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**In search of convexity:
Diagonals and numerical ranges**

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IN SEARCH OF CONVEXITY: DIAGONALS AND NUMERICAL RANGES

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ABSTRACT. We show that the set of all possible constant diagonals of a bounded Hilbert space operator is always convex. This, in particular, answers an open question of J.-C. Bourin (2003). Moreover, we show that the joint numerical range of a commuting operator tuple is, in general, not convex, which fills a gap in the literature. We also prove that the Asplund-Ptak numerical range (which is convex for pairs of operators) is, in general, not convex for tuples of operators.

1. INTRODUCTION

Let H be a separable (complex) Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let T be a bounded linear operator acting on H . By the classical result of Hausdorff and Toeplitz, the numerical range $W(T) = \{\langle Tx, x \rangle : x \in H, \|x\| = 1\}$ is always a convex set. This is one of the most important properties of the numerical range and convexity of various sets related to the numerical range is the basic issue in the theory, and underlying many of its developments.

Unfortunately, the convexity of $W(T)$ fails if a single operator T is replaced by a tuple $\mathcal{T} = (T_1, \dots, T_n)$ of bounded linear operators on H . It is well known that the joint numerical range

$$W(\mathcal{T}) := W(T_1, \dots, T_n) := \{(\langle T_1 x, x \rangle, \dots, \langle T_n x, x \rangle) : x \in H, \|x\| = 1\}$$

is, in general, not convex for $n \geq 2$. Apparently, Hausdorff knew this already in 1918; for a simple example see e.g. [8, p. 138] or [27]. However, $W(\mathcal{T})$ still have some traces of convexity. In particular, as shown in [29], the set $W(\mathcal{T})$ is star-shaped if $\dim H \geq \lfloor \frac{2n+1}{2} \rfloor (2n+1)^2$, where $\lfloor \cdot \rfloor$ stands for the integer part. Thus $W(\mathcal{T})$ is always star-shaped if $\dim H = \infty$, [29, Proposition 4.1]. (To relate $W(\mathcal{T})$ to the setting of [29], one should identify $W(\mathcal{T})$ with $W(\operatorname{Re} T_1, \operatorname{Im} T_1, \dots, \operatorname{Re} T_n, \operatorname{Im} T_n) \subset \mathbb{R}^{2n}$.)

In this note we study the convexity of numerical ranges in three related situations, which surprisingly escaped the attention of experts. First, we

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show that the joint numerical range of commuting tuples is not necessarily convex. Such an example seems not to exist in the literature. Next we demonstrate that the version of the numerical range introduced by E. Asplund and V. Ptak [1], which is convex for all pairs of Hilbert space operators, is in general not convex for triples of operators even if the operators commute.

Finally, we address the convexity of the set of constant diagonals of operators and operator tuples. The study of the structure for diagonals of operators in infinite dimensions has a long history, and we refer to the recent survey [33] for its finer details. In the beginning of 2000's it received an impetus due to the works by R. Kadison and W. Arveson, and has attracted a considerable attention over the last years. For a good introduction into Kadison's theory one may consult [2], see also [3]. For a comprehensive account of the latest developments in this developing area of research, see again [33].

For $T \in B(H)$ acting on a separable space H , its set of diagonals is defined as

$$\mathcal{D}(T) := \{(\langle Te_n, e_n \rangle)_{n=1}^N\}$$

when $(e_n)_{n=1}^N$ varies through all orthonormal bases of H and $N = \dim H$. While $\mathcal{D}(T)$ is rarely convex as a subset of l^∞ (see Section 4), we show that the set of all $\lambda \in \mathbb{C}$ such that $\langle Te_n, e_n \rangle = \lambda, 1 \leq n \leq N$, for some orthonormal basis $(e_n)_{n=1}^N$ in H is always convex. This set is naturally identified with a subset $\mathcal{D}_{\text{const}}(T)$ of $\mathcal{D}(T)$ consisting of constant diagonals. The result gives a positive answer to a question of J.-C. Bourin from [9, p. 213]. In case of operator tuples, the convexity of $\mathcal{D}_{\text{const}}(T_1, \dots, T_n)$ remains an open problem.

2. NON-CONVEXITY OF NUMERICAL RANGES FOR COMMUTING TUPLES

Let $B(H)$ denote the space of all bounded linear operators on a Hilbert space H , and let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. While the joint numerical range $W(\mathcal{T})$ is not in general convex, it can of course be convex for particular classes of \mathcal{T} , and moreover, one may define other useful joint numerical ranges associated to \mathcal{T} having sometimes better geometric properties.

The convexity of various types of (joint) numerical ranges has been studied intensively, see e.g. [7], [19], [27], [28], and [38]-[40] and the references therein. In many of these results, the commutativity of the operators plays an important role. For example, it is well known that the joint numerical range of each commuting tuple of normal operators is convex (see e.g. [8, Chapter 7.35, Theorem 5] or [12, Theorem 2.5]), while there are non-commuting tuples of selfadjoint operators with non-convex joint numerical range even in a two-dimensional space, see e.g. [27, Example 1.1]. In [7, Theorem 3.1], the convexity of the joint numerical range of doubly commuting matrices was proved. It is also well known that spectral properties of commuting tuples are much better than those of non-commuting tuples.

This allows one to show that if $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ and T_1, \dots, T_n commute, then

$$\text{Int conv } \sigma(\mathcal{T}) \subset W(\mathcal{T}),$$

where $\text{Int conv } \sigma(\mathcal{T})$ stands for the interior of the convex hull of the spectrum $\sigma(\mathcal{T})$, see [39, Corollary 4.3] for more details and proof.

So the joint numerical range of a commuting n -tuple $\mathcal{T} = (T_1, \dots, T_n)$ exhibits some additional convexity properties, and there was some hope that it might be convex for all commuting tuples. Apparently, this question remained open for a long time. (To our knowledge, [13, p. 522] is the earliest reference where the question has been mentioned explicitly.)

The next example fills this gap and shows that the joint numerical range of commuting tuples is not convex, in general. The example appeared first in [37]. Inspired by [37], very recently, another example of even two commuting matrices with non-convex numerical range was given in [26]. Nevertheless, the present example has merit of being much simpler than the one in [26], and moreover, it can be applied to the situation of Asplund-Ptak numerical range without any changes, as we show below.

Theorem 2.1. *For any Hilbert space H , $\dim H \geq 4$, there exists a triple $\mathcal{T} = (T_1, T_2, T_3)$ of mutually commuting bounded operators on H , such that their joint numerical range $W(\mathcal{T})$ is not convex.*

Proof. First, let $H = \mathbb{C}^4$ with the standard basis e_1, e_2, e_3 and e_4 , and for $x \in \mathbb{C}^4$ write $x = (x_1, x_2, x_3, x_4)$. Let the linear operators T_1, T_2 and T_3 on H be given by

$$T_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let H_0 be the two-dimensional subspace spanned by e_1 and e_2 . Clearly $\text{Im } T_j \subset H_0$ and $\text{Ker } T_j \supset H_0$ for all $j = 1, 2, 3$ (where $\text{Im } T_j$ denotes the range and $\text{Ker } T_j$ the kernel of T_j , respectively). So we have $T_j T_k = 0$ for all $j, k = 1, 2, 3$. In particular, the operators T_1, T_2 , and T_3 are mutually commuting.

We show that the numerical range of the triple $\mathcal{T} = (T_1, T_2, T_3)$ is not convex. A direct computation shows that

$$W(\mathcal{T}) = \{(x_3 \bar{x}_1, x_4 \bar{x}_1, x_3 \bar{x}_2) : x \in \mathbb{C}^4, \|x\|_{\mathbb{C}^4} = 1\}.$$

In particular, for $x_1 = 0, x_2 = \sqrt{2}/2, x_3 = \sqrt{2}/2, x_4 = 0$ we have $\alpha := (0, 0, 1/2) \in W(\mathcal{T})$. Similarly, for $x_1 = \sqrt{2}/2, x_2 = x_3 = 0, x_4 = \sqrt{2}/2$ we have $\beta := (0, 1/2, 0) \in W(\mathcal{T})$.

We show that the midpoint $\frac{\alpha + \beta}{2} := (0, 1/4, 1/4)$ does not belong to $W(\mathcal{T})$. Suppose on the contrary that there exist $x \in \mathbb{C}^4$ with $\|x\|_{\mathbb{C}^4} = 1$ such that $x_3 \bar{x}_1 = 0, x_4 \bar{x}_1 = 1/4$ and $x_3 \bar{x}_2 = 1/4$. So either $x_3 = 0$ or $x_1 = 0$. If $x_3 = 0$

then $x_3\bar{x}_2 = 0$, a contradiction. If $x_1 = 0$ then $x_4\bar{x}_1 = 0$, a contradiction again. Therefore $\frac{\alpha+\beta}{2} \notin W(\mathcal{T})$, and $W(\mathcal{T})$ is not convex.

Note that the argument above also shows that if a and b are complex numbers such that $a \neq 0 \neq b$, then $(0, a, b) \notin W(\mathcal{T})$. (It is enough to repeat the same reasoning for any $(0, a, b)$ instead of the mid-point $(0, 1/4, 1/4)$.) This observation can be used to show that a triple of commuting operators with non-convex numerical range exist in any Hilbert space with dimension greater than 4.

Indeed, let now F be any nontrivial Hilbert space, and consider the operators $T'_1 = T_1 \oplus 0_F$, $T'_2 = T_2 \oplus 0_F$, and $T'_3 = T_3 \oplus 0_F$ on a Hilbert space $H = \mathbb{C}^4 \oplus F$. Then clearly T'_1 , T'_2 and T'_3 commute, and for $\mathcal{T}' = (T'_1, T'_2, T'_3)$ the points $\alpha := (0, 0, 1/2)$ and $\beta := (0, 1/2, 0)$ belong to $W(\mathcal{T}')$. On the other hand, for every $h = (x, f) \in H$, $\|h\| = 1$, we have

$$(\langle T'_1 h, h \rangle, \langle T'_2 h, h \rangle, \langle T'_3 h, h \rangle) \in \|x\|_{\mathbb{C}^4}^2 W(\mathcal{T}).$$

So, using the observation above, one infers that $\frac{\alpha+\beta}{2} \notin W(\mathcal{T})$, and gets a contradiction again. (It is instructive to note that, in fact, $W(\mathcal{T}') = \{(\langle T_1 x, x \rangle, \langle T_2 x, x \rangle, \langle T_3 x, x \rangle) : x \in \mathbb{C}^4, \|x\|_{\mathbb{C}^4} \leq 1\}$. \square)

In [1], E. Asplund and V. Ptak considered another type of numerical range. For $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ define

$$W_{\text{AP}}(\mathcal{T}) = \{(\langle T_1 x, y \rangle, \dots, \langle T_n x, y \rangle) : x, y \in H, \|x\| \leq 1, \|y\| \leq 1\}.$$

It was proved in [1] that $W_{\text{AP}}(T_1, T_2)$ is convex for each pair (T_1, T_2) .

In fact the matrices T_1, T_2 and T_3 constructed in Theorem 2.1 can be used to show that, in general, $W_{\text{AP}}(\mathcal{T})$ is not convex, even for $\mathcal{T} = (T_1, T_2, T_3)$ with mutually commuting operators T_1, T_2 and T_3 .

Theorem 2.2. *For any Hilbert space H , $\dim H \geq 4$, there exists a triple $\mathcal{T} = (T_1, T_2, T_3)$ of mutually commuting bounded operators on H , such that $W_{\text{AP}}(\mathcal{T})$ is not convex.*

Proof. The proof is analogous to the proof of Theorem 2.1. If $H = \mathbb{C}^4$ and T_1, T_2 and T_3 are the operators on H defined in this proof, then for $\mathcal{T} = (T_1, T_2, T_3)$ one has

$$W_{\text{AP}}(\mathcal{T}) = \{(x_3\bar{y}_1, x_4\bar{y}_1, x_3\bar{y}_2) : x, y \in \mathbb{C}^4, \|x\|_{\mathbb{C}^4} \leq 1, \|y\|_{\mathbb{C}^4} \leq 1\},$$

and one checks as before that $(0, 0, 1/2) \in W_{\text{AP}}(\mathcal{T})$, $(0, 1/2, 0) \in W_{\text{AP}}(\mathcal{T})$ and $(0, 1/4, 1/4) \notin W_{\text{AP}}(\mathcal{T})$. So the numerical range $W_{\text{AP}}(\mathcal{T})$ is not convex. The general case can be considered precisely as in the proof of Theorem 2.1, and we omit easy details. \square

3. CONVEXITY OF THE SET OF CONSTANT DIAGONALS

Let H be a separable Hilbert space. For an n -tuple $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ denote by $\mathcal{D}_{\text{const}}(\mathcal{T})$ the set of all n -tuples $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such

that \mathcal{T} has constant diagonal $(\lambda_1, \dots, \lambda_n)$, i.e., there exists an orthonormal basis $(u_j)_{j=1}^N, 1 \leq N \leq \infty$, in H with

$$\langle T_k u_j, u_j \rangle = \lambda_k$$

for all $j \in \mathbb{N}$ and $k = 1, \dots, n$.

If the space H is finite dimensional and $T \in B(H)$, then it is easy to see that $\mathcal{D}_{\text{const}}(T)$ is a singleton — the normalized trace of T . This is a classical result due to A. Parker, see [21, Theorem 1.3.4 and p. 28] or [17]. We give its proof for completeness, and since the argument is instructive for our subsequent considerations. For $T \in B(H)$ denote by $\text{tr} T$ the trace of T , whenever it is well defined.

Proposition 3.1. *Let $\dim H = N < \infty$ and $T \in B(H)$. Then $\mathcal{D}_{\text{const}}(T) = \{N^{-1}\text{tr} T\}$.*

Proof. If $\lambda \in \mathcal{D}_{\text{const}}(T)$ then $\text{tr} T = N\lambda$, and so $\lambda = N^{-1}\text{tr} T$.

Conversely, let $\lambda = N^{-1}\text{tr} T$ and $(u_j)_{j=1}^N$ be any orthonormal basis in H . Then

$$\lambda = N^{-1} \sum_{j=1}^N \langle T u_j, u_j \rangle \in W(T)$$

since $W(T)$ is convex. Let $u \in H$ be a unit vector such that $\langle T u, u \rangle = \lambda$. Decomposing H as $H = \mathbb{C}u \oplus \{u\}^\perp$, one infers that T is of the form

$$T = \begin{pmatrix} \lambda & * \\ * & T' \end{pmatrix},$$

where $\text{tr} T' = (N-1)\lambda$. Hence, the induction on the dimension of H yields a constant diagonal for T equal to λ . \square

Corollary 3.2. *Let $\dim H = N < \infty$ and $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Then $\mathcal{D}_{\text{const}}(\mathcal{T})$ is either a singleton $\{(N^{-1}\text{tr} T_1, \dots, N^{-1}\text{tr} T_n)\}$, or it is empty. Hence $\mathcal{D}_{\text{const}}(\mathcal{T})$ is convex.*

The next example shows that $\mathcal{D}_{\text{const}}(T_1, T_2)$ may be empty even for pairs of operators (T_1, T_2) in a finite-dimensional space.

Example 3.3. Let $T_1, T_2 \in B(\mathbb{C}^2)$ be given by

$$T_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\text{tr} T_1 = \text{tr} T_2 = 0$. We show that $(0, 0) \notin W(T_1, T_2)$. Consequently, $(0, 0) \notin \mathcal{D}_{\text{const}}(T_1, T_2)$ and $\mathcal{D}_{\text{const}}(T_1, T_2) = \emptyset$.

Suppose on the contrary that $(0, 0) \in W(T_1, T_2)$. So there exist $x_1, x_2 \in \mathbb{C}$ with $|x_1|^2 + |x_2|^2 = 1$ such that $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ satisfies $\langle T_1 x, x \rangle = \langle T_2 x, x \rangle = 0$.

We have

$$\langle T_1 x, x \rangle = x_1 \bar{x}_2,$$

so either $x_1 = 0$ or $x_2 = 0$. If $x_1 = 0$ then $|x_2| = 1$ and

$$\langle T_2 x, x \rangle = |x_1|^2 - |x_2|^2 = -1 \neq 0,$$

a contradiction. Similarly, if $x_2 = 0$ then $|x_1| = 1$ and

$$\langle T_2 x, x \rangle = |x_1|^2 - |x_2|^2 = 1,$$

a contradiction again. Hence $(0, 0) \notin W(T_1, T_2)$.

The situation is much more involved in infinite-dimensional spaces. Let H be a separable infinite-dimensional Hilbert space, and $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Recall that the joint essential numerical range $W_e(\mathcal{T})$ of \mathcal{T} is defined as the set of all n -tuples $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that there exists an orthonormal sequence $(u_j)_{j=1}^\infty$ in H satisfying

$$\lim_{j \rightarrow \infty} \langle T_k u_j, u_j \rangle = \lambda_k$$

for all $k = 1, \dots, n$. An important property of the joint essential numerical range is that it is always non-empty, closed and convex, see [6, Lemma 3.1] or [27]. Moreover, by [28, Theorem 3.1], each point of $W_e(\mathcal{T})$ is a star-center for the star-shaped set $\overline{W(\mathcal{T})}$. It is also crucial that

$$(3.1) \quad \text{conv} \left(\overline{W(\mathcal{T})} \right) = \text{conv} \left(W(\mathcal{T}) \cup W_e(\mathcal{T}) \right),$$

see [39, Theorem 5.1] for a simple proof, or alternatively, [43] or [28, Theorem 5.2]. Clearly $W_e(\mathcal{T})$ is invariant under compact perturbations, and, in particular, for any finite-rank projection P on H ,

$$(3.2) \quad W_e(\mathcal{T}) = W_e((I - P)T_1(I - P), \dots, (I - P)T_n(I - P)).$$

This fact is very useful in various inductive arguments.

Recall that

$$(3.3) \quad \text{Int } W_e(\mathcal{T}) \subset \mathcal{D}_{\text{const}}(\mathcal{T}) \subset W_e(\mathcal{T}),$$

thus $\mathcal{D}_{\text{const}}(\mathcal{T})$ is a union of $\text{Int } W_e(\mathcal{T})$ and a part of $\partial W_e(\mathcal{T})$. The second inclusion in (3.3) follows from the definition. The first inclusion is non-trivial and follows from [40, Corollary 4.2], see also [40, Theorem 1.1] (and [9, Theorem 1.2] and [20, Theorem 1(i)] for $n = 1$). Moreover, by [40, Corollary 4.2], one can replace in (3.3) the interior $W_e(\mathcal{T})$ by the relative interior of $W_e(\mathcal{T})$ in the smallest affine subspace containing $W(\mathcal{T})$. This is relevant, if e.g. $T_i, 1 \leq i \leq n$, are selfadjoint. For other relations between $\mathcal{D}_{\text{const}}(\mathcal{T})$ and $W_e(\mathcal{T})$ see [40, Propositions 5.3 and 5.4].

Since both $W_e(\mathcal{T})$ and $\text{Int } W_e(\mathcal{T})$ are convex sets, the inclusions (3.3) suggest that $\mathcal{D}_{\text{const}}(\mathcal{T})$ is always close to a convex set. So it is reasonable to ask whether $\mathcal{D}_{\text{const}}(\mathcal{T})$ is convex itself. Recall that the sets $S \subset \mathbb{C}^n$ satisfying $\text{Int } C \subset S \subset C$ for a convex set $C \subset \mathbb{C}^n$ are called *almost convex* in the literature. They share some properties of convex sets, such as e.g. separation properties. For interesting spectral conditions for almost convexity of joint numerical ranges as well as a pertinent discussion of almost convex sets, see [34], [35] and [36].

In Theorem 3.8 below, we give a positive answer to this question for $n = 1$. This solves a problem posed by J.-C. Bourin in [9, p. 213]. Our proof is based on the following criterion for existence of zero diagonals due to P. Fan, [15, Theorem 1]. The criterion has a “Tauberian” character expressing the property $0 \in \mathcal{D}_{\text{const}}(T)$ for $T \in B(H)$ in terms of the limit behavior of partial sums of diagonal entries of T .

Theorem 3.4. *Let H be a separable infinite-dimensional Hilbert space, and $T \in B(H)$. Then $0 \in \mathcal{D}_{\text{const}}(T)$ if and only if there exists an orthonormal basis $(u_j)_{j=1}^{\infty} \subset H$ such that the sequence $\left\{ \sum_{j=1}^k \langle Tu_j, u_j \rangle : k \geq 1 \right\}$ has a subsequence converging to zero.*

It is worth to mention that the original proof of Theorem 3.4 in [15] contained a gap, which was recently corrected in [30, Appendix B].

The proof of convexity for $\mathcal{D}_{\text{const}}(T)$ is based on two lemmas. The first one addresses the continuity of the Gram-Schmidt procedure, and it is surely known. However, we were not able to find an appropriate reference, especially for our infinite-dimensional setting.

Lemma 3.5. *For all $m \in \mathbb{N}$ and $\eta \in (0, 1)$ there exists $\delta_{m,\eta} > 0$ with the following property: if $\{e_j : 1 \leq j \leq m\} \subset H$ is an orthonormal system of vectors and vectors $e'_1, \dots, e'_m \in H$ satisfy $\max_{1 \leq j \leq m} \|e'_j - e_j\| \leq \delta_{m,\eta}$, then there exists an orthonormal basis $\{f_j : 1 \leq j \leq m\}$ in $\bigvee_{j=1}^m e'_j$ such that*

$$\max_{1 \leq j \leq m} \|f_j - e_j\| \leq \eta.$$

Remark 3.6. Note that the set $\{e'_j : 1 \leq j \leq m\}$ may be at a positive distance from $\bigvee_{j=1}^m e_j$. A posteriori, due to our choice of δ , it consists of linearly independent vectors, and $\dim(\bigvee_{j=1}^m e'_j) = m$.

Proof. We prove the statement by induction on m . The statement is clear for $m = 1$: set $\delta_{1,\eta} = \eta/2$. If $\|e'_1 - e_1\| \leq \delta_{1,\eta}$ then let $f_1 = \frac{e'_1}{\|e'_1\|}$. We have

$$\|f_1 - e_1\| \leq \|f_1 - e'_1\| + \|e'_1 - e_1\| \leq |1 - \|e'_1\|| + \delta_{1,\eta} \leq 2\delta_{1,\eta} = \eta.$$

Let $m \geq 2$ and suppose that the statement is true for $m - 1$. Let $\eta' = \frac{\eta}{2m}$. Fix $\delta_{m,\eta} > 0$ such that $\delta_{m,\eta} \leq \delta_{m-1,\eta'}$ and

$$2(m-1)(\delta_{m,\eta} + (1 + \delta_{m,\eta})\eta') + \delta_{m,\eta} \leq \eta.$$

Let e_1, \dots, e_m be orthonormal vectors in H and let e'_1, \dots, e'_m be any vectors in H satisfying $\max_{1 \leq j \leq m} \|e'_j - e_j\| \leq \delta_{m,\eta}$.

By the induction assumption, there exist orthonormal vectors $f_1, \dots, f_{m-1} \in \bigvee_{j=1}^{m-1} e'_j$ such that $\max_{1 \leq j \leq m-1} \|f_j - e_j\| \leq \eta'$. Set

$$\tilde{f}_m = e'_m - \sum_{j=1}^{m-1} \langle e'_m, f_j \rangle f_j.$$

We have

$$\|e'_m\| \leq \|e_m\| + \|e'_m - e_m\| \leq 1 + \delta_{m,\eta},$$

and for $1 \leq j \leq m-1$,

$$\begin{aligned} |\langle e'_m, f_j \rangle| &\leq |\langle e'_m, e_j \rangle| + |\langle e'_m, f_j - e_j \rangle| \\ &\leq \|e'_m - e_m\| + \|e'_m\| \cdot \|f_j - e_j\| \leq \delta_{m,\eta} + (1 + \delta_{m,\eta})\eta'. \end{aligned}$$

Thus,

$$\|\tilde{f}_m - e'_m\| \leq (m-1)(\delta_{m,\eta} + (1 + \delta_{m,\eta})\eta')$$

and

$$|1 - \|\tilde{f}_m\|| \leq \|\tilde{f}_m - e'_m\| + \|e'_m - e_m\| \leq (m-1)(\delta_{m,\eta} + (1 + \delta_{m,\eta})\eta') + \delta_{m,\eta}.$$

Note that by the choice of $\delta_{m,\eta}$ we have $\tilde{f}_m \neq 0$, and set $f_m = \frac{\tilde{f}_m}{\|\tilde{f}_m\|}$. Then, by construction, the vectors f_1, \dots, f_m are orthonormal, $\bigvee_{j=1}^m f_j = \bigvee_{j=1}^m e'_j$, and moreover

$$\begin{aligned} \|f_m - e_m\| &\leq \|f_m - \tilde{f}_m\| + \|\tilde{f}_m - e'_m\| + \|e'_m - e_m\| \\ &\leq 2(m-1)(\delta_{m,\eta} + (1 + \delta_{m,\eta})\eta') + \delta_{m,\eta} \leq \eta. \end{aligned}$$

□

The second, approximation lemma allows one to reduce the convexity property of $\mathcal{D}_{\text{const}}(T)$ to (essentially) Theorem 3.4.

Lemma 3.7. *Let H be a separable infinite-dimensional Hilbert space, and let $T \in B(H)$. Suppose there exist $\alpha, \beta \in \mathbb{R}$, $\alpha < 0 < \beta$, satisfying $\alpha \in \mathcal{D}_{\text{const}}(T)$ and $\beta \in W_e(T)$. Then for every subspace $M \subset H$ with $\dim M < \infty$ and every $\varepsilon > 0$, there exists a subspace $M' \subset H$ such that $M \subset M'$, $\dim M' < \infty$, and*

$$|\text{tr}(P_{M'} T P_{M'})| \leq \varepsilon,$$

where $P_{M'}$ denotes the orthogonal projection onto M' .

Proof. By the assumption, there exists an orthonormal basis $(u_j)_{j=1}^\infty$ in H such that $\langle T u_j, u_j \rangle = \alpha$ for all $j \in \mathbb{N}$.

Let $\dim M = m$, and let e_1, \dots, e_m be an orthonormal basis in M . For fixed $\varepsilon > 0$, let $\eta = \frac{\varepsilon}{4m\|T\|}$.

Find $k \in \mathbb{N}$ so large that

$$(3.4) \quad \text{dist} \left\{ e_j, \bigvee_{j=1}^k u_j \right\} < \delta, \quad j = 1, \dots, m,$$

where $\delta = \delta_{m,\eta}$ be the number given by Lemma 3.5.

Let $L = \bigvee_{j=1}^k u_j$. Clearly $\|P_L e_j - e_j\| \leq \delta$ for all $j = 1, \dots, m$. Let

$$\tilde{M} := P_L M = \bigvee_{j=1}^m P_L e_j.$$

By applying Lemma 3.5 to the set $\{e_j : 1 \leq j \leq m\}$ and its ‘‘perturbation’’ $\{e'_j := P_L e_j : 1 \leq j \leq m\}$, we infer that there exists an orthonormal basis $\{f_j : 1 \leq j \leq m\} \subset \widetilde{M}$ such that

$$\max_{1 \leq j \leq m} \|f_j - e_j\| \leq \frac{\epsilon}{4m\|T\|}.$$

Note that for any $x \in L$ and $1 \leq j \leq m$ one has

$$\langle x, e_j \rangle = \langle P_L x, e_j \rangle = \langle x, P_L e_j \rangle,$$

so that $x \in M^\perp$ if and only if $x \in \widetilde{M}^\perp$, that is $L \cap M^\perp = L \cap \widetilde{M}^\perp$. Hence $L = \widetilde{M} \oplus (L \cap M^\perp)$. Let

$$K = M \oplus (L \cap M^\perp).$$

We have

$$\text{tr}(P_K T P_K) = \text{tr}(P_M T P_M) + \text{tr}(P_{L \cap M^\perp} T P_{L \cap M^\perp}),$$

and

$$\alpha k = \text{tr}(P_L T P_L) = \text{tr}(P_{\widetilde{M}} T P_{\widetilde{M}}) + \text{tr}(P_{L \cap M^\perp} T P_{L \cap M^\perp}).$$

Thus

$$\begin{aligned} |\text{tr}(P_K T P_K) - \alpha k| &= |\text{tr}(P_{\widetilde{M}} T P_{\widetilde{M}}) - \text{tr}(P_M T P_M)| \\ &\leq \sum_{j=1}^m |\langle T f_j, f_j \rangle - \langle T e_j, e_j \rangle| \\ &\leq \sum_{j=1}^m 2\|T\| \cdot \|f_j - e_j\| \leq \frac{\epsilon}{2}. \end{aligned}$$

Recalling the notation $[\cdot]$ for the integer part, set

$$n := \left[\frac{|\alpha k|}{\beta} \right] \quad \text{and} \quad \gamma = |\alpha k| - n\beta.$$

Then $0 \leq \gamma < \beta$ and so $\gamma \in [\alpha, \beta] \subset W_\epsilon(T)$ in view of convexity of $W_\epsilon(T)$. Moreover,

$$\alpha k + n\beta + \gamma = 0$$

by the choice of n and γ . Using (3.2), choose inductively orthonormal vectors $x_1, \dots, x_{n+1} \in K^\perp$ such that

$$|\langle T x_j, x_j \rangle - \beta| < \frac{\epsilon}{2(n+1)}, \quad 1 \leq j \leq n, \quad \text{and} \quad |\langle T x_{n+1}, x_{n+1} \rangle - \gamma| < \frac{\epsilon}{2(n+1)}.$$

Let $M' = K \oplus \bigvee_{j=1}^{n+1} x_j$. Then $M \subset K \subset M'$, $\dim M' < \infty$ and

$$\begin{aligned} |\operatorname{tr}(P_{M'}TP_{M'})| &= \left| \operatorname{tr}(P_KTP_K) + \sum_{j=1}^{n+1} \langle Tx_j, x_j \rangle \right| \\ &\leq \frac{\varepsilon}{2} + \left| \alpha k + \sum_{j=1}^{n+1} \langle Tx_j, x_j \rangle \right| \\ &\leq \frac{\varepsilon}{2} + |\alpha k + n\beta + \gamma| + (n+1) \cdot \frac{\varepsilon}{2(n+1)} \\ &\leq \varepsilon. \end{aligned}$$

□

Now the convexity of $\mathcal{D}_{\text{const}}(T)$ is a direct consequence of Lemma 3.7. However, we prove a property of $\mathcal{D}_{\text{const}}(T)$ slightly stronger than convexity, which is the main result of this note.

Theorem 3.8. *Let H be a separable Hilbert space and let $T \in B(H)$. If $\alpha \in \mathcal{D}_{\text{const}}(T)$ and $\beta \in W_e(T)$, then*

$$(3.5) \quad t\alpha + (1-t)\beta \in \mathcal{D}_{\text{const}}(T), \quad t \in (0, 1).$$

Thus, for a Hilbert space H of any dimension and $T \in B(H)$, the set $\mathcal{D}_{\text{const}}(T)$ is convex.

Proof. To prove (3.5), without loss of generality, it is sufficient to show that if $\alpha < 0 < \beta$, $\alpha \in \mathcal{D}_{\text{const}}(T)$, and $\beta \in W_e(T)$, then $0 \in \mathcal{D}_{\text{const}}(T)$. Otherwise, we can replace T by a suitable linear combination $aT + bI$, $a, b \in \mathbb{C}$, if necessary.

Fix an orthonormal basis $(e_j)_{j=1}^{\infty}$ in H . Set $M_0 = \{0\}$. Using Lemma 3.7 inductively, construct the family of finite-dimensional subspaces $M_k \subset H$, $k \geq 1$, such that

$$M_{k+1} \supset (M_k \vee e_{k+1}) \quad \text{and} \quad |\operatorname{tr}(P_{M_{k+1}}TP_{M_{k+1}})| < (k+1)^{-1}$$

for all $k \geq 0$. Choose inductively an orthonormal sequence $(u_j)_{j=1}^{\infty}$ such that $\{u_j : 1 \leq j \leq \dim M_k\}$ is an orthonormal basis in M_k . By construction,

$$\overline{\bigcup_{k=1}^{\infty} M_k} = H,$$

for all $k \in \mathbb{N}$, hence $(u_j)_{j=1}^{\infty}$ is an orthonormal basis in H . Since

$$\lim_{k \rightarrow \infty} \sum_{j=1}^{\dim M_k} \langle Tu_j, u_j \rangle = 0,$$

from Theorem 3.4 it follows that $0 \in \mathcal{D}_{\text{const}}(T)$. In particular, the set $\mathcal{D}_{\text{const}}(T)$ is convex.

If $\dim H < \infty$, then the convexity of $\mathcal{D}_{\text{const}}(T)$ is noted in Corollary 3.2. □

Observe that by [46, Theorem 2.3.4], if $K \subset \mathbb{C}^n$ is convex, then for any x from the interior of K and $y \in \overline{K}$ the points $tx + (1-t)y, t \in (0, 1]$, belong to the interior of K . Thus, in view of (3.3), Theorem 3.8 has new operator-theoretical content only if $\alpha, \beta \in \partial W_e(\mathcal{T})$.

For any n -tuple $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$ we have clearly $\mathcal{D}_{\text{const}}(\mathcal{T}) \subset W(\mathcal{T})$. Next we characterize those n -tuples of operators for which the set $\mathcal{D}_{\text{const}}(\mathcal{T})$ is maximal.

To this aim, recall that a subset $A \subset \mathbb{R}^k$ is said to be an affine subspace if $M = u + L$ for some $u \in \mathbb{R}^k$ and a subspace $L \subset \mathbb{R}^k$. **The smallest affine subspace containing a set $A \subset \mathbb{R}^k$ is called the affine hull of A . A nonempty subset $A \subset \mathbb{R}^k$ is called relatively open if it is relatively open in the affine hull of A . Denote by $\text{rInt } A$ the relative interior of A in the affine hull of A .**

The above definitions can be applied also for subsets of \mathbb{C}^n if we identify \mathbb{C}^n with \mathbb{R}^{2n} in the usual way.

Theorem 3.9. *Let $\mathcal{T} = (T_1, \dots, T_n) \in B(H)^n$. Then the following conditions are equivalent:*

- (i) $W(\mathcal{T}) = \mathcal{D}_{\text{const}}(\mathcal{T})$;
- (ii) $W(\mathcal{T}) = \text{rInt } W_e(\mathcal{T})$;
- (iii) $W(\mathcal{T})$ is convex and relatively open.

Proof. (i) \Rightarrow (ii): **First, we show that $W(\mathcal{T})$ is relatively open.**

The proof relies on several convenient reductions of the general set-up.

Instead of the n -tuple (T_1, \dots, T_n) we may consider the $(2n)$ -tuple of selfadjoint operators $(\text{Re } T_1, \text{Im } T_1, \dots, \text{Re } T_n, \text{Im } T_n)$ and deal with its joint numerical and essential numerical ranges contained in \mathbb{R}^{2n} . As far as, we are concerned with the *relative interior* of $W_e(\mathcal{T})$, without loss of generality, we may assume that $\mathcal{T} = (T_1, \dots, T_k)$ is a k -tuple of selfadjoint operators such that $\mathcal{D}_{\text{const}}(\mathcal{T}) = W(\mathcal{T})$.

Suppose on the contrary that there exists $\lambda = (\lambda_1, \dots, \lambda_k) \in W(\mathcal{T}) \setminus \text{rInt } W(\mathcal{T})$. We may assume that $\lambda = (0, \dots, 0) := (0)_k$. (If not, then replace \mathcal{T} by the k -tuple of operators $\mathcal{T} - \lambda = (T_1 - \lambda_1, \dots, T_k - \lambda_k)$).

Suppose that $\mathcal{T} = (T_1, \dots, T_k) \in B(H)^k$ is a k -tuple of selfadjoint operators such that $\mathcal{D}_{\text{const}}(\mathcal{T}) = W(\mathcal{T})$ and $(0)_k \in W(\mathcal{T}) \setminus \text{rInt } W(\mathcal{T})$.

After a rotation with real coefficient in \mathbb{R}^k we can obtain $m \leq k$ and an k -tuple $\mathcal{T}' = (T'_1, \dots, T'_m, 0, \dots, 0)$ of selfadjoint operators such that T'_1, \dots, T'_m are linearly independent and the properties $W(\mathcal{T}') = \mathcal{D}_{\text{const}}(\mathcal{T}')$ and $(0)_k \in W(\mathcal{T}') \setminus \text{rInt } W(\mathcal{T}')$ are still preserved. Note that this implies that $m \geq 1$.

Write $\mathcal{T}'' = (T''_1, \dots, T''_m)$. We have $W(\mathcal{T}'') = \mathcal{D}_{\text{const}}(\mathcal{T}'')$ and $(0)_m \in W(\mathcal{T}'') \setminus \text{rInt } W(\mathcal{T}'')$.

Note that $\overline{W(\mathcal{T}'')} = \overline{\mathcal{D}_{\text{const}}(\mathcal{T}'')} \subset W_e(\mathcal{T}'')$. So $\overline{W(\mathcal{T}'')} = W_e(\mathcal{T}'')$. In particular, $\overline{W(\mathcal{T}'')}$ is convex. Since $\overline{W(\mathcal{T}'')}$ is convex and $(0)_m$

lies on its boundary, there is a supporting hyperplane of $\overline{W(\mathcal{T}'')}$ passing through $(0)_m$, and after another rotation in \mathbb{R}^m with real coefficients we can obtain an m -tuple $\mathcal{T}''' = (T_1''', \dots, T_m''')$ of self-adjoint operators such that T_1''', \dots, T_m''' are linearly independent, $(0)_m \in W(\mathcal{T}''') = \mathcal{D}_{\text{const}}(\mathcal{T}''')$ and

$$W(\mathcal{T}''') \subset \{(r_1, \dots, r_m) \in \mathbb{R}^m : r_m \geq 0\}.$$

So $T_m''' \geq 0$ and $(0)_m \in \mathcal{D}_{\text{const}}(T_m''')$. Then $T_m''' = 0$, a contradiction with the assumption that the operators T_1''', \dots, T_m''' are linearly independent. So $W(\mathcal{T}''')$ is relatively open and so is $W(\mathcal{T})$.

Since $W(\mathcal{T})$ is relatively open and $\overline{W(\mathcal{T})} = W_e(\mathcal{T})$, we conclude that $W(\mathcal{T}) = \text{rInt } \overline{W(\mathcal{T})} = \text{rInt } W_e(\mathcal{T})$.

(ii) \Rightarrow (iii): Clear.

(iii) \Rightarrow (i): We show that $\overline{W(\mathcal{T})} = W_e(\mathcal{T})$. Suppose on the contrary that $\overline{W(\mathcal{T})} \setminus W_e(\mathcal{T}) \neq \emptyset$. Then there exists λ in the relative topological boundary of $W(\mathcal{T})$ such that $\lambda \notin W_e(\mathcal{T})$. Since $W(\mathcal{T})$ is relatively open, $\lambda \notin W(\mathcal{T})$ either, and therefore $\lambda \in \overline{W(\mathcal{T})} \setminus (W(\mathcal{T}) \cup W_e(\mathcal{T}))$. By (3.1), we have

$$\lambda \in \overline{W(\mathcal{T})} \subset \text{conv}(W(\mathcal{T}) \cup W_e(\mathcal{T})).$$

Since both $W(\mathcal{T})$ and $W_e(\mathcal{T})$ are convex, there exist $\mu \in W(\mathcal{T})$, $\nu \in W_e(\mathcal{T})$, and $t \in (0, 1)$ such that

$$\lambda = t\mu + (1-t)\nu.$$

Let L be the smallest affine subspace containing $W(\mathcal{T})$. Since $W(\mathcal{T})$ is relatively open in L , there exists $\varepsilon > 0$ such that $\mu' \in W(\mathcal{T})$ whenever $\mu' \in L$ and $|\mu' - \mu| < \varepsilon$.

We have $\nu \in W_e(\mathcal{T}) \subset \overline{W(\mathcal{T})}$. So there exists $\nu' \in W(\mathcal{T})$ such that $|\nu' - \nu| < \varepsilon t$. Let

$$\mu' = \mu - \frac{\nu' - \nu}{t}(1-t).$$

Then $\mu' \in W(\mathcal{T})$ and

$$\lambda = t\mu' + (1-t)\nu'.$$

Hence λ is a convex combination of elements of $W(\mathcal{T})$, and so $\lambda \in W(\mathcal{T})$, a contradiction

Thus $\overline{W(\mathcal{T})} = W_e(\mathcal{T})$ and, as in the proof of (i) \Rightarrow (ii), we infer that (??) holds by [40, Corollary 4.2] (cf. (3.3) and comments following it again). Therefore, $\mathcal{D}_{\text{const}}(\mathcal{T}) = W(\mathcal{T})$. \square

The examples of n -tuples of operators \mathcal{T} with convex and relatively open $W(\mathcal{T})$ are, in particular, provided by n -tuples of Toeplitz operators on the Hardy space $H^2(\mathbb{D})$, where \mathbb{D} is the unit disc, see [10, Proposition 3] (and also [25] for $n = 1$). In fact, in this case $W(\mathcal{T})$ is open unless it is a single point.

For a single operator we get the following corollary. Note that it was stated in [9, Proposition 1.4] without proof.

Corollary 3.10. *Let $T \in B(H)$. Then $\mathcal{D}_{\text{const}}(T) = W(T)$ if and only if $W(T)$ is relatively open.*

Apparently, the simplest example of $T \in B(H)$ with open $W(T)$ is provided by selfadjoint T such that $m := \min \sigma(T)$ and $M := \max \sigma(T)$ are not eigenvalues of T and $M > m$. Indeed, it is well-known that in this case m and M do not belong to $W(T)$. Since $\overline{W(T)} = \text{conv } \sigma(T) = [m, M]$ and $W(T)$ is an interval, it follows that $W(T) = (m, M)$. Apart from Toeplitz operators mentioned above, the examples of $T \in B(H)$ with open $W(T)$ include, in particular, weighted shifts with periodic weights, see [42, Proposition 6]. Several more general classes of weighted shifts with open numerical ranges were described in [44] and [45]. Remark that the numerical range of a weighted shift is an open or closed disc centered at the origin, so this class of operators fits very well into the framework of Corollary 3.10. Unfortunately, the numerical ranges of tuples of weighted shifts have not been studied in the literature.

Remark 3.11. Note that if $T \in B(H)$ and $\lambda \in \mathcal{D}_{\text{const}}(T)$, then λ is an element of $W(T)$ which is attained on a set spanning H . Thus from [14, Theorem 1] (or from the argument given in the proof of (i) \Rightarrow (ii), Theorem 3.9), it follows that $\mathcal{D}_{\text{const}}(T) \subset \text{rInt } W(T)$ for any $T \in B(H)$. So $\text{rInt } W(T)$ arises naturally in the study of $\mathcal{D}_{\text{const}}(T)$ even when $W(T)$ is not relatively open.

Corollary 3.10 describes the situation when $\mathcal{D}_{\text{const}}(T)$ is maximal. On the other hand, $\mathcal{D}_{\text{const}}(T)$ can be empty.

Example 3.12. Let $H = \ell_2(\mathbb{N})$ and let $T \in B(H)$ be the diagonal operator given by $T := \text{diag}(1, \frac{1}{2}, \frac{1}{4}, \dots)$. Then T is compact and $\mathcal{D}_{\text{const}}(T) \subset W_e(T) = \{0\}$. However, $0 \notin W(T)$, so $\mathcal{D}_{\text{const}}(T) = \emptyset$.

4. FINAL REMARKS

The convexity of $\mathcal{D}_{\text{const}}(T_1, \dots, T_n)$ for $n \geq 2$ is an open problem, even for commuting operator tuples. However, the next example shows that Fan's Theorem 3.4 is not true for n -tuples of operators.

Example 4.1. Let $n = 2$ and T_1, T_2 be the operators on \mathbb{C}^2 considered in Example 3.3. Let $\{e_1, e_2\}$ be the standard basis in \mathbb{C}^2 . Let F be the separable infinite-dimensional Hilbert space with an orthonormal basis $(e_j)_{j=3}^\infty$. Let $H = \mathbb{C}^2 \oplus F$ and let $S_1, S_2 \in B(H)$ be defined by $S_1 = T_1 \oplus 0_F$ and $S_2 = T_2 \oplus 0_F$. Then

$$\sum_{j=1}^k \langle S_1 e_j, e_j \rangle = \sum_{j=1}^k \langle S_2 e_j, e_j \rangle = 0$$

for all $k \geq 2$, but $(0, 0) \notin \mathcal{D}_{\text{const}}(S_1, S_2)$. Indeed, let us show that if

$$\langle S_1 h, h \rangle = \langle S_2 h, h \rangle = 0$$

for a unit vector $h = (x, f) \in H$, then $x = 0$.

Suppose on the contrary that $x \neq 0$. Then

$$(\langle S_1 h, h \rangle, \langle S_2 h, h \rangle) \in \|x\|_{\mathbb{C}^2}^2 \cdot W(T_1, T_2),$$

but $(0, 0) \notin \|x\|_{\mathbb{C}^2}^2 \cdot W(T_1, T_2)$ in view of Example 3.3, a contradiction. Hence $(0, 0) \notin \mathcal{D}_{\text{const}}(S_1, S_2)$.

Remark 4.2. The pair of operators (S_1, S_2) in the previous example can be identified with the triple of selfadjoint operators $(\text{Re } S_1, \text{Im } S_1, S_2)$. Thus the example shows that Theorem 3.4 is not true even for triples of selfadjoint operators (in spite of the fact that the joint numerical range of any triple of selfadjoint operators on a Hilbert space of dimension at least 3 is convex, see [16, Theorem 1] and [19, Theorem 5.4]).

Naturally, given $T \in B(H)$ with $\dim H = N$, $1 \leq N \leq \infty$, one may attempt to study the convexity of the set $\mathcal{D}(T)$ of all diagonals of T , i.e., the convexity of the subset of ℓ^∞ given by

$$\mathcal{D}(T) := \{(\langle T e_n, e_n \rangle)_{n=1}^N : (e_n)_{n=1}^N \text{ is an orthonormal basis of } H\}.$$

Note that since the unitary group in $B(H)$ is path connected ([11]), the set $\mathcal{D}(T)$ is path-connected as well. However, this direction seems to be much more demanding, at least in our general setting. If $N < \infty$ then $\mathcal{D}(T)$ coincides with the “ N -dimensional” numerical range $\mathcal{W}(T)$ defined in [18]. It was noted in [18] that while $\mathcal{D}(T)$ is convex for selfadjoint T (by an old result due to Horn), the convexity of $\mathcal{W}(T)$ may fail if T is normal. Later on, it was proved in [4] that $\mathcal{W}(T)$ is convex if and only if there exist $\alpha \in \mathbb{C}, \alpha \neq 0$, and $\beta \in \mathbb{C}$ such that $\alpha T + \beta$ is selfadjoint, implying that $\mathcal{W}(T)$ is not convex for most of normal T .

If $N = \infty$, then it was discovered in [41] that if $T \in B(H)$ is selfadjoint, then the ℓ^∞ -closure of $\mathcal{D}(T)$ is convex. This result may lead to a hope that $\mathcal{D}(T)$ is convex for such a T , as in the case $\dim H < \infty$. Slightly later, R. Kadison proved in [22, 23] that a sequence $d = (d_n)_{n=1}^\infty$ is a diagonal of some *selfadjoint* projection in H if and only if it takes values in $[0, 1]$ and if the sums $a(d) := \sum_{d_j < 1/2} d_j$ and $b(d) := \sum_{d_j \geq 1/2} (1 - d_j)$ satisfy either $a(d) + b(d) = \infty$, or $a(d) + b(d) < \infty$ and $a(d) - b(d) \in \mathbb{Z}$. Using this description of $\mathcal{D}(P)$, it is easy to show that $\mathcal{D}(P)$ is not, in general, convex even in this, comparatively simple case. (See [2] and [3] for more details on Kadison’s result and its improvements by W. Arveson). While Kadison’s theorem concerns the set

$$\bigcup \{\mathcal{D}(P) : P \text{ is a selfadjoint projection on } H\},$$

it is easy to adopt it to our framework of fixed P (as observed in [32, p. 94]). It suffices to note that the selfadjoint projections P and Q are unitary equivalent if and only if $\text{tr } P = \text{tr } Q$ and $\text{tr}(I - P) = \text{tr}(I - Q)$, and for $(d_k)_{k=1}^\infty$ as above, $\text{tr } P = \sum_{k \geq 1} d_k$ and $\text{tr}(I - P) = \sum_{k \geq 1} (1 - d_k)$.

If $d^1 = (0, 1, 0, 1, \dots)$ and $d^2 = (1/2, 1, 1/4, 1, 1/8, \dots)$, then both sequences can be realized as diagonals of the same projection P_0 , since $a(d^1) =$

$b(d^1) = 0$, and $a(d^2) = 1, b(d^2) = 0$, and the corresponding traces are infinite. On the other hand, for $d^0 = (d^1 + d^2)/2 = (1/4, 1, 1/8, 1, \dots)$, one has $a(d^0) = 1/2$ and $b(d^0) = 0$. Hence d^0 is not a diagonal of a projection by Kadison's theorem, and $\mathcal{D}(P_0)$ is not convex. A version of this example has already appeared in [30, Example 3.0.1], but we feel that the details given above would nicely supplement our discussion here.

Despite the convexity of $\mathcal{D}(T)$ may, in general, fail even for selfadjoint T , it was proved that compact positive operators, which are either of finite rank or of infinite rank and with infinite-dimensional kernel, have convex sets of diagonals. See [24, Corollary 6.7] and [31, Corollary 4.3] for these results.

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