam} W_e(T)$. Without loss of generality, we may assume that $|\lambda| \ge |\nu|0$. Set $\mu = \frac{\lambda+\nu}{2}$. Then $\mu \in W_e(T)$ since it is a convex set.

We have

$$|\lambda - \mu| = \frac{|\lambda - \nu|}{2} = \frac{\operatorname{diam} W_e(T)}{2}$$

and

$$|\lambda + \mu| = |\frac{3}{2}\lambda + \frac{1}{2}\nu| \ge |\lambda| \ge \frac{\operatorname{diam} W_e(T)}{2}.$$

Let C and D such that

$$0 < D < \frac{\operatorname{diam} W_e(T)}{2} \qquad \text{and} \qquad 0 < C < \frac{\operatorname{diam} W_e(T)}{2\sqrt{2}}.$$

Let $M \subset H$ be a subspace of a finite codimension.

Let $\varepsilon > 0$ satisfy $\varepsilon < \frac{|\lambda+\mu|}{2} - D$ and $\varepsilon < \frac{|\lambda-\mu|}{2} - C\sqrt{2}$.

Find a unit vector $x \in M$ such that $|\langle Tx, x\rangle - \lambda| < \varepsilon$. Let $M' = M \cap \{x, Tx, T^*x\}^\perp$. Then $\operatorname{codim} M' < \infty$ and there exits a unit vector $y \in M'$ such that $|\langle Ty, y\rangle - \mu| < \varepsilon$.

Set
$$u = \frac{x+y}{\sqrt{2}}.$$
Clearly $u \in M$. Since $y \perp x$, we have $\|u\| = 1$.

We have
$$|\langle Tu, u\rangle| = \frac{|\langle Tx, x\rangle + \langle Ty, y\rangle|}{2} \geq \frac{|\lambda+\mu|}{2} - \varepsilon > D.$$
Furthermore,
$$\begin{aligned}\|Tu - \langle Tu, u\rangle u\| \geq & \big|\langle Tu, x\rangle - \langle Tu, u\rangle \cdot \langle u, x\rangle\big| \geq \Big|\frac{\langle Tx, x\rangle}{\sqrt{2}} - \frac{\langle Tu, u\rangle}{\sqrt{2}}\Big| \\ \geq & \frac{1}{\sqrt{2}} \frac{\big|\langle Tx, x\rangle - \langle Ty, y\rangle\big|}{2} \geq \frac{|\lambda-\mu|}{2\sqrt{2}} - \frac{\varepsilon}{\sqrt{2}} > C,\end{aligned}$$
and similarly,
$$\begin{aligned}\|T^*u - \langle T^*u, u\rangle u\| \geq & \big|\langle T^*u, x\rangle - \langle T^*u, u\rangle \cdot \langle u, x\rangle\big| \geq \Big|\frac{\langle T^*x, x\rangle}{\sqrt{2}} - \frac{\langle T^*u, u\rangle}{\sqrt{2}}\Big| \\ \geq & \frac{1}{\sqrt{2}} \frac{\big|\langle T^*x, x\rangle - \langle T^*y, y\rangle\big|}{2} \geq \frac{|\bar\lambda-\bar\mu|}{2\sqrt{2}} - \frac{\varepsilon}{\sqrt{2}} > C.\end{aligned}$$
□

Next we proceed with the proof of Theorem 2.8 producing large matrix entries for $T \in B(H)$ with $W_e(T)$ containing more than two points.

Proof of Theorem 2.8 Without loss of generality we may assume that $\|T\| = 1$.

Fix a sequence $(y_m)_{m=0}^\infty$ of unit vectors in H such that $\bigvee_{m=1}^\infty y_m = H$.

For $n \in \mathbb{N}$ denote by $m(n)$ the unique non-negative integer such that $n = 2^{m(n)}(2k-1)$ for some $k \in \mathbb{N}$.

Let C and D be the constants given by Lemma 5.1, i.e., for every subspace $M \subset H$ of finite codimension there exists a unit vector $u \in M$ with
$$|\langle Tu, u\rangle| \geq D, \quad \|Tu - \langle Tu, u\rangle u\| \geq C \quad \text{and} \quad \|T^*u - \langle T^*u, u\rangle u\| \geq C,$$
where C and D are independent of M.

Let $d = \frac{D}{2}$. Fix $a \geq 1$ such that
$$\frac{4}{a} \leq D \qquad \text{and} \qquad \frac{54}{\sqrt{a}} \leq C^2.$$
Set
$$c_1 = \frac{C}{3\sqrt{a}} \qquad \text{and} \qquad c_2 = \frac{1}{\sqrt{a}}.$$

We construct the vectors $u_n, n \geq 1$, inductively. Let $n \geq 1$ and suppose we have constructed mutually orthogonal vectors $u_1, \ldots, u_{n-1}$ satisfying

$$|\langle Tu_k, u_k\rangle| \geq d, \qquad (1 \leq k \leq n), \tag{5.1}$$

and

(5.2)
$$\frac{c_1 \min\{n,j\}^{1/2}}{\max\{n,j\}^{3/2}} \leq |\langle Tu_n, u_j\rangle| \leq \frac{c_2}{\max\{n,j\}^{1/2}} \qquad (1 \leq k, j \leq n, k \neq j)$$

as well as,

$$\|(I - P_k)Tu_j\|^2 \geq \frac{C^2 j}{2k}, \qquad 1 \leq j \leq k \leq n-1, \tag{5.3}$$

$$\|(I - P_k)T^*u_j\|^2 \geq \frac{C^2 j}{2k}, \qquad 1 \leq j \leq k \leq n-1, \tag{5.4}$$

and

$$\|(I - P_{n-1})y_m\|^2 \leq \prod_{\substack{j \leq n-1 \\ m(j)=m}} \Big(1 - \frac{1}{2aj}\Big), \qquad m \geq 0, \tag{5.5}$$

where $P_k, 1 \leq k \leq n-1$, is the orthogonal projection onto the subspace $M_k := \bigvee\{u_1, \ldots, u_k\}$. (Note that if $2^m > n-1$ then the product in (5.5) is over the empty set and, by definition, it is equal to 1. The inequality $\|(I - P_{n-1})y_m\|^2 \leq \|y_m\|^2 = 1$ is then satisfied automatically).

We construct the vector u_n satisfying (5.1)–(5.5) in the following way.

Using Lemma 5.1, find a unit vector

$$v_n \perp \big\{u_j, Tu_j, T^*u_j, y_{m(j)},\ 1 \leq j \leq n-1; Ty_{m(j)}, T^*y_{m(j)},\ 1 \leq j \leq n\big\}$$

such that

$$\begin{aligned} |\langle Tv_n, v_n\rangle| \geq& D, \\ \|Tv_n - \langle Tv_n, v_n\rangle v_n\| \geq& C, \end{aligned}$$

and

$$\|T^*v_n - \langle T^*v_n, v_n\rangle v_n\| \geq C.$$

Consider now the $(2n-1)$-tuple of vectors $(I - P_{n-1})Tu_j$, $(I - P_{n-1})T^*u_j$, $j = 1, \ldots, n-1$, and $(I - P_{n-1})y_{m(n)}$.

By Theorem 2.1, there exists a unit vector

$$z_n \in \bigvee \big\{(I - P_{n-1})Tu_j, (I - P_{n-1})T^*u_j,\ 1 \leq j \leq n-1; (I - P_{n-1})y_{m(n)}\big\}$$

such that

$$\begin{aligned} \big|\langle (I - P_{n-1})y_{m(n)}, z_n\rangle\big| \geq& \frac{1}{\sqrt{2}} \cdot \|(I - P_{n-1})y_{m(n)}\|, \\ \big|\langle (I - P_{n-1})Tu_j, z_n\rangle\big| \geq& \frac{1}{2\sqrt{n}} \cdot \|(I - P_{n-1})Tu_j\|, \\ \big|\langle (I - P_{n-1})T^*u_j, z_n\rangle\big| \geq& \frac{1}{2\sqrt{n}} \cdot \|(I - P_{n-1})T^*u_j\|. \end{aligned}$$

for all $1 \le j \le n-1$ (since $\left(\frac{1}{\sqrt{2}}\right)^2 + 2(n-1)\left(\frac{1}{2\sqrt{n}}\right)^2 < 1$).

Note that $z_n \in M_{n-1}^\perp$ and $v_n \perp z_n$.

Set

$$u_n = \frac{1}{\sqrt{an}} z_n + \sqrt{1 - \frac{1}{an}} v_n.$$

Clearly $\|u_n\| = 1$ and $u_n \perp u_1, \dots, u_{n-1}$.

Let us show that u_n satisfies conditions (5.1)–(5.5).

We have

$$\begin{aligned} |\langle Tu_n, u_n\rangle| =& \left|\left(1 - \frac{1}{an}\right)\langle Tv_n, v_n\rangle + \frac{1}{an}\langle Tz_n, z_n\rangle\right| \\ \ge& |\langle Tv_n, v_n\rangle| - \frac{1}{an} - \frac{1}{an} \ge |\langle Tv_n, v_n\rangle| - \frac{2}{an} \\ \ge& D - \frac{2}{a} \ge \frac{D}{2} = d. \end{aligned}$$

So u_n satisfies (1).

For $j = 1, \dots, n-1$ we have

$$\begin{aligned} |\langle Tu_j, u_n\rangle| =& \frac{1}{\sqrt{an}} |\langle Tu_j, z_n\rangle| = \frac{1}{\sqrt{an}} \left|\langle (I - P_{n-1})Tu_j, z_n\rangle\right| \\ \ge& \frac{1}{\sqrt{an}} \cdot \frac{1}{2\sqrt{n}} \cdot \|(I - P_{n-1})Tu_j\| \\ \ge& \frac{1}{2n\sqrt{a}} \cdot \frac{Cj^{1/2}}{\sqrt{2(n-1)}} \end{aligned}$$

by the induction assumption. So

$$|\langle Tu_j, u_n\rangle| \ge \frac{Cj^{1/2}}{3\sqrt{a}n^{3/2}} = \frac{c_1 j^{1/2}}{n^{3/2}}.$$

Obviously

$$|\langle Tu_j, u_n\rangle| = \frac{1}{\sqrt{an}} |\langle Tu_j, z_n\rangle| \le \frac{1}{\sqrt{an}} = \frac{c_2}{n^{1/2}}.$$

The inequalities

$$\frac{c_1 j^{1/2}}{n^{3/2}} \le |\langle T^* u_j, u_n\rangle| = |\langle Tu_n, u_j\rangle| \le \frac{c_2}{n^{1/2}}$$

for $j = 1, \dots, n-1$ can be proved analogously. So u_n satisfies (5.2).

Let $j = 1, \ldots, n-1$. We have

$$\begin{aligned}\|(I-P_n)Tu_j\|^2 =&\big\|(I-P_{n-1})Tu_j\big\|^2 - \big|\langle (I-P_{n-1})Tu_j, u_n\rangle\big|^2 \\ =&\|(I-P_{n-1})Tu_j\|^2 - \frac{1}{an}\big|\langle (I-P_{n-1})Tu_j, z_n\rangle\big|^2 \\ \geq&\|(I-P_{n-1})Tu_j\|^2\Big(1-\frac{1}{an}\Big) \geq \frac{C^2 j}{2(n-1)}\cdot\frac{an-1}{an} \\ \geq&\frac{C^2 j}{2(n-1)}\cdot\frac{n-1}{n} = \frac{C^2 j}{2n}.\end{aligned}$$

Similarly one can prove that

$$\|(I-P_n)T^*u_j\|^2 = \big\|(I-P_{n-1})T^*u_j\big\|^2 - \big|\langle (I-P_{n-1})T^*u_j, u_n\rangle\big|^2 \geq \frac{C^2 j}{2n}.$$

To prove (5.3), it remains to estimate

$$\|(I-P_n)Tu_n\|^2 = \|(I-P_{n-1})Tu_n\|^2 - |\langle (I-P_{n-1})Tu_n, u_n\rangle|^2.$$

We have

$$\begin{aligned}&\|(I-P_{n-1})Tu_n\|^2 \\ =&\Big\|\sqrt{1-\frac{1}{an}}Tv_n + (I-P_{n-1})Tz_n\Big\|^2 \\ \geq&\Big(1-\frac{1}{an}\Big)\|Tv_n\|^2 - \frac{2\sqrt{1-\frac{1}{an}}}{\sqrt{an}}\|Tv_n\|\cdot\|Tz_n\| + \frac{1}{an}\|(I-P_{n-1})Tz_n\|^2 \\ \geq&\|Tv_n\|^2 - \frac{1}{an} - \frac{2}{\sqrt{an}} \geq \|Tv_n\|^2 - \frac{3}{\sqrt{an}} \\ \geq&\|Tv_n\|^2 - \frac{3}{\sqrt{a}}\end{aligned}$$

and

$$\begin{aligned}&\big|\langle (I-P_{n-1})Tu_n, u_n\rangle\big| = |\langle Tu_n, u_n\rangle| \\ =&\Big|\Big(1-\frac{1}{an}\Big)\langle Tv_n, v_n\rangle + \frac{\sqrt{1-\frac{1}{an}}}{\sqrt{an}}\Big(\langle Tv_n, z_n\rangle + \langle Tz_n, v_n\rangle\Big) + \frac{1}{an}\langle Tz_n, z_n\rangle\Big| \\ \leq&|\langle Tv_n, v_n\rangle| + \frac{1}{an} + \frac{2}{\sqrt{an}} + \frac{1}{an} \\ \leq&|\langle Tv_n, v_n\rangle| + \frac{4}{\sqrt{a}}.\end{aligned}$$

So

$$\begin{aligned}\|(I-P_n)Tu_n\|^2 =&\|(I-P_{n-1})Tu_n\|^2 - \bigl|\langle (I-P_{n-1})Tu_n,u_n\rangle\bigr|^2\\ \geq&\|Tv_n\|^2-\frac{3}{\sqrt{a}}-\bigl(|\langle Tv_n,v_n\rangle|+\frac{4}{\sqrt{a}}\bigr)^2\\ \geq&\bigl(\|Tv_n\|^2-|\langle Tv_n,v_n\rangle|^2\bigr)-\frac{3}{\sqrt{a}}-\frac{8}{\sqrt{a}}-\frac{16}{a}\geq C^2-\frac{27}{\sqrt{a}}\geq\frac{C^2}{2}.\end{aligned}$$

The inequality

$$\|(I-P_n)T^*u_n\|^2\geq\frac{C^2}{2}$$

can be proved similarly. So u_n satisfies (5.3) and (5.4).

Inequality (5.5) is trivial for all $m\neq m(n)$. For $m(n)$ we have

$$\begin{aligned}\|(I-P_n)y_{m(n)}\|^2 =&\|(I-P_{n-1})y_{m(n)}\|^2-\bigl|\langle (I-P_{n-1})y_{m(n)},u_n\rangle\bigr|^2\\ =&\|(I-P_{n-1})y_{m(n)}\|^2-\frac{1}{an}\bigl|\langle (I-P_{n-1})y_{m(n)},z_n\rangle\bigr|^2\\ \leq&\|(I-P_{n-1})y_{m(n)}\|^2\Bigl(1-\frac{1}{an}\Bigr),\end{aligned}$$

which is (5.5).

So the vectors $u_1,\dots,u_n$ satisfy all conditions (1)–(5).

If we continue the construction inductively, we construct an orthonormal system $(u_n)_{n=1}^\infty$ satisfying (1) and (2). It remains to show that (u_n) is a basis.

Let $m\geq 0$. We have

$$\begin{aligned}\lim_{n\to\infty}\ln\|(I-P_n)y_m\|^2\leq&\lim_{n\to\infty}\sum_{\substack{j\leq n\\ m(j)=m}}\ln\Bigl(1-\frac{1}{aj}\Bigr)\\ =&\lim_{k\to\infty}\sum_{j=1}^{k}\ln\Bigl(1-\frac{1}{a\cdot 2^m(2j-1)}\Bigr)\\ \leq&-\sum_{j=1}^{\infty}\frac{1}{2^m a(2j-1)}=-\infty.\end{aligned}$$

Hence

$$\lim_{n\to\infty}\|(I-P_n)y_m\|^2=0$$

and $y_m\in\bigvee_{j=1}^\infty u_j$. Since $\bigvee_{m=0}^\infty y_m=H$, we conclude that $(u_n)_{n=1}^\infty$ is an orthonormal basis.

This finishes the proof. □

6. Final remarks

Note that Theorems 2.2, 2.4 [IS IT SO ???] and 2.5 have their counterparts for tuples of bounded linear operators $\mathcal{T}=(T_1,\dots,T_n)\in B(H)^n$ and for tuples of selfadjoint operators if one replaces the interior of the essential numerical range with an appropriate relative interior. We have decided to present their single operator versions to simplify the presentation and to illustrate the method rather than its technicalities.

Recall that for $\mathcal{T} \in B(H)^n$ the essential numerical range $W_e(\mathcal{T})$ of $\mathcal{T}$ is defined as the set of all n-tuples $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that there exists an orthonormal sequence $(u_j)_{j=1}^\infty$ in H satisfying

$$\lim_{j\to\infty} \langle T_k u_j, u_j\rangle = \lambda_k$$

for all $k = 1, \dots, n$. This definition is completely analogous to the one given in Section 2 for $n = 1$. Arguing as in the case $n = 1$, cf. [40] for a "tuple argument", one gets the following statement.

Theorem 6.1. *Let $\mathcal{T} \in B(H)^n$, let $(\lambda)_{n=1}^\infty \subset \operatorname{Int} W_e(\mathcal{T})$ be such that $\sum_{n=1}^\infty \operatorname{dist}\{\lambda_n, \partial W_e(\mathcal{T})\} = \infty$, and let $K \in \mathbb{N}$ be fixed. Then there exists an orthonormal basis $(u_n)_{n=1}^\infty \subset H$ such that*

$$\langle \mathcal{T} u_n, u_n\rangle = \lambda_n, \qquad n \in \mathbb{N},$$

and

$$\langle \mathcal{T} u_j, u_k\rangle = 0, \qquad 1 \le |j-k| \le K.$$

The formulation of a tuples analogue of Theorem 2.5 is also straightforward.

Theorem 6.2. *Let $\mathcal{T} \in B(H)^n$ be such that $0 \in W_e(\mathcal{T})$, and let $(a_j)_{j=1}^\infty \subset \mathbb{R}_+$ satisfy $(a_j)_{j=1}^\infty \notin \ell^1$. Then there exists an orthonormal basis $(u_n)_{n=1}^\infty$ in H such that*

$$|\langle \mathcal{T} u_n, u_j\rangle| \le \sqrt{a_n a_j}$$

for all $n, j \in \mathbb{N}$.

[SHOULD ONE ALSO GIVE AN EXAMPLE OF THEOREM WITH RELATIVE INTERIOR ? COULD BE A BIT LESS TRIVIAL]

Remarks on relations to the Kadison-Singer problem.

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