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On Bilinear Hardy Inequality and Corresponding Geometric Mean Inequality

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Abstract. The main aim of this paper to provide several scales of equivalent conditions for the bilinear Hardy inequalities in the case $1 < q, p_1, p_2 < \infty$ with $q \geq \max(p_1, p_2)$.

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1 Introduction

Let $\mathfrak{M}$ denote the set of all Lebesgue measurable functions on (a, b), $-\infty \leq a < b \leq \infty$, $\mathfrak{M}^+ \subset \mathfrak{M}$ is the subset of all non-negative functions.

Let $u, v, \in \mathfrak{M}^+$, $0 < p, q \leq \infty$, $p \geq 1$. Denote $p' := \frac{p}{p-1}$ and write

$$U(x) = \int_x^b u(t)\,dt, \quad V(x) = \int_a^x v^{1-p'}(t)\,dt, \tag{1.1}$$

and assume that $U(x) < \infty$, $V(x) < \infty$ for almost everywhere (a.e.) $x \in (a, b)$. Consider the one dimensional Hardy inequality

$$\left(\int_a^b \left(\int_a^x f(t)\,dt\right)^q u(x)\,dx\right)^{\frac{1}{q}} \leq C \left(\int_a^b f^p(x) v(x)\,dx\right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}^+. \tag{1.2}$$

It is known that the inequality (1.2) is characterized by the Muckenhoupt condition [9] in the case $1 < p \leq q < \infty$ which is given by

$$A_M := \sup_{a<x<b} A_M(x) = \sup_{a<x<b} U^{\frac{1}{q}}(x) V^{\frac{1}{p'}}(x) < \infty. \tag{1.3}$$

It is noted that the Muckenhoupt condition $A_M < \infty$ is not unique. It can be replaced by several scales of equivalent conditions (see [2], [3]). Preciously, the following result was proved in [3]:

Theorem A. *Let* $-\infty \le a < b \le \infty$ *,* α, β, s *be positive numbers and* $f, g, h \in \mathfrak{M}^+$ *. Denote*

$$F(x) := \int_x^b f(t)\,dt, \quad G(x) := \int_a^x g(t)\,dt, \tag{1.4}$$

and suppose that $F(x) < \infty$ *,* $G(x) < \infty$ *for every* $x \in (a, b)$ *. Furthermore, denote*

$$\begin{aligned}
&B_1(x;\alpha,\beta) := F^\alpha(x) G^\beta(x);\\
&B_2(x;\alpha,\beta,s) := \left(\int_x^b f(t) G^{\frac{\beta-s}{\alpha}}(t)\,dt\right)^\alpha G^s(x);\\
&B_3(x;\alpha,\beta,s) := \left(\int_a^x g(t) F^{\frac{\alpha-s}{\beta}}(t)\,dt\right)^\beta F^s(x);\\
&B_4(x;\alpha,\beta,s) := \left(\int_a^x f(t) G^{\frac{\beta+s}{\alpha}}\,dt\right)^\alpha G^{-s}(x);\\
&B_5(x;\alpha,\beta,s) := \left(\int_x^b g(t) F^{\frac{\alpha+s}{\beta}}(t)\,dt\right)^\beta F^{-s}(x);\\
&B_6(x;\alpha,\beta,s) := \left(\int_x^b f(t) G^{\frac{\beta}{\alpha+s}}(t)\,dt\right)^{\alpha+s} F^{-s}(x);\\
&B_7(x;\alpha,\beta,s) := \left(\int_a^x g(t) F^{\frac{\alpha}{\beta+s}}(t)\,dt\right)^{\beta+s} G^{-s}(x);\\
&B_8(x;\alpha,\beta,s) := \left(\int_a^x f(t) G^{\frac{\beta}{\alpha-s}}(t)\,dt\right)^{\alpha-s} F^s(x), \quad \alpha > s;\\
&B_9(x;\alpha,\beta,s) := \left(\int_x^b f(t) G^{\frac{\beta}{\alpha-s}}(t)\,dt\right)^{\alpha-s} F^s(x), \quad \alpha < s;\\
&B_{10}(x;\alpha,\beta,s) := \left(\int_x^b g(t) F^{\frac{\alpha}{\beta-s}}(t)\,dt\right)^{\beta-s} G^s(x), \quad \beta > s;\\
&B_{11}(x;\alpha,\beta,s) := \left(\int_a^x g(t) F^{\frac{\alpha}{\beta-s}}(t)\,dt\right)^{\beta-s} G^s(x), \quad \beta < s;\\
&B_{12}(x;\alpha,\beta,s;h) := \left(\int_x^b f(t) h^{\frac{\beta-s}{\alpha}}(t)\,dt\right)^\alpha \Big(h(x)+G(x)\Big)^s, \quad \beta < s;\\
&B_{13}(x;\alpha,\beta,s;h) := \left(\int_a^x g(t) h^{\frac{\alpha-s}{\beta}}(t)\,dt\right)^\beta \Big(h(x)+F(x)\Big)^s, \quad \alpha < s;\\
&B_{14}(x;\alpha,\beta,s;h) := \left(\int_a^x f(t)\Big(h(t)+G(t)\Big)^{\frac{\beta+s}{\alpha}}\,dt\right)^\alpha h^{-s}(x);\\
&B_{15}(x;\alpha,\beta,s;h) := \left(\int_x^b g(t)\Big(h(t)+F(t)\Big)^{\frac{\alpha+s}{\beta}}\,dt\right)^\beta h^{-s}(x).
\end{aligned}$$

The numbers $B_1(\alpha,\beta) := \sup\limits_{a<x<b} B_1(x;\alpha,\beta)$ *,* $B_i(\alpha,\beta,s) := \sup\limits_{a<x<b} B_i(x;\alpha,\beta,s)$ $(i =$

$2, 3, ..., 11)$ *and* $B_i(\alpha, \beta, s) := \inf_{h \geq 0} \sup_{a<x<b} B_i(x; \alpha, \beta, s; h)$ $(i = 12, 13, 14, 15)$ *are mutually equivalent.*

For $\alpha = \frac{1}{q}$, $\beta = \frac{1}{p'}$, $F(x) = \int_x^b u(t)\, dt$ and $G(x) = \int_a^x v^{1-p'}(t)\, dt$, we find that

$$B_1\left(x; \frac{1}{q}, \frac{1}{p'}\right) = A_M(x)$$

so that the condition

$$\sup_{a<x<b} B_1\left(x; \frac{1}{q}, \frac{1}{p'}\right) < \infty$$

in Theorem A is the Muckenhoupt condition (1.3). Consequently, the other conditions in Theorem A are equivalent to (1.3). Let us mention that similar equivalent conditions for the inequality (1.2) for the case $1 < q < p < \infty$ were obtained by Persson, Stepanov and Wall [12].

Towards the first aim of this paper, we provide a stronger version of Theorem A by proving that, in Theorem A, the supremum over the interval (a, b) can be considered in certain truncated intervals. This is done in Section 2.

Recently, Cañestro *et al.* [1] (see also [8]) considered the weighted bilinear Hardy inequality

$$\left(\int_a^b \left(\int_a^x f(t)\, dt\right)^q \left(\int_a^x g(t)\, dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leq C \left(\int_a^b f^{p_1}(x) v_1(x)\, dt\right)^{\frac{1}{p_1}} \times$$
$$\times \left(\int_a^b g^{p_2}(x) v_2(x)\right)^{\frac{1}{p_2}} \qquad (1.5)$$

and proved the following:

Theorem B. *Let* $1 < q, p_1, p_2 < \infty$ *with* $q \geq \max(p_1, p_2)$ *. Let* $u, v_1, v_2 \in \mathfrak{M}^+$ *. Then there exits a positive constant* C *such that the inequality* (1.5) *holds for all* $f, g \in \mathfrak{M}^+$ *if and only if* $\mathcal{D} < \infty$ *, where*

$$\mathcal{D} := \sup_{a<x<b} \left(\int_x^b u(t)\, dt\right)^{\frac{1}{q}} \left(\int_a^x v_1^{1-p_1'}(t)\, dt\right)^{\frac{1}{p_1'}} \left(\int_a^x v_2^{1-p_2'}(t)\, dt\right)^{\frac{1}{p_2'}}. \qquad (1.6)$$

The next aim of this paper is to prove a result of the type of Theorem A in respect of the bilinear Hardy inequality (1.5). Some of these equivalent conditions have recently been proved in [7]. In this paper, we give different proofs of some of the conditions proved in [7] and moreover, we provide several new equivalent conditions. This is done in Section 3.

Finally, in Section 4, we give a characterization for the bilinear type geometric mean inequality

$$\left(\int_0^\infty (Tf)^q(x)\, (Tg)^q(x) u(x)\, dx\right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f^{p_1}(x) v_1(x)\, dx\right)^{\frac{1}{p_1}}$$

$$\times \left(\int_0^\infty f^{p_2}(x) v_2(x)\, dx \right)^{\frac{1}{p_2}},$$

where

$$(Tf)(x) := \exp\left(\frac{1}{x} \int_0^x \ln(f(t))\, dt \right), \quad f \in \mathfrak{M}^+.$$

2 Improvement of the Theorem A

We prove the following theorem:

Theorem 2.1. *Under the setting of Theorem A, for* $x \in (a,b)$ *, the following hold:*

(i) $\sup\limits_{x<y<b} B_1(y;\alpha,\beta) \approx \sup\limits_{x<y<b} B_2(y;\alpha,\beta,s);$

(ii) $\sup\limits_{x<y<b} B_1(y;\alpha,\beta) \approx \sup\limits_{x<y<b} \left(\int_x^y f(t) G^{\frac{\beta+s}{\alpha}}(t)\, dt \right)^{\alpha} G^{-s}(y);$

(iii) $\sup\limits_{x<y<b} B_1(y;\alpha,\beta) \approx \sup\limits_{x<y<b} B_6(y;\alpha,\beta,s);$

(iv) $\sup\limits_{x<y<b} B_1(y;\alpha,\beta) \approx \sup\limits_{x<y<b} \left(\int_x^y f(t) G^{\frac{\beta}{\alpha-s}}(t)\, dt \right)^{\alpha-s} F^{s}(y), \ \ \alpha > s;$

(v) $\sup\limits_{x<y<b} B_1(y;\alpha,\beta) \approx \sup\limits_{x<y<b} B_9(y;\alpha,\beta,s), \ \ \alpha < s;$

(vi) $\sup\limits_{x<y<b} B_1(y;\alpha,\beta) \approx \inf\limits_{h\geq 0} \sup\limits_{x<y<b} B_{12}(y;\alpha,\beta,s;h), \ \ \beta < s;$

(vii) $\sup\limits_{x<y<b} B_1(y;\alpha,\beta) \approx \inf\limits_{h\geq 0} \sup\limits_{x<y<b} h(y)^{-s} \left(\int_x^y f(z) \Big(h(z) + G(z) \Big)^{\frac{\beta+s}{\alpha}} dz \right)^{\alpha};$

(viii) $\sup\limits_{a<y<x} B_1(y;\alpha,\beta) \approx \sup\limits_{a<y<x} B_3(y;\alpha,\beta,s);$

(ix) $\sup\limits_{a<y<x} B_1(y;\alpha,\beta) \approx \sup\limits_{a<y<x} \left(\int_y^x g(t) F^{\frac{\alpha+s}{\beta}}(t)\, dt \right)^{\beta} F^{-s}(y);$

(x) $\sup\limits_{a<y<x} B_1(y;\alpha,\beta) \approx \sup\limits_{a<y<x} B_7(y;\alpha,\beta,s);$

(xi) $\sup\limits_{a<y<x} B_1(y;\alpha,\beta) \approx \sup\limits_{a<y<x} \left(\int_y^x g(t) F^{\frac{\alpha}{\beta-s}}(t)\, dt \right)^{\beta-s} G^{s}(y), \ \ \beta > s;$

(xii) $\sup\limits_{a<y<x} B_1(y;\alpha,\beta) \approx \sup\limits_{a<y<x} B_{11}(y;\alpha,\beta,s), \ \ \beta < s;$

(xiii) $\sup\limits_{a<y<x} B_1(y;\alpha,\beta) \approx \inf\limits_{h\geq 0} \sup\limits_{a<y<x} B_{13}(y;\alpha,\beta,s;h), \ \ \alpha < s;$

(xiv) $\sup_{a<y<x} B_1(y;\alpha,\beta) \approx \inf_{h\geq 0} \sup_{a<y<x} h(y)^{-s} \left(\int_y^x g(z) \Big(h(z) + F(z) \Big)^{\frac{\alpha+s}{\beta}} dz \right)^{\beta}.$

Proof. (i) Let us consider the function

$$f_x(y) := \chi_{(x,b)}(y) f(y).$$

By using the equivalence $B_1(\alpha,\beta) \approx B_2(\alpha,\beta,s)$ from Theorem A, we get

$$\begin{aligned}
\sup_{x<y<b} B_1(y;\alpha,\beta) &= \sup_{x<y<b} F^{\alpha}(y) G^{\beta}(y) \\
&= \sup_{a<y<b} \left(\int_y^b f_x(t)\,dt \right)^{\alpha} \left(\int_a^y g(t)\,dt \right)^{\beta} \\
&\approx \sup_{a<y<b} \left(\int_y^b f_x(t) G^{\frac{\beta-s}{\alpha}}(t)\,dt \right)^{\alpha} G^s(y) \\
&= \sup_{x<y<b} \left(\int_y^b f(t) G^{\frac{\beta-s}{\alpha}}(t)\,dt \right)^{\alpha} G^s(y) \\
&= \sup_{x<y<b} B_2(y;\alpha,\beta,s).
\end{aligned}$$

(ii) Let us consider the function

$$f_x(y) := \chi_{(x,b)}(y) f(y).$$

By using the equivalence $B_1(\alpha,\beta) \approx B_4(\alpha,\beta,s)$ from Theorem A, we get

$$\begin{aligned}
\sup_{x<y<b} B_1(y;\alpha,\beta) &= \sup_{x<y<b} F^{\alpha}(y) G^{\beta}(y) \\
&= \sup_{a<y<b} \left(\int_y^b f_x(t)\,dt \right)^{\alpha} \left(\int_a^y g(t)\,dt \right)^{\beta} \\
&\approx \sup_{a<y<b} \left(\int_a^y f_x(t) G^{\frac{\beta+s}{\alpha}}(t)\,dt \right)^{\alpha} G^{-s}(y) \\
&= \sup_{x<y<b} \left(\int_x^y f(t) G^{\frac{\beta+s}{\alpha}}(t)\,dt \right)^{\alpha} G^{-s}(y).
\end{aligned}$$

(iii) Let us consider the function

$$f_x(y) := \chi_{(x,b)}(y) f(y).$$

By using the equivalence $B_1(\alpha,\beta) \approx B_6(\alpha,\beta,s)$ from Theorem A, we get

$$\begin{aligned}
\sup_{x<y<b} B_1(y;\alpha,\beta) &= \sup_{x<y<b} F^{\alpha}(y) G^{\beta}(y) \\
&= \sup_{a<y<b} \left(\int_y^b f_x(t)\,dt \right)^{\alpha} \left(\int_a^y g(t)\,dt \right)^{\beta}
\end{aligned}$$

$$
\begin{aligned}
&\approx \sup_{a<y<b} \left(\int_y^b f_x(t) G^{\frac{\beta}{\alpha+s}}(t)\,dt \right)^{\alpha+s} F^{-s}(y) \\
&= \sup_{x<y<b} \left(\int_y^b f(t) G^{\frac{\beta}{\alpha+s}}(t)\,dt \right)^{\alpha+s} F^{-s}(y) \\
&= \sup_{x<y<b} B_6(y;\alpha,\beta,s).
\end{aligned}
$$

(iv) Let us consider the function

$$
f_x(y) := \chi_{(x,b)}(y) f(y).
$$

By using the equivalence $B_1(\alpha,\beta) \approx B_8(\alpha,\beta,s)$ from Theorem A, we get

$$
\begin{aligned}
\sup_{x<y<b} B_1(y;\alpha,\beta) &= \sup_{x<y<b} F^{\alpha}(y) G^{\beta}(y) \\
&= \sup_{a<y<b} \left(\int_y^b f_x(t)\,dt \right)^{\alpha} \left(\int_a^y g(t)\,dt \right)^{\beta} \\
&\approx \sup_{a<y<b} \left(\int_a^y f_x(t) G^{\frac{\beta}{\alpha-s}}(t)\,dt \right)^{\alpha-s} F^{s}(y) \\
&= \sup_{x<y<b} \left(\int_x^y f(t) G^{\frac{\beta}{\alpha-s}}(t)\,dt \right)^{\alpha-s} F^{s}(y).
\end{aligned}
$$

(v) Let us consider the function

$$
f_x(y) := \chi_{(x,b)}(y) f(y).
$$

By using the equivalence $B_1(\alpha,\beta) \approx B_9(\alpha,\beta,s)$ from Theorem A, we get

$$
\begin{aligned}
\sup_{x<y<b} B_1(y;\alpha,\beta) &= \sup_{x<y<b} F^{\alpha}(y) G^{\beta}(y) \\
&= \sup_{a<y<b} \left(\int_y^b f_x(t)\,dt \right)^{\alpha} \left(\int_a^y g(t)\,dt \right)^{\beta} \\
&\approx \sup_{a<y<b} \left(\int_y^b f_x(t) G^{\frac{\beta}{\alpha-s}}(t)\,dt \right)^{\alpha-s} F^{s}(y) \\
&= \sup_{x<y<b} \left(\int_y^b f(t) G^{\frac{\beta}{\alpha-s}}(t)\,dt \right)^{\alpha-s} F^{s}(y) \\
&= \sup_{x<y<b} B_9(y;\alpha,\beta,s).
\end{aligned}
$$

(vi) Let us consider the function

$$
f_x(y) := \chi_{(x,b)}(y) f(y).
$$

By using the equivalence $B_1(\alpha,\beta) \approx B_{12}(\alpha,\beta,s)$ from Theorem A, we get

$$\begin{aligned}
\sup_{x<y<b} B_1(y;\alpha,\beta) &= \sup_{x<y<b} F^\alpha(y) G^\beta(y) \\
&= \sup_{a<y<b} \left(\int_y^b f_x(t)\,dt\right)^\alpha \left(\int_a^y g(t)\,dt\right)^\beta \\
&\approx \inf_{h\geq 0} \sup_{a<y<b} \Big(h(y)+G(y)\Big)^s \left(\int_y^b f_x(z) h(z)^{\frac{\beta-s}{\alpha}}\,dz\right)^\alpha \\
&= \inf_{h\geq 0} \sup_{x<y<b} \Big(h(y)+G(y)\Big)^s \left(\int_y^b f(z) h(z)^{\frac{\beta-s}{\alpha}}\,dz\right)^\alpha \\
&= \inf_{h\geq 0} \sup_{x<y<b} B_{12}(y;\alpha,\beta,s;h).
\end{aligned}$$

(vii) Let us consider the function

$$f_x(y) := \chi_{(x,b)}(y) f(y).$$

By using the equivalence $B_1(\alpha,\beta) \approx B_{14}(\alpha,\beta,s)$ from Theorem A, we get

$$\begin{aligned}
\sup_{x<y<b} B_1(y;\alpha,\beta) &= \sup_{x<y<b} F^\alpha(y) G^\beta(y) \\
&= \sup_{a<y<b} \left(\int_y^b f_x(t)\,dt\right)^\alpha \left(\int_a^y g(t)\,dt\right)^\beta \\
&\approx \inf_{h\geq 0} \sup_{a<y<b} h(y)^{-s} \left(\int_a^y f_x(z) \Big(h(z)+G(z)\Big)^{\frac{\beta+s}{\alpha}}\,dz\right)^\alpha \\
&= \inf_{h\geq 0} \sup_{x<y<b} h(y)^{-s} \left(\int_x^y f(z) \Big(h(z)+G(z)\Big)^{\frac{\beta+s}{\alpha}}\,dz\right)^\alpha.
\end{aligned}$$

$(viii)$ Let us consider the function

$$g_x(y) := \chi_{(a,x)}(y) g(y).$$

By using the equivalence $B_1(\alpha,\beta) \approx B_3(\alpha,\beta,s)$ from Theorem A, we get

$$\begin{aligned}
\sup_{a<y<x} B_1(y;\alpha,\beta) &= \sup_{a<y<x} F^\alpha(y) G^\beta(y) \\
&= \sup_{a<y<b} \left(\int_y^b f(t)\,dt\right)^\alpha \left(\int_a^y g_x(t)\,dt\right)^\beta \\
&\approx \sup_{a<y<b} \left(\int_a^y g_x(t) F^{\frac{\alpha-s}{\beta}}(t)\,dt\right)^\beta F^s(y) \\
&= \sup_{a<y<x} \left(\int_a^y g(t) F^{\frac{\alpha-s}{\beta}}(t)\,dt\right)^\beta F^s(y) \\
&= \sup_{a<y<x} B_3(y;\alpha,\beta,s).
\end{aligned}$$

(ix) Let us consider the function

$$g_x(y) := \chi_{(a,x)}(y)g(y).$$

By using the equivalence $B_1(\alpha,\beta) \approx B_5(\alpha,\beta,s)$ from Theorem A, we get

$$\begin{aligned}\sup_{a<y<x} B_1(y;\alpha,\beta) &= \sup_{a<y<x} F^{\alpha}(y)G^{\beta}(y)\\ &= \sup_{a<y<b}\left(\int_y^b f(t)\,dt\right)^{\alpha}\left(\int_a^y g_x(t)\,dt\right)^{\beta}\\ &\approx \sup_{a<y<b}\left(\int_y^b g_x(t)F^{\frac{\alpha+s}{\beta}}(t)\,dt\right)^{\beta} F^{-s}(y)\\ &= \sup_{a<y<x}\left(\int_y^x g(t)F^{\frac{\alpha+s}{\beta}}(t)\,dt\right)^{\beta} F^{-s}(y).\end{aligned}$$

(x) Let us consider the function

$$g_x(y) := \chi_{(a,x)}(y)g(y).$$

By using the equivalence $B_1(\alpha,\beta) \approx B_7(\alpha,\beta,s)$ from Theorem A, we get

$$\begin{aligned}\sup_{a<y<x} B_1(y;\alpha,\beta) &= \sup_{a<y<x} F^{\alpha}(y)G^{\beta}(y)\\ &= \sup_{a<y<b}\left(\int_y^b f(t)\,dt\right)^{\alpha}\left(\int_a^y g_x(t)\,dt\right)^{\beta}\\ &\approx \sup_{a<y<b}\left(\int_a^y g_x(t)F^{\frac{\alpha}{\beta+s}}(t)\,dt\right)^{\beta+s} G^{-s}(y)\\ &= \sup_{a<y<x}\left(\int_a^y g(t)F^{\frac{\alpha}{\beta+s}}(t)\,dt\right)^{\beta+s} G^{-s}(y)\\ &= \sup_{a<y<x} B_7(y;\alpha,\beta,s).\end{aligned}$$

(xi) Let us consider the function

$$g_x(y) := \chi_{(a,x)}(y)g(y).$$

By using the equivalence $B_1(\alpha,\beta) \approx B_{10}(\alpha,\beta,s)$ from Theorem A, we get

$$\begin{aligned}\sup_{a<y<x} B_1(y;\alpha,\beta) &= \sup_{a<y<x} F^{\alpha}(y)G^{\beta}(y)\\ &= \sup_{a<y<b}\left(\int_y^b f(t)\,dt\right)^{\alpha}\left(\int_a^y g_x(t)\,dt\right)^{\beta}\\ &\approx \sup_{a<y<b}\left(\int_y^b g_x(t)F^{\frac{\alpha}{\beta-s}}(t)\,dt\right)^{\beta-s} G^{s}(y)\end{aligned}$$

$$= \sup_{a<y<x} \left(\int_y^x g(t) F^{\frac{\alpha}{\beta-s}}(t)\, dt \right)^{\beta-s} G^s(y).$$

(xii) Let us consider the function

$$g_x(y) := \chi_{(a,x)}(y) g(y).$$

By using the equivalence $B_1(\alpha,\beta) \approx B_{11}(\alpha,\beta,s)$ from Theorem A, we get

$$\begin{aligned}
\sup_{a<y<x} B_1(y;\alpha,\beta) &= \sup_{a<y<x} F^{\alpha}(y) G^{\beta}(y) \\
&= \sup_{a<y<b} \left(\int_y^b f(t)\, dt \right)^{\alpha} \left(\int_a^y g_x(t)\, dt \right)^{\beta} \\
&\approx \sup_{a<y<b} \left(\int_a^y g_x(t) F^{\frac{\alpha}{\beta-s}}(t)\, dt \right)^{\beta-s} G^s(y) \\
&= \sup_{a<y<x} \left(\int_a^y g(t) F^{\frac{\alpha}{\beta-s}}(t)\, dt \right)^{\beta-s} G^s(y) \\
&= \sup_{a<y<x} B_{11}(y;\alpha,\beta,s).
\end{aligned}$$

$(xiii)$ Let us consider the function

$$g_x(y) := \chi_{(a,x)}(y) g(y).$$

By using the equivalence $B_1(\alpha,\beta) \approx B_{13}(\alpha,\beta,s)$ from Theorem A, we get

$$\begin{aligned}
\sup_{a<y<x} B_1(y;\alpha,\beta) &= \sup_{x<y<b} F^{\alpha}(y) G^{\beta}(y) \\
&= \sup_{a<y<b} \left(\int_y^b f(t)\, dt \right)^{\alpha} \left(\int_a^y g_x(t)\, dt \right)^{\beta} \\
&\approx \inf_{h\geq 0} \sup_{a<y<b} \Big(h(y) + F(y) \Big)^{s} \left(\int_a^y g_x(z) h(z)^{\frac{\alpha-s}{\beta}}\, dz \right)^{\beta} \\
&= \inf_{h\geq 0} \sup_{a<y<x} \Big(h(y) + F(y) \Big)^{s} \left(\int_a^y g(z) h(z)^{\frac{\alpha-s}{\beta}}\, dz \right)^{\beta} \\
&= \inf_{h\geq 0} \sup_{a<y<x} B_{13}(y;\alpha,\beta,s;h).
\end{aligned}$$

(xiv) Let us consider the function

$$g_x(y) := \chi_{(a,x)}(y) g(y).$$

By using the equivalence $B_1(\alpha,\beta) \approx B_{15}(\alpha,\beta,s)$ from Theorem A, we get

$$\sup_{a<y<x} B_1(y;\alpha,\beta) = \sup_{a<y<x} F^{\alpha}(y) G^{\beta}(y)$$

$$= \sup_{a<y<b} \left(\int_y^b f(t)\,dt \right)^{\alpha} \left(\int_a^y g_x(t)\,dt \right)^{\beta}$$

$$\approx \inf_{h\ge 0} \sup_{a<y<b} h(y)^{-s} \left(\int_y^b g_x(z) \Big(h(z) + F(z) \Big)^{\frac{\alpha+s}{\beta}} dz \right)^{\beta}$$

$$= \inf_{h\ge 0} \sup_{a<y<x} h(y)^{-s} \left(\int_y^x g(z) \Big(h(z) + F(z) \Big)^{\frac{\alpha+s}{\beta}} dz \right)^{\beta}.$$

□

Remark 2.2. Theorem 2.1 suggests that the equivalence is independent of x.

Remark 2.3. Theorem 2.1 demonstrates that, in Theorem A, the supremum over the interval (a,b) can be considered in certain truncated intervals. This consideration will be useful in the results of next section. Moreover, Theorem A can be obtained by Theorem 2.1 by taking $x=a$ or $x=b$ appropriately.

3 Equivalence Theorem for Bilinear Hardy Inequality

We prove the following:

Theorem 3.1. *Let* $-\infty \le a < b \le \infty$, $\alpha, \beta, \gamma, s, s_1, s_2$ *be positive numbers and* $f, g, h, h_1, h_2 \in \mathfrak{M}^+$. *Let* F, G *be as in* (1.4) *and denote*

$$H(x) := \int_a^x h(t)\,dt. \tag{3.1}$$

Assume that $F(x)$, $G(x)$ *and* $H(x)$ *are finite for all* $x \in (a,b)$. *Consider*

$\tilde{B}_1(x;\alpha,\beta,\gamma) := F^{\alpha}(x) G^{\beta}(x) H^{\gamma}(x);$

$\tilde{B}_2(x;\alpha,\beta,\gamma,s_1,s_2) := \left(\int_x^b f(t) G^{\frac{\beta-s_1}{\alpha}}(t) H^{\frac{\gamma-s_2}{\alpha}}(t)\,dt \right)^{\alpha} G^{s_1}(x) H^{s_2}(x);$

$\tilde{B}_3(x;\alpha,\beta,\gamma,s_1,s_2) := \left(\int_a^x f(t) G^{\frac{\beta+s_1}{\alpha}}(t) H^{\frac{\gamma+s_2}{\alpha}}(t)\,dt \right)^{\alpha} G^{-s_1}(x) H^{-s_2}(x);$

$\tilde{B}_4(x;\alpha,\beta,\gamma,s_1,s_2) := \left(\int_a^x f(t) G^{\frac{\beta}{\alpha+s_1}}(t) H^{\frac{\gamma-s_2}{\alpha+s_1}}(t)\,dt \right)^{\alpha+s_1} F^{-s_1}(x) H^{s_2}(x);$

$\tilde{B}_5(x;\alpha,\beta,\gamma,s_1,s_2) := \left(\int_a^x f(t) G^{\frac{\beta-s_2}{\alpha+s_1}}(t) H^{\frac{\gamma}{\alpha+s_1}}(t)\,dt \right)^{\alpha+s_1} F^{-s_1}(x) G^{s_2}(x);$

$\tilde{B}_6(x;\alpha,\beta,\gamma,s_1,s_2) := \left(\int_a^x f(t) G^{\frac{\beta}{\alpha-s_1}}(t) H^{\frac{\gamma+s_2}{\alpha-s_1}}(t)\,dt \right)^{\alpha-s_1} F^{s_1}(x) H^{-s_2}(x), \quad \alpha > s_1;$

$\tilde{B}_7(x;\alpha,\beta,\gamma,s_1,s_2) := \left(\int_a^x f(t) G^{\frac{\beta+s_2}{\alpha-s_1}}(t) H^{\frac{\gamma}{\alpha-s_1}}(t)\,dt \right)^{\alpha-s_1} F^{s_1}(x) G^{-s_2}(x), \quad \alpha > s_1;$

$\tilde{B}_8(x;\alpha,\beta,\gamma,s_1,s_2) := \left(\int_x^b f(t) G^{\frac{\beta}{\alpha-s_1}}(t)\,dt \right)^{\alpha-s_1} H^{s_2}(x) \left(\int_x^b f(t) H^{\frac{\gamma-s_2}{s_1}}(t)\,dt \right)^{s_1}, \quad \alpha < s_1;$

$$\tilde{B}_9(x;\alpha,\beta,\gamma,s_1,s_2) := \left(\int_x^b f(t)H^{\frac{\gamma}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1} G^{s_2}(x)\left(\int_x^b f(t)G^{\frac{\beta-s_2}{s_1}}(t)\,dt\right)^{s_1}, \quad \alpha<s_1;$$

$$\tilde{B}_{10}(x;\alpha,\beta,\gamma,s_1,s_2) := \left(\int_a^x g(t)F^{\frac{\alpha/2}{\beta+s_1}}(t)\,dt\right)^{\beta+s_1}\left(\int_a^x h(t)F^{\frac{\alpha/2}{\gamma+s_2}}(t)\,dt\right)^{\gamma+s_2} G^{-s_1}(x)H^{-s_2}(x);$$

$$\tilde{B}_{11}(x;\alpha,\beta,\gamma,s) := \left(\int_x^b f(t)G^{\frac{\beta(1-s)}{\alpha}}(t)H^{\frac{\gamma(1-s)}{\alpha}}(t)\,dt\right)^{\alpha} G^{\beta s}(x)H^{\gamma s}(x);$$

$$\tilde{B}_{12}(x;\alpha,\beta,\gamma,s) := \left(\int_a^x f(t)G^{\frac{\beta(1+s)}{\alpha}}(t)H^{\frac{\gamma(1+s)}{\alpha}}(t)\,dt\right)^{\alpha} G^{-\beta s}(x)H^{-\gamma s};$$

$$\tilde{B}_{13}(x;\alpha,\beta,\gamma,s) := \left(\int_x^b f(t)G^{\frac{\beta}{\alpha+s}}(t)H^{\frac{\gamma}{\alpha+s}}(t)\,dt\right)^{\alpha+s} F^{-s}(x);$$

$$\tilde{B}_{14}(x;\alpha,\beta,\gamma,s) := \left(\int_a^x f(t)G^{\frac{\beta}{\alpha-s}}(t)H^{\frac{\gamma}{\alpha-s}}(t)\,dt\right)^{\alpha-s} F^{s}(x), \quad \alpha>s;$$

$$\tilde{B}_{15}(x;\alpha,\beta,\gamma,s) := \left(\int_x^b f(t)G^{\frac{\beta}{\alpha-s}}(t)H^{\frac{\gamma}{\alpha-s}}(t)\,dt\right)^{\alpha-s} F^{s}(x), \quad \alpha<s;$$

$$\tilde{B}_{16}(x;\alpha,\beta,\gamma,s;h_1) := \left(\int_x^b f(t)h_1^{\frac{\gamma-s}{\alpha}}(t)\,dt\right)^{\alpha} G^{\beta}(x)\Big(h_1(x)+H(x)\Big)^{s}, \quad \gamma<s;$$

$$\tilde{B}_{17}(x;\alpha,\beta,\gamma,s;h_1) := \left(\int_x^b f(t)h_1^{\frac{\beta-s}{\alpha}}(t)\,dt\right)^{\alpha} H^{\gamma}(x)\Big(h_1(x)+G(x)\Big)^{s}, \quad \beta<s;$$

$$\tilde{B}_{18}(x;\alpha,\beta,\gamma,s_1,s_2;h_1,h_2) := \left(\int_x^b f(t)h_1^{\frac{\beta-s_1}{\alpha}}(t)h_2^{\frac{\gamma-s_2}{\alpha}}(t)\,dt\right)^{\alpha}\left\{\sup_{a<y<x}\Big(h_1(y)+G(y)\Big)^{s_1}\right\}\times$$
$$\times\Big(h_2(x)+H(x)\Big)^{s_2}, \quad \beta<s_1,\ \gamma<s_2;$$

$$\tilde{B}_{19}(x;\alpha,\beta,\gamma,s_1,s_2;h_1,h_2) := \left(\int_x^b f(t)h_1^{\frac{\beta-s_1}{\alpha}}(t)h_2^{\frac{\gamma-s_2}{\alpha}}(t)\,dt\right)^{\alpha}\Big(h_1(x)+G(x)\Big)^{s_1}\times$$
$$\times\left\{\sup_{a<y<x}\Big(h_2(y)+H(y)\Big)^{s_2}\right\}, \quad \beta<s_1,\ \gamma<s_2;$$

$$\tilde{B}_{20}(x;\alpha,\beta,\gamma,s_1,s_2;h_1,h_2) := \left(\int_x^b f(t)h_1^{\frac{\beta-s_1}{\alpha}}(t)h_2^{\frac{\gamma-s_2}{\alpha}}(t)\,dt\right)^{\alpha}\left\{\sup_{a<y<x}\Big(h_1(y)+G(y)\Big)^{s_1}\right\}\times$$
$$\times\left\{\sup_{a<y<x}\Big(h_2(y)+H(y)\Big)^{s_2}\right\}, \quad \beta<s_1,\ \gamma<s_2;$$

$$\tilde{B}_{21}(x;\alpha,\beta,\gamma,s_1,s_2;h_1,h_2) := \left(\int_a^x f(t)\Big(h_1(t)+G(t)\Big)^{\frac{\beta+s_1}{\alpha}}\Big(h_2(t)+H(t)\Big)^{\frac{\gamma+s_2}{\alpha}}dt\right)^{\alpha}\times$$
$$\times\left\{\sup_{x<y<b} h_1^{-s_1}(y)\right\}h_2^{-s_2}(x);$$

$$\tilde{B}_{22}(x;\alpha,\beta,\gamma,s_1,s_2;h_1,h_2) := \left(\int_a^x f(t)\Big(h_1(t)+G(t)\Big)^{\frac{\beta+s_1}{\alpha}}\Big(h_2(t)+H(t)\Big)^{\frac{\gamma+s_2}{\alpha}}dt\right)^{\alpha}\times$$

$$\times h_1^{-s_1}(x)\left\{\sup_{x<y<b} h_2^{-s_2}(y)\right\};$$

$$\tilde{B}_{23}(x;\alpha,\beta,\gamma,s_1,s_2;h_1,h_2) := \left(\int_a^x f(t)\Big(h_1(t)+G(t)\Big)^{\frac{\beta+s_1}{\alpha}}\Big(h_2(t)+H(t)\Big)^{\frac{\gamma+s_2}{\alpha}} dt\right)^{\alpha} \times$$

$$\times \left\{\sup_{x<y<b} h_1^{-s_1}(y)\right\}\left\{\sup_{x<y<b} h_2^{-s_2}(y)\right\}.$$

The numbers

$$\tilde{B}_1(\alpha,\beta,\gamma) := \sup_{a<x<b} \tilde{B}_1(x;\alpha,\beta,\gamma),$$

$$\tilde{B}_i(\alpha,\beta,\gamma,s_1,s_2) := \sup_{a<x<b} \tilde{B}_i(x;\alpha,\beta,\gamma,s_1,s_2); \quad (i=2,3,4,5,6,7,8,9,10),$$

$$\tilde{B}_i(\alpha,\beta,\gamma,s) := \sup_{a<x<b} \tilde{B}_i(x;\alpha,\beta,\gamma,s); \quad (i=11,12,13,14,15),$$

$$\tilde{B}_i(\alpha,\beta,\gamma,s) := \inf_{h_1\geq 0}\sup_{a<x<b} \tilde{B}_i(x;\alpha,\beta,\gamma,s;h_1); \quad (i=16,17)$$

and

$$\tilde{B}_i(\alpha,\beta,\gamma,s_1,s_2) := \inf_{h_1,h_2\geq 0}\sup_{a<x<b} \tilde{B}_i(x;\alpha,\beta,\gamma,s_1,s_2;h_1,h_2); \quad (i=18,19,20,21,22,23)$$

are mutually equivalent in the sense that if one number is finite then the others are so.

Proof. The equivalence $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_i(\alpha,\beta,\gamma,s_1,s_2) \approx \tilde{B}_j(\alpha,\beta,\gamma,s)$, $(i = 2,3,10)$, $(j = 13,14,15)$ has been proved in [7]. Here, we give different proofs for the equivalence

$$\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_k(\alpha,\beta,\gamma,s_1,s_2),\ (k=2,3)$$

and prove the remaining ones.

(i) $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_2(\alpha,\beta,\gamma,s_1,s_2)$

By using the equivalence $B_1(\alpha,\beta) \approx B_2(\alpha,\beta,s)$ from Theorem A and in view of Theorem 2.1 (i)

$$\sup_{x<y<b} F^{\alpha}(y)G^{\beta}(y) \approx \sup_{x<y<b}\left(\int_y^b f(t)G^{\frac{\beta-s_1}{\alpha}}(t)\,dt\right)^{\alpha} G^{s_1}(y),$$

we get

$$\begin{aligned}\tilde{B}_1(\alpha,\beta,\gamma) &= \sup_{a<x<b} \tilde{B}_1(x;\alpha,\beta,\gamma)\\ &= \sup_{a<x<b} F^{\alpha}(x)G^{\beta}(x)H^{\gamma}(x)\end{aligned}$$

$$
\begin{aligned}
&= \sup_{a<x<b} H^{\gamma}(x) \sup_{x<y<b} F^{\alpha}(y) G^{\beta}(y) \\
&\approx \sup_{a<x<b} H^{\gamma}(x) \sup_{x<y<b} G^{s_1}(y) \left(\int_y^b f(t) G^{\frac{\beta-s_1}{\alpha}}(t)\, dt \right)^{\alpha} \\
&= \sup_{a<x<b} G^{s_1}(x) \sup_{x<y<b} H^{\gamma}(y) \left(\int_y^b f(t) G^{\frac{\beta-s_1}{\alpha}}(t)\, dt \right)^{\alpha} \\
&\approx \sup_{a<x<b} G^{s_1}(y) \sup_{x<y<b} H^{s_2}(y) \left(\int_y^b f(t) G^{\frac{\beta-s_1}{\alpha}}(t) H^{\frac{\gamma-s_2}{\alpha}}(t)\, dt \right)^{\alpha} \\
&= \sup_{a<x<b} \left(\int_x^b f(t) G^{\frac{\beta-s_1}{\alpha}}(t) H^{\frac{\gamma-s_2}{\alpha}}(t)\, dt \right)^{\alpha} G^{s_1}(x) H^{s_2}(x) \\
&= \sup_{a<x<b} \tilde{B}_2(x; \alpha, \beta, \gamma, s_1, s_2) \\
&= \tilde{B}_2(\alpha, \beta, \gamma, s_1, s_2).
\end{aligned}
$$

(ii) $\tilde{B}_1(\alpha, \beta, \gamma) \approx \tilde{B}_3(\alpha, \beta, \gamma, s_1, s_2)$

By using the equivalence $B_1(\alpha, \beta) \approx B_4(\alpha, \beta, s)$ from Theorem A and in view of Theorem 2.1 (ii)

$$
\sup_{x<y<b} F^{\alpha}(y) G^{\beta}(y) \approx \sup_{x<y<b} \left(\int_x^y f(t) G^{\frac{\beta+s_1}{\alpha}}(t)\, dt \right)^{\alpha} G^{-s_1}(y),
$$

we get

$$
\begin{aligned}
\tilde{B}_1(\alpha, \beta, \gamma) &= \sup_{a<x<b} \tilde{B}_1(x; \alpha, \beta, \gamma) \\
&= \sup_{a<x<b} F^{\alpha}(x) G^{\beta}(x) H^{\gamma}(x) \\
&= \sup_{a<x<b} H^{\gamma}(x) \sup_{x<y<b} F^{\alpha}(y) G^{\beta}(y) \\
&\approx \sup_{a<x<b} H^{\gamma}(x) \sup_{x<y<b} G^{-s_1}(y) \left(\int_x^y f(t) G^{\frac{\beta+s_1}{\alpha}}(t)\, dt \right)^{\alpha} \\
&= \sup_{a<y<b} G^{-s_1}(y) \sup_{a<x<y} H^{\gamma}(x) \left(\int_x^y f(t) G^{\frac{\beta+s_1}{\alpha}}(t)\, dt \right)^{\alpha} \\
&\approx \sup_{a<y<b} G^{-s_1}(y) \sup_{a<x<y} H^{-s_2}(x) \left(\int_a^x f(t) G^{\frac{\beta+s_1}{\alpha}}(t) H^{\frac{\gamma+s_2}{\alpha}}(t)\, dt \right)^{\alpha} \\
&= \sup_{a<y<b} \left(\int_a^y f(t) G^{\frac{\beta+s_1}{\alpha}}(t) H^{\frac{\gamma+s_2}{\alpha}}(t)\, dt \right)^{\alpha} G^{-s_1}(y) H^{-s_2}(y) \\
&= \sup_{a<x<b} \left(\int_a^x f(t) G^{\frac{\beta+s_1}{\alpha}}(t) H^{\frac{\gamma+s_2}{\alpha}}(t)\, dt \right)^{\alpha} G^{-s_1}(x) H^{-s_2}(x)
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{a<x<b} \tilde{B}_3(x;\alpha,\beta,\gamma,s_1,s_2)\\
&=\tilde{B}_3(\alpha,\beta,\gamma,s_1,s_2).
\end{aligned}
$$

(iii) $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_4(\alpha,\beta,\gamma,s_1,s_2)$

By using the equivalences $B_1(\alpha,\beta) \approx B_6(\alpha,\beta,s)$, $B_1(\alpha,\beta) \approx B_2(\alpha,\beta,s)$ from Theorem A and in view of Theorem 2.1 (iii)

$$
\sup_{x<y<b} F^{\alpha}(y)G^{\beta}(y) \approx \sup_{x<y<b} \left(\int_y^b f(t) G^{\frac{\beta}{\alpha+s_1}}(t)\,dt \right)^{\alpha+s_1} F^{-s_1}(y),
$$

we get

$$
\begin{aligned}
\tilde{B}_1(\alpha,\beta,\gamma) &= \sup_{a<x<b} \tilde{B}_1(x;\alpha,\beta,\gamma)\\
&= \sup_{a<x<b} F^{\alpha}(x)G^{\beta}(x)H^{\gamma}(x)\\
&= \sup_{a<x<b} H^{\gamma}(x) \sup_{x<y<b} F^{\alpha}(y)G^{\beta}(y)\\
&\approx \sup_{a<x<b} H^{\gamma}(x) \sup_{x<y<b} \left(\int_y^b f(t) G^{\frac{\beta}{\alpha+s_1}}(t)\,dt \right)^{\alpha+s_1} F^{-s_1}(y)\\
&= \sup_{a<x<b} F^{-s_1}(x) \sup_{x<y<b} \left(\int_y^b f(t) G^{\frac{\beta}{\alpha+s_1}}(t)\,dt \right)^{\alpha+s_1} H^{\gamma}(y)\\
&\approx \sup_{a<x<b} F^{-s_1}(x) \sup_{x<y<b} \left(\int_y^b f(t) G^{\frac{\beta}{\alpha+s_1}}(t) H^{\frac{\gamma-s_2}{\alpha+s_1}}(t)\,dt \right)^{\alpha+s_1} H^{s_2}(y)\\
&= \sup_{a<x<b} F^{-s_1}(x) H^{s_2}(x) \left(\int_x^b f(t) G^{\frac{\beta}{\alpha+s_1}}(t) H^{\frac{\gamma-s_2}{\alpha+s_1}}(t)\,dt \right)^{\alpha+s_1}\\
&= \sup_{a<x<b} \tilde{B}_4(x;\alpha,\beta,\gamma,s_1,s_2)\\
&=\tilde{B}_4(\alpha,\beta,\gamma,s_1,s_2).
\end{aligned}
$$

(iv) $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_5(\alpha,\beta,\gamma,s_1,s_2)$

By using the equivalences $B_1(\alpha,\beta) \approx B_6(\alpha,\beta,s)$, $B_1(\alpha,\beta) \approx B_2(\alpha,\beta,s)$ from Theorem A and in view of Theorem 2.1 (iii)

$$
\sup_{x<y<b} F^{\alpha}(y)H^{\gamma}(y) \approx \sup_{x<y<b} \left(\int_y^b f(t) H^{\frac{\gamma}{\alpha+s_1}}(t)\,dt \right)^{\alpha+s_1} F^{-s_1}(y),
$$

we get

$$\begin{aligned}
\tilde{B}_1(\alpha,\beta,\gamma) &= \sup_{a<x<b} \tilde{B}_1(x;\alpha,\beta,\gamma) \\
&= \sup_{a<x<b} F^{\alpha}(x)G^{\beta}(x)H^{\gamma}(x) \\
&= \sup_{a<x<b} G^{\beta}(x) \sup_{x<y<b} F^{\alpha}(y)H^{\gamma}(y) \\
&\approx \sup_{a<x<b} G^{\beta}(x) \sup_{x<y<b} \left(\int_y^b f(t)H^{\frac{\gamma}{\alpha+s_1}}(t)\,dt\right)^{\alpha+s_1} F^{-s_1}(y) \\
&= \sup_{a<x<b} F^{-s_1}(x) \sup_{x<y<b} \left(\int_y^b f(t)H^{\frac{\gamma}{\alpha+s_1}}(t)\,dt\right)^{\alpha+s_1} G^{\beta}(y) \\
&\approx \sup_{a<x<b} F^{-s_1}(x) \sup_{x<y<b} \left(\int_y^b f(t)H^{\frac{\gamma}{\alpha+s_1}}(t)G^{\frac{\beta-s_2}{\alpha+s_1}}(t)\,dt\right)^{\alpha+s_1} G^{s_2}(y) \\
&= \sup_{a<x<b} F^{-s_1}(x)G^{s_2}(x) \left(\int_x^b f(t)H^{\frac{\gamma}{\alpha+s_1}}(t)G^{\frac{\beta-s_2}{\alpha+s_1}}(t)\,dt\right)^{\alpha+s_1} \\
&= \sup_{a<x<b} \tilde{B}_5(x;\alpha,\beta,\gamma,s_1,s_2) \\
&= \tilde{B}_5(\alpha,\beta,\gamma,s_1,s_2).
\end{aligned}$$

(v) $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_6(\alpha,\beta,\gamma,s_1,s_2)$

By using the equivalences $B_1(\alpha,\beta) \approx B_8(\alpha,\beta,s)$, $B_1(\alpha,\beta) \approx B_4(\alpha,\beta,s)$ from Theorem A and in view of Theorem 2.1 (iv)

$$\sup_{x<y<b} G^{\beta}(y)F^{\alpha}(y) \approx \sup_{x<y<b} \left(\int_x^y f(t)G^{\frac{\beta}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1} F^{s_1}(y); \;\; \alpha > s_1,$$

we have

$$\begin{aligned}
\tilde{B}_1(\alpha,\beta,\gamma) &= \sup_{a<x<b} \tilde{B}_1(x;\alpha,\beta,\gamma) \\
&= \sup_{a<x<b} F^{\alpha}(x)G^{\beta}(x)H^{\gamma}(x) \\
&= \sup_{a<x<b} H^{\gamma}(x) \sup_{x<y<b} F^{\alpha}(y)G^{\beta}(y) \\
&\approx \sup_{a<x<b} H^{\gamma}(x) \sup_{x<y<b} \left(\int_x^y f(t)G^{\frac{\beta}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1} F^{s_1}(y) \\
&= \sup_{a<x<b} F^{s_1}(x) \sup_{a<y<x} H^{\gamma}(y) \left(\int_y^x f(t)G^{\frac{\beta}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1} \\
&\approx \sup_{a<x<b} F^{s_1}(x) \sup_{a<y<x} H^{-s_2}(y) \left(\int_a^y f(t)H^{\frac{\gamma+s_2}{\alpha-s_1}}(t)G^{\frac{\beta}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1}
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{a<x<b} F^{s_1}(x) H^{-s_2}(x) \left(\int_a^x f(t) H^{\frac{\gamma+s_2}{\alpha-s_1}}(t) G^{\frac{\beta}{\alpha-s_1}}(t)\, dt \right)^{\alpha-s_1} \\
&= \sup_{a<x<b} \tilde{B}_6(x;\alpha,\beta,\gamma,s_1,s_2) \\
&= \tilde{B}_6(\alpha,\beta,\gamma,s_1,s_2).
\end{aligned}
$$

(vi) $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_7(\alpha,\beta,\gamma,s_1,s_2)$

By using the equivalences $B_1(\alpha,\beta) \approx B_8(\alpha,\beta,s)$, $B_1(\alpha,\beta) \approx B_4(\alpha,\beta,s)$ from Theorem A and in view of Theorem 2.1 (iv)

$$
\sup_{x<y<b} H^{\gamma}(y) F^{\alpha}(y) \approx \sup_{x<y<b} \left(\int_x^y f(t) H^{\frac{\gamma}{\alpha-s_1}}(t)\, dt \right)^{\alpha-s_1} F^{s_1}(y); \;\; \alpha > s_1,
$$

we have

$$
\begin{aligned}
\tilde{B}_1(\alpha,\beta,\gamma) &= \sup_{a<x<b} \tilde{B}_1(x;\alpha,\beta,\gamma) \\
&= \sup_{a<x<b} F^{\alpha}(x) G^{\beta}(x) H^{\gamma}(x) \\
&= \sup_{a<x<b} G^{\beta}(x) \sup_{x<y<b} F^{\alpha}(y) H^{\gamma}(y) \\
&\approx \sup_{a<x<b} G^{\beta}(x) \sup_{x<y<b} \left(\int_x^y f(t) H^{\frac{\gamma}{\alpha-s_1}}(t)\, dt \right)^{\alpha-s_1} F^{s_1}(y) \\
&= \sup_{a<x<b} F^{s_1}(x) \sup_{a<y<x} G^{\beta}(y) \left(\int_y^x f(t) H^{\frac{\gamma}{\alpha-s_1}}(t)\, dt \right)^{\alpha-s_1} \\
&\approx \sup_{a<x<b} F^{s_1}(x) \sup_{a<y<x} G^{-s_2}(y) \left(\int_a^y f(t) G^{\frac{\beta+s_2}{\alpha-s_1}}(t) H^{\frac{\gamma}{\alpha-s_1}}(t)\, dt \right)^{\alpha-s_1} \\
&= \sup_{a<x<b} F^{s_1}(x) G^{-s_2}(x) \left(\int_a^x f(t) G^{\frac{\beta+s_2}{\alpha-s_1}}(t) H^{\frac{\gamma}{\alpha-s_1}}(t)\, dt \right)^{\alpha-s_1} \\
&= \sup_{a<x<b} \tilde{B}_7(x;\alpha,\beta,\gamma,s_1,s_2) \\
&= \tilde{B}_7(\alpha,\beta,\gamma,s_1,s_2).
\end{aligned}
$$

(vii) $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_8(\alpha,\beta,\gamma,s_1,s_2)$

By using the equivalences $B_1(\alpha,\beta) \approx B_9(\alpha,\beta,s)$, $B_1(\alpha,\beta) \approx B_2(\alpha,\beta,s)$ from Theorem A and in view of Theorem 2.1 (v)

$$
\sup_{x<y<b} G^{\beta}(y) F^{\alpha}(y) \approx \sup_{x<y<b} \left(\int_y^b f(t) G^{\frac{\beta}{\alpha-s_1}}(t)\, dt \right)^{\alpha-s_1} F^{s_1}(y); \;\; \alpha < s_1,
$$

we have

$$\begin{aligned}
\tilde{B}_1(\alpha,\beta,\gamma) &= \sup_{a<x<b} \tilde{B}_1(x;\alpha,\beta,\gamma)\\
&= \sup_{a<x<b} F^{\alpha}(x)G^{\beta}(x)H^{\gamma}(x)\\
&= \sup_{a<x<b} H^{\gamma}(x) \sup_{x<y<b} F^{\alpha}(y)G^{\beta}(y)\\
&\approx \sup_{a<x<b} H^{\gamma}(x) \sup_{x<y<b} \left(\int_y^b f(t)G^{\frac{\beta}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1} F^{s_1}(y)\\
&= \sup_{a<x<b} \left(\int_x^b f(t)G^{\frac{\beta}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1} F^{s_1}(x)H^{\gamma}(x)\\
&= \sup_{a<x<b} \left(\int_x^b f(t)G^{\frac{\beta}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1} \sup_{x<y<b} F^{s_1}(y)H^{\gamma}(y)\\
&\approx \sup_{a<x<b} \left(\int_x^b f(t)G^{\frac{\beta}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1} \sup_{x<y<b} H^{s_2}(y)\left(\int_y^b f(t)H^{\frac{\gamma-s_2}{s_1}}(t)\,dt\right)^{s_1}\\
&= \sup_{a<x<b} \left(\int_x^b f(t)G^{\frac{\beta}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1} H^{s_2}(x)\left(\int_x^b f(t)H^{\frac{\gamma-s_2}{s_1}}(t)\,dt\right)^{s_1}\\
&= \sup_{a<x<b} \tilde{B}_8(x;\alpha,\beta,\gamma,s_1,s_2)\\
&=\tilde{B}_8(\alpha,\beta,\gamma,s_1,s_2).
\end{aligned}$$

(viii) $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_9(\alpha,\beta,\gamma,s_1,s_2)$

By using the equivalences $B_1(\alpha,\beta) \approx B_9(\alpha,\beta,s)$, $B_1(\alpha,\beta) \approx B_2(\alpha,\beta,s)$ from Theorem A and in view of Theorem 2.1 (v)

$$\sup_{x<y<b} H^{\gamma}(y)F^{\alpha}(y) \approx \sup_{x<y<b} \left(\int_y^b f(t)H^{\frac{\gamma}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1} F^{s_1}(y);\;\; \alpha < s_1,$$

we have

$$\begin{aligned}
\tilde{B}_1(\alpha,\beta,\gamma) &= \sup_{a<x<b} \tilde{B}_1(x;\alpha,\beta,\gamma)\\
&= \sup_{a<x<b} F^{\alpha}(x)G^{\beta}(x)H^{\gamma}(x)\\
&= \sup_{a<x<b} G^{\beta}(x) \sup_{x<y<b} F^{\alpha}(y)H^{\gamma}(y)\\
&\approx \sup_{a<x<b} G^{\beta}(x) \sup_{x<y<b} \left(\int_y^b f(t)H^{\frac{\gamma}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1} F^{s_1}(y)
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{a<x<b} G^{\beta}(x)F^{s_1}(x)\left(\int_x^b f(t)H^{\frac{\gamma}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1}\\
&= \sup_{a<x<b}\left(\int_x^b f(t)H^{\frac{\gamma}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1}\sup_{x<y<b}F^{s_1}(y)G^{\beta}(y)\\
&\approx \sup_{a<x<b}\left(\int_x^b f(t)H^{\frac{\gamma}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1}\sup_{x<y<b}G^{s_2}(y)\left(\int_y^b f(t)G^{\frac{\beta-s_2}{s_1}}(t)\,dt\right)^{s_1}\\
&= \sup_{a<x<b}\left(\int_x^b f(t)H^{\frac{\gamma}{\alpha-s_1}}(t)\,dt\right)^{\alpha-s_1} G^{s_2}(x)\left(\int_x^b f(t)G^{\frac{\beta-s_2}{s_1}}(t)\,dt\right)^{s_1}\\
&= \sup_{a<x<b}\tilde{B}_9(x;\alpha,\beta,\gamma,s_1,s_2)\\
&= \tilde{B}_9(\alpha,\beta,\gamma,s_1,s_2).
\end{aligned}
$$

(ix) $\tilde{B}_1(\alpha,\beta,\gamma)\approx\tilde{B}_{11}(\alpha,\beta,\gamma,s)$

Easily follows from $B_1(\alpha,\beta)\approx B_2(\alpha,\beta,s)$.

(x) $\tilde{B}_1(\alpha,\beta,\gamma)\approx\tilde{B}_{12}(\alpha,\beta,\gamma,s)$

Similarly from $B_1(\alpha,\beta)\approx B_4(\alpha,\beta,s)$.

(xi) $\tilde{B}_1(\alpha,\beta,\gamma)\approx\tilde{B}_{16}(\alpha,\beta,\gamma,s)$

By using the equivalences $B_1(\alpha,\beta)\approx B_{12}(\alpha,\beta,s)$ from Theorem A and in view of Theorem 2.1 (vi)

$$
\sup_{x<y<b} H^{\gamma}(y)F^{\alpha}(y)\approx\inf_{h_1\geq 0}\sup_{x<y<b}\Big(h_1(y)+H(y)\Big)^{s}\left(\int_y^b f(t)h_1^{\frac{\gamma-s}{\alpha}}(t)\,dt\right)^{\alpha};\quad \gamma<s,
$$

we have

$$
\begin{aligned}
\tilde{B}_1(\alpha,\beta,\gamma) &= \sup_{a<x<b}\tilde{B}_1(x;\alpha,\beta,\gamma)\\
&= \sup_{a<x<b}F^{\alpha}(x)G^{\beta}(x)H^{\gamma}(x)\\
&= \sup_{a<x<b}G^{\beta}(x)\sup_{x<y<b}F^{\alpha}(y)H^{\gamma}(y)\\
&\approx \inf_{h_1\geq 0}\sup_{a<x<b}G^{\beta}(x)\sup_{x<y<b}\Big(h_1(y)+H(y)\Big)^{s}\left(\int_y^b f(t)h_1^{\frac{\gamma-s}{\alpha}}(t)\,dt\right)^{\alpha}\\
&= \inf_{h_1\geq 0}\sup_{a<x<b}\left(\int_x^b f(t)h_1^{\frac{\gamma-s}{\alpha}}(t)\,dt\right)^{\alpha}G^{\beta}(x)\Big(h_1(x)+H(x)\Big)^{s}\\
&= \inf_{h_1\geq 0}\sup_{a<x<b}\tilde{B}_{16}(x;\alpha,\beta,\gamma,s;h_1)
\end{aligned}
$$

$$=\tilde{B}_{16}(\alpha,\beta,\gamma,s).$$

(xii) $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_{17}(\alpha,\beta,\beta,s)$

By using the equivalences $B_1(\alpha,\beta) \approx B_{12}(\alpha,\beta,s)$ from Theorem A and in view of Theorem 2.1 (vi)

$$\sup_{x<y<b} G^{\beta}(y)F^{\alpha}(y) \approx \inf_{h_1\geq 0} \sup_{x<y<b} \Big(h_1(y)+G(y)\Big)^s \left(\int_y^b f(t)h_1^{\frac{\beta-s}{\alpha}}(t)\,dt\right)^{\alpha}; \quad \beta<s,$$

we have

$$\begin{aligned}
\tilde{B}_1(\alpha,\beta,\gamma) &= \sup_{a<x<b} \tilde{B}_1(x;\alpha,\beta,\gamma)\\
&= \sup_{a<x<b} F^{\alpha}(x)G^{\beta}(x)H^{\gamma}(x)\\
&= \sup_{a<x<b} H^{\gamma}(x) \sup_{x<y<b} F^{\alpha}(y)G^{\beta}(y)\\
&\approx \inf_{h_1\geq 0} \sup_{a<x<b} H^{\gamma}(x) \sup_{x<y<b} \Big(h_1(y)+G(y)\Big)^s \left(\int_y^b f(t)h_1^{\frac{\beta-s}{\alpha}}(t)\,dt\right)^{\alpha}\\
&= \inf_{h_1\geq 0} \sup_{a<x<b} \left(\int_x^b f(t)h_1^{\frac{\beta-s}{\alpha}}(t)\,dt\right)^{\alpha} H^{\gamma}(x)\Big(h_1(x)+G(x)\Big)^s\\
&= \inf_{h_1\geq 0} \sup_{a<x<b} \tilde{B}_{17}(x;\alpha,\beta,\gamma,s;h_1)\\
&=\tilde{B}_{17}(\alpha,\beta,\gamma,s).
\end{aligned}$$

$(xiii)$ $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_{18}(\alpha,\beta,\gamma,s_1,s_2)$

By using the equivalence $B_1(\alpha,\beta) \approx B_{12}(\alpha,\beta,s)$ from Theorem A and in view of Theorem 2.1 (vi)

$$\sup_{x<y<b} F^{\alpha}(y)G^{\beta}(y) \approx \inf_{h_1>0} \sup_{x<y<b} \Big(h_1(y)+G(y)\Big)^s \left(\int_y^b f(z)h_1^{\frac{\beta-s}{\alpha}}(z)dz\right)^{\alpha}; \quad \beta<s$$

and also the fact that $H^{\gamma}(x)\geq 0$ as well as $\Big(h_1(x)+G(x)\Big)^{s_1}\geq 0$, we get

$$\begin{aligned}
\tilde{B}_1(\alpha,\beta,\gamma) &= \sup_{a<x<b} \tilde{B}_1(x;\alpha,\beta,\gamma)\\
&= \sup_{a<x<b} F^{\alpha}(x)G^{\beta}(x)H^{\gamma}(x)\\
&= \sup_{a<x<b} H^{\gamma}(x) \sup_{x<y<b} F^{\alpha}(y)G^{\beta}(y)
\end{aligned}$$

$$
\begin{aligned}
&\approx \sup_{a<x<b} H^\gamma(x) \inf_{h_1\ge 0} \sup_{x<y<b} \left(\int_y^b f(t) h_1^{\frac{\beta-s_1}{\alpha}}(t)\,dt\right)^\alpha \Big(h_1(y)+G(y)\Big)^{s_1}\\
&= \inf_{h_1\ge 0} \sup_{a<x<b} H^\gamma(x) \sup_{x<y<b} \left(\int_y^b f(t) h_1^{\frac{\beta-s_1}{\alpha}}(t)\,dt\right)^\alpha \Big(h_1(y)+G(y)\Big)^{s_1}\\
&\le \inf_{h_1\ge 0} \sup_{a<x<b} \left\{\sup_{a<y<x} \Big(h_1(y)+G(y)\Big)^{s_1}\right\} \left\{\sup_{x<y<b} \left(\int_y^b f(t) h_1^{\frac{\beta-s_1}{\alpha}}(t)\,dt\right)^\alpha H^\gamma(y)\right\}\\
&\approx \inf_{h_1\ge 0} \sup_{a<x<b} \left\{\sup_{a<y<x} \Big(h_1(y)+G(y)\Big)^{s_1}\right\} \times\\
&\qquad\qquad \times \left\{\inf_{h_2\ge 0} \sup_{x<y<b} \left(\int_y^b f(t) h_1^{\frac{\beta-s_1}{\alpha}}(t) h_2^{\frac{\gamma-s_2}{\alpha}}(t)\,dt\right)^\alpha \Big(h_2(y)+H(y)\Big)^{s_2}\right\}\\
&= \inf_{h_1\ge 0} \inf_{h_2\ge 0} \sup_{a<x<b} \left\{\sup_{a<y<x} \Big(h_1(y)+G(y)\Big)^{s_1}\right\} \times\\
&\qquad\qquad \times \left\{\sup_{x<y<b} \left(\int_y^b f(t) h_1^{\frac{\beta-s_1}{\alpha}}(t) h_2^{\frac{\gamma-s_2}{\alpha}}(t)\,dt\right)^\alpha \Big(h_2(y)+H(y)\Big)^{s_2}\right\}\\
&= \inf_{h_1,h_2\ge 0} \sup_{a<x<b} \left(\int_x^b f(t) h_1^{\frac{\beta-s_1}{\alpha}}(t) h_2^{\frac{\gamma-s_2}{\alpha}}(t)\,dt\right)^\alpha \times\\
&\qquad\qquad\qquad \times \left\{\sup_{a<y<x} \Big(h_1(y)+G(y)\Big)^{s_1}\right\} \Big(h_2(x)+H(x)\Big)^{s_2}\\
&= \inf_{h_1,h_2\ge 0} \sup_{a<x<b} \tilde{B}_{18}(x;\alpha,\beta,\gamma,s_1,s_2;h_1,h_2)\\
&=\tilde{B}_{18}(\alpha,\beta,\gamma,s_1,s_2).
\end{aligned}
$$

Thus

$$
\tilde{B}_1(\alpha,\beta,\gamma) \lesssim \tilde{B}_{18}(\alpha,\beta,\gamma,s_1,s_2). \tag{3.2}
$$

Since for $h_1(x)=G(x)$ and $h_2(x)=H(x)$, we have

$$
\begin{aligned}
\tilde{B}_{18}(x;\alpha,\beta,\gamma,s_1,s_2;G,H) &= 2^{s_1+s_2} \left(\int_x^b f(t) G^{\frac{\beta-s_1}{\alpha}}(t) H^{\frac{\gamma-s_2}{\alpha}}(t)\,dt\right)^\alpha G^{s_1}(x) H^{s_2}(x)\\
&= 2^{s_1+s_2} \tilde{B}_2(x;\alpha,\beta,\gamma,s_1,s_2)\\
&\le 2^{s_1+s_2} \sup_{a<x<b} \tilde{B}_2(x;\alpha,\beta,\gamma,s_1,s_2)\\
&= 2^{s_1+s_2} \tilde{B}_2(\alpha,\beta,\gamma,s_1,s_2)\\
&\approx 2^{s_1+s_2} \tilde{B}_1(\alpha,\beta,\gamma).
\end{aligned}
$$

The last relation holds by using $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_2(\alpha,\beta,\gamma,s_1,s_2)$. Clearly we get that

$$
\tilde{B}_{18}(\alpha,\beta,\gamma,s_1,s_2) \lesssim \tilde{B}_1(\alpha,\beta,\gamma). \tag{3.3}
$$

In view of (3.2) and (3.3), we obtain

$$\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_{18}(\alpha,\beta,\gamma,s_1,s_2).$$

(xiv) $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_{19}(\alpha,\beta,\gamma,s_1,s_2)$

Using the similar reasoning as in $(xiii)$.

(xv) $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_{20}(\alpha,\beta,\gamma,s_1,s_2)$

Using the similar reasoning as in $(xiii)$.

(xvi) $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_{21}(\alpha,\beta,\gamma,s_1,s_2)$

By using the equivalence $B_1(\alpha,\beta) \approx B_{14}(\alpha,\beta,s)$ from Theorem A and in view of Theorem 2.1 (vii)

$$\sup_{x<y<b} F^\alpha(y)G^\beta(y) \approx \inf_{h_1>0}\sup_{x<y<b} h_1^{-s_1}(y)\left(\int_x^y f(z)\Big(h_1(z)+G(z)\Big)^{\frac{\beta+s_1}{\alpha}}dz\right)^\alpha$$

and also the fact that $H^\gamma(x)\geq 0$ as well as $h_1^{-s_1}(x)\geq 0$, we get

$$\begin{aligned}
\tilde{B}_1(\alpha,\beta,\gamma) &= \sup_{a<x<b}\tilde{B}_1(x;\alpha,\beta,\gamma)\\
&= \sup_{a<x<b} F^\alpha(x)G^\beta(x)H^\gamma(x)\\
&= \sup_{a<x<b} H^\gamma(x)\sup_{x<y<b}F^\alpha(y)G^\beta(y)\\
&\approx \sup_{a<x<b} H^\gamma(x)\inf_{h_1\geq 0}\sup_{x<y<b}\left(\int_x^y f(t)\Big(h_1(t)+G(t)\Big)^{\frac{\beta+s_1}{\alpha}}dt\right)^\alpha h_1^{-s_1}(y)\\
&= \inf_{h_1\geq 0}\sup_{a<x<b} H^\gamma(x)\sup_{x<y<b} h_1^{-s_1}(y)\left(\int_x^y f(t)\Big(h_1(t)+G(t)\Big)^{\frac{\beta+s_1}{\alpha}}dt\right)^\alpha\\
&\leq \inf_{h_1\geq 0}\sup_{a<y<b}\left\{\sup_{y<x<b} h_1^{-s_1}(x)\right\}\left\{\sup_{a<x<y}\left(\int_x^y f(t)\Big(h_1(t)+G(t)\Big)^{\frac{\beta+s_1}{\alpha}}dt\right)^\alpha H^\gamma(x)\right\}\\
&\approx \inf_{h_1\geq 0}\sup_{a<y<b}\left\{\sup_{y<x<b} h_1^{-s_1}(x)\right\}\times\\
&\qquad\times\left\{\inf_{h_2\geq 0}\sup_{a<x<y} h_2^{-s_2}(x)\left(\int_a^x f(t)\Big(h_1(t)+G(t)\Big)^{\frac{\beta+s_1}{\alpha}}\Big(h_2(t)+H(t)\Big)^{\frac{\gamma+s_2}{\alpha}}dt\right)^\alpha\right\}\\
&= \inf_{h_1\geq 0}\inf_{h_2\geq 0}\sup_{a<y<b}\left\{\sup_{y<x<b} h_1^{-s_1}(x)\right\}\times
\end{aligned}$$

$$\times \left\{ \sup_{a<x<y} h_2^{-s_2}(x) \left(\int_a^x f(t)\Big(h_1(t)+G(t)\Big)^{\frac{\beta+s_1}{\alpha}} \Big(h_2(t)+H(t)\Big)^{\frac{\gamma+s_2}{\alpha}} dt \right)^{\alpha} \right\}$$

$$= \inf_{h_1,h_2\geq 0} \sup_{a<x<b} \left(\int_a^x f(t)\Big(h_1(t)+G(t)\Big)^{\frac{\beta+s_1}{\alpha}} \Big(h_2(t)+H(t)\Big)^{\frac{\gamma+s_2}{\alpha}} dt \right)^{\alpha} \times$$

$$\times \left\{ \sup_{x<y<b} h_1^{-s_1}(y) \right\} h_2^{-s_2}(x)$$

$$= \inf_{h_1,h_2\geq 0} \sup_{a<x<b} \tilde{B}_{21}(x;\alpha,\beta,\gamma,s_1,s_2;h_1,h_2)$$

$$= \tilde{B}_{21}(\alpha,\beta,\gamma,s_1,s_2).$$

Thus

$$\tilde{B}_1(\alpha,\beta,\gamma) \lesssim \tilde{B}_{21}(\alpha,\beta,\gamma,s_1,s_2). \tag{3.4}$$

Since for $h_1(x) = G(x)$ and $h_2(x) = H(x)$, we have

$$\begin{aligned}
\tilde{B}_{21}(x;\alpha,\beta,\gamma,s_1,s_2;G,H) &= 2^{\beta+\gamma+s_1+s_2} \left(\int_a^x f(t) G^{\frac{\beta+s_1}{\alpha}}(t) H^{\frac{\gamma+s_2}{\alpha}}(t)\, dt \right)^{\alpha} G^{-s_1}(x) H^{-s_2}(x) \\
&= 2^{\beta+\gamma+s_1+s_2} \tilde{B}_3(x;\alpha,\beta,\gamma,s_1,s_2) \\
&\leq 2^{\beta+\gamma+s_1+s_2} \sup_{a<x<b} \tilde{B}_3(x;\alpha,\beta,\gamma,s_1,s_2) \\
&= 2^{\beta+\gamma+s_1+s_2} \tilde{B}_3(\alpha,\beta,\gamma,s_1,s_2) \\
&\approx 2^{\beta+\gamma+s_1+s_2} \tilde{B}_1(\alpha,\beta,\gamma).
\end{aligned}$$

The last relation holds by using $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_3(\alpha,\beta,\gamma,s_1,s_2)$. Clearly we get that

$$\tilde{B}_{21}(\alpha,\beta,\gamma,s_1,s_2) \lesssim \tilde{B}_1(\alpha,\beta,\gamma). \tag{3.5}$$

Therefore, in view of (3.4) and (3.5), we find

$$\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_{21}(\alpha,\beta,\gamma,s_1,s_2).$$

(xvii) $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_{22}(\alpha,\beta,\gamma,s_1,s_2)$

Using the similar reasoning as in *(xvi)*.

(xviii) $\tilde{B}_1(\alpha,\beta,\gamma) \approx \tilde{B}_{23}(\alpha,\beta,\gamma,s_1,s_2)$

Using the similar reasoning as in *(xvi)*. □

We apply Theorem 3.1 and provide equivalent conditions for the bilinear inequality (1.5) to hold.

Theorem 3.2. *Let* $1 < \max(p_1,p_2) \le q < \infty$ *with* s, s_1, s_2 *be positive numbers. Define* U *as in* (1.1) *and*

$$V_i(x) = \int_a^x v_i^{1-p_i'}(t)\,dt, \quad i = 1,2.$$

Moreover define

$$\tilde{A}_1(s_1,s_2) := \sup_{a<x<b} \left(\int_x^b u(t) V_1^{\left(\frac{\frac{1}{p_1'}-s_1}{\frac{1}{q}}\right)}(t) V_2^{\left(\frac{\frac{1}{p_2'}-s_2}{\frac{1}{q}}\right)}(t)\,dt \right)^{\frac{1}{q}} V_1^{s_1}(x) V_2^{s_2}(x);$$

$$\tilde{A}_2(s_1,s_2) := \sup_{a<x<b} \left(\int_a^x u(t) V_1^{\left(\frac{\frac{1}{p_1'}+s_1}{\frac{1}{q}}\right)}(t) V_2^{\left(\frac{\frac{1}{p_2'}+s_2}{\frac{1}{q}}\right)}(t)\,dt \right)^{\frac{1}{q}} V_1^{-s_1}(x) V_2^{-s_2}(x);$$

$$\tilde{A}_3(s_1,s_2) := \sup_{a<x<b} \left(\int_a^x u(t) V_1^{\left(\frac{\frac{1}{p_1'}}{\frac{1}{q}+s_1}\right)}(t) V_2^{\left(\frac{\frac{1}{p_2'}-s_2}{\frac{1}{q}+s_1}\right)}(t)\,dt \right)^{\frac{1}{q}+s_1} U^{-s_1}(x) V_2^{s_2}(x);$$

$$\tilde{A}_4(s_1,s_2) := \sup_{a<x<b} \left(\int_a^x u(t) V_1^{\left(\frac{\frac{1}{p_1'}-s_2}{\frac{1}{q}+s_1}\right)}(t) V_2^{\left(\frac{\frac{1}{p_2'}}{\frac{1}{q}+s_1}\right)}(t)\,dt \right)^{\frac{1}{q}+s_1} U^{-s_1}(x) V_1^{s_2}(x);$$

$$\tilde{A}_5(s_1,s_2) := \sup_{a<x<b} \left(\int_a^x u(t) V_1^{\left(\frac{\frac{1}{p_1'}}{\frac{1}{q}-s_1}\right)}(t) V_2^{\left(\frac{\frac{1}{p_2'}+s_2}{\frac{1}{q}-s_1}\right)}(t)\,dt \right)^{\frac{1}{q}-s_1} U^{s_1}(x) V_2^{-s_2}(x), \quad \frac{1}{q} > s_1;$$

$$\tilde{A}_6(s_1,s_2) := \sup_{a<x<b} \left(\int_a^x u(t) V_1^{\left(\frac{\frac{1}{p_1'}+s_2}{\frac{1}{q}-s_1}\right)}(t) V_2^{\left(\frac{\frac{1}{p_2'}}{\frac{1}{q}-s_1}\right)}(t)\,dt \right)^{\frac{1}{q}-s_1} U^{s_1}(x) V_1^{-s_2}(x), \quad \frac{1}{q} > s_1;$$

$$\tilde{A}_7(s_1,s_2) := \sup_{a<x<b} \left(\int_x^b u(t) V_1^{\left(\frac{\frac{1}{p_1'}}{\frac{1}{q}-s_1}\right)}(t)\,dt \right)^{\frac{1}{q}-s_1} V_2^{s_2}(x) \left(\int_x^b u(t) V_2^{\left(\frac{\frac{1}{p_2'}-s_2}{s_1}\right)}(t)\,dt \right)^{s_1}, \quad \frac{1}{q} < s_1;$$

$$\tilde{A}_8(s_1,s_2) := \sup_{a<x<b} \left(\int_x^b u(t) V_2^{\left(\frac{\frac{1}{p_2'}}{\frac{1}{q}-s_1}\right)}(t)\,dt \right)^{\frac{1}{q}-s_1} V_1^{s_2}(x) \left(\int_x^b u(t) V_1^{\left(\frac{\frac{1}{p_1'}-s_2}{s_1}\right)}(t)\,dt \right)^{s_1}, \quad \frac{1}{q} < s_1;$$

$$\tilde{A}_9(s_1,s_2) := \sup_{a<x<b} \left(\int_a^x v_1^{1-p_1'}(t) U^{\left(\frac{\frac{1}{2q}}{\frac{1}{p_1'}+s_1}\right)}(t)\,dt \right)^{\frac{1}{p_1'}+s_1} \left(\int_a^x v_2^{1-p_2'}(t) U^{\left(\frac{\frac{1}{2q}}{\frac{1}{p_2'}+s_2}\right)}(t)\,dt \right)^{\frac{1}{p_2'}+s_2} \times$$
$$\times V_1^{-s_1}(x) V_2^{-s_2}(x);$$

$$\tilde{A}_{10}(s) := \sup_{a<x<b} \left(\int_x^b u(t) V_1^{\left(\frac{\frac{1}{p_1'}(1-s)}{\frac{1}{q}}\right)}(t) V_2^{\left(\frac{\frac{1}{p_2'}(1-s)}{\frac{1}{q}}\right)}(t)\,dt \right)^{\frac{1}{q}} V_1^{\frac{s}{p_1'}}(x) V_2^{\frac{s}{p_2'}}(x);$$

$$\tilde{A}_{11}(s) := \sup_{a<x<b} \left(\int_a^x u(t) V_1^{\left(\frac{\frac{1}{p_1'}(1+s)}{\frac{1}{q}}\right)}(t) V_2^{\left(\frac{\frac{1}{p_2'}(1+s)}{\frac{1}{q}}\right)}(t)\,dt \right)^{\frac{1}{q}} V_1^{-\frac{s}{p_1'}}(x) V_2^{-\frac{s}{p_2'}}(x);$$

$$\tilde{A}_{12}(s) := \sup_{a<x<b} \left(\int_x^b u(t) V_1^{\left(\frac{\frac{1}{p_1'}}{\frac{1}{q}+s}\right)}(t) V_2^{\left(\frac{\frac{1}{p_2'}}{\frac{1}{q}+s}\right)}(t)\,dt \right)^{\frac{1}{q}+s} U^{-s}(x);$$

$$\tilde{A}_{13}(s) := \sup_{a<x<b} \left(\int_a^x u(t) V_1^{\left(\frac{\frac{1}{p_1'}}{\frac{1}{q}-s}\right)}(t) V_2^{\left(\frac{\frac{1}{p_2'}}{\frac{1}{q}-s}\right)}(t)\,dt \right)^{\frac{1}{q}-s} U^{s}(x), \quad \frac{1}{q} > s;$$

$$\tilde{A}_{14}(s) := \sup_{a<x<b} \left(\int_x^b u(t) V_1^{\left(\frac{\frac{1}{p_1'}}{\frac{1}{q}-s}\right)}(t) V_2^{\left(\frac{\frac{1}{p_2'}}{\frac{1}{q}-s}\right)}(t)\,dt \right)^{\frac{1}{q}-s} U^{s}(x), \quad \frac{1}{q} < s;$$

$$\tilde{A}_{15}(s) := \inf_{h_1\ge 0} \sup_{a<x<b} \left(\int_x^b u(t) h_1^{\left(\frac{\frac{1}{p_2'}-s}{\frac{1}{q}}\right)}(t)\,dt \right)^{\frac{1}{q}} V_1^{\frac{1}{p_1'}}(x) \Big(h_1(x) + V_2(x) \Big)^{s}, \quad \frac{1}{p_2'} < s;$$

$$\tilde{A}_{16}(s) := \inf_{h_1 \geq 0} \sup_{a<x<b} \left(\int_x^b u(t) h_1^{\left(\frac{\frac{1}{p_1'} - s}{\frac{1}{q}}\right)}(t)\, dt \right)^{\frac{1}{q}} V_2^{\frac{1}{p_2'}}(x) \left(h_1(x) + V_1(x) \right)^s, \quad \frac{1}{p_1'} < s;$$

$$\tilde{A}_{17}(s_1, s_2) := \inf_{h_1, h_2 \geq 0} \sup_{a<x<b} \left(\int_x^b u(t) h_1^{\left(\frac{\frac{1}{p_1'} - s_1}{\frac{1}{q}}\right)}(t) h_2^{\left(\frac{\frac{1}{p_2'} - s_2}{\frac{1}{q}}\right)}(t)\, dt \right)^{\frac{1}{q}} \times$$

$$\times \left\{ \sup_{a<y<x} \left(h_1(y) + V_1(y) \right)^{s_1} \right\} \left(h_2(x) + V_2(x) \right)^{s_2}, \quad \frac{1}{p_1'} < s_1,\ \frac{1}{p_2'} < s_2;$$

$$\tilde{A}_{18}(s_1, s_2) := \inf_{h_1, h_2 \geq 0} \sup_{a<x<b} \left(\int_x^b u(t) h_1^{\left(\frac{\frac{1}{p_1'} - s_1}{\frac{1}{q}}\right)}(t) h_2^{\left(\frac{\frac{1}{p_2'} - s_2}{\frac{1}{q}}\right)}(t)\, dt \right)^{\frac{1}{q}} \times$$

$$\times \left(h_1(x) + V_1(x) \right)^{s_1} \left\{ \sup_{a<y<x} \left(h_2(y) + V_2(y) \right)^{s_2} \right\}, \quad \frac{1}{p_1'} < s_1,\ \frac{1}{p_2'} < s_2;$$

$$\tilde{A}_{19}(s_1, s_2) := \inf_{h_1, h_2 \geq 0} \sup_{a<x<b} \left(\int_x^b u(t) h_1^{\left(\frac{\frac{1}{p_1'} - s_1}{\frac{1}{q}}\right)}(t) h_2^{\left(\frac{\frac{1}{p_2'} - s_2}{\frac{1}{q}}\right)}(t)\, dt \right)^{\frac{1}{q}} \times$$

$$\times \left\{ \sup_{a<y<x} \left(h_1(y) + V_1(y) \right)^{s_1} \right\} \left\{ \sup_{a<y<x} \left(h_2(y) + V_2(y) \right)^{s_2} \right\}, \quad \frac{1}{p_1'} < s_1,\ \frac{1}{p_2'} < s_2;$$

$$\tilde{A}_{20}(s_1, s_2) := \inf_{h_1, h_2 \geq 0} \sup_{a<x<b} \left(\int_a^x u(t) \left(h_1(t) + V_1(t) \right)^{\left(\frac{\frac{1}{p_1'} + s_1}{\frac{1}{q}}\right)} \left(h_2(t) + V_2(t) \right)^{\left(\frac{\frac{1}{p_2'} + s_2}{\frac{1}{q}}\right)} dt \right)^{\frac{1}{q}} \times$$

$$\times \left\{ \sup_{x<y<b} h_1^{-s_1}(y) \right\} h_2^{-s_2}(x);$$

$$\tilde{A}_{21}(s_1, s_2) := \inf_{h_1, h_2 \geq 0} \sup_{a<x<b} \left(\int_a^x u(t) \left(h_1(t) + V_1(t) \right)^{\left(\frac{\frac{1}{p_1'} + s_1}{\frac{1}{q}}\right)} \left(h_2(t) + V_2(t) \right)^{\left(\frac{\frac{1}{p_2'} + s_2}{\frac{1}{q}}\right)} dt \right)^{\frac{1}{q}} \times$$

$$\times h_1^{-s_1}(x) \left\{ \sup_{x<y<b} h_2^{-s_2}(y) \right\};$$

$$\tilde{A}_{22}(s_1,s_2) := \inf_{h_1,h_2\geq 0} \sup_{a<x<b} \left(\int_a^x u(t) \left(h_1(t)+V_1(t) \right)^{\left(\frac{\frac{1}{p_1'}+s_1}{\frac{1}{q}}\right)} \left(h_2(t)+V_2(t) \right)^{\left(\frac{\frac{1}{p_2'}+s_2}{\frac{1}{q}}\right)} dt \right)^{\frac{1}{q}} \times$$

$$\times \left\{ \sup_{x<y<b} h_1^{-s_1}(y) \right\} \left\{ \sup_{x<y<b} h_2^{-s_2}(y) \right\}.$$

Then the inequality (1.5) *holds for all* $f, g \in \mathfrak{M}^+$ *if and only if any of the* $\tilde{A}_i(s_1,s_2)$, $\tilde{A}_j(s)$, $(i = 1,2,...,9,17,18,...,22)$, $(j = 10,11,...,16)$ *is finite. Also the best constant* C *in* (1.5) *satisfies* $C \approx \tilde{A}_i(s_1,s_2) \approx \tilde{A}_j(s)$, $(i = 1,2,...,9,17,18,...,22)$, $(j = 10,11,...,16)$.

Proof. By putting $f(x) = u(x)$, $g(x) = v_1^{1-p_1'}(x)$ in (1.4) and $h(x) = v_2^{1-p_2'}(x)$ in (3.1), we get

$$F(x) = U(x),\ G(x) = V_1(x),\ H(x) = V_2(x).$$

Consequently for $\alpha = \frac{1}{q}$, $\beta = \frac{1}{p_1'}$ and $\gamma = \frac{1}{p_2'}$, we have

$$\tilde{A}_i(s_1,s_2) = \tilde{B}_{i+1}\left(\frac{1}{q}, \frac{1}{p_1'}, \frac{1}{p_2'}, s_1, s_2\right); \quad (i = 1,2,...,9,17,18,...,22),$$
$$\tilde{A}_j(s) = \tilde{B}_{j+1}\left(\frac{1}{q}, \frac{1}{p_1'}, \frac{1}{p_2'}, s\right); \quad (j = 10,11,...,16).$$

Now by Theorem 3.1, these are all equivalent to $\tilde{B}_1\left(\frac{1}{q}, \frac{1}{p_1'}, \frac{1}{p_2'}\right) \overset{(1.6)}{=} \mathcal{D}$ and $\mathcal{D}$ is necessary and sufficient condition for the inequality (1.5) to hold.

Since the constant C in (1.5) verifies $C \approx \mathcal{D} = \tilde{B}_1\left(\frac{1}{q}, \frac{1}{p_1'}, \frac{1}{p_2'}\right)$, it follows that $C \approx \tilde{A}_i(s_1,s_2) \approx \tilde{A}_j(s)$, $(i = 1,2,...,9,17,18,...,22)$, $(j = 10,11,...,16)$. □

4 Bilinear Type Geometric Mean Operator

Consider the Hardy averaging operator A :

$$(Af)(x) := \frac{1}{x} \int_0^x f(t)\,dt$$

and the geometric mean operator T which is defined by

$$(Tf)(x) := \exp\left(\frac{1}{x} \int_0^x \ln(f(t))\,dt \right), \quad f \in \mathfrak{M}^+.$$

Note that

$$\lim_{\alpha\to 0} \left\{ (Af^\alpha)(x) \right\}^{\frac{1}{\alpha}} = Tf(x). \tag{4.1}$$

Therefore, it seems reasonable to study the inequality

$$\left(\int_0^\infty (Tf)^q(x)u(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(x)v(x)\,dx\right)^{\frac{1}{p}} \tag{4.2}$$

via (1.2) with suitable modifications. However, it was observed in [5], [6] that the Muckenhoupt condition (1.5) is not suitable for these modifications. To this end, Persson and Stepanov [11] obtained an alternate condition for the inequality (1.2) and by suitable modifications studied the inequality (4.2). Precisely, they proved the following

Theorem C. *Let* $0 < p \le q < \infty$. *Then the inequality* (4.2) *with the best constant* C *holds for* $f \in \mathfrak{M}^+$ *if and only if following holds*

$$\mathcal{B} := \sup_{t>0} t^{-\frac{1}{p}} \left(\int_0^t \left[T\left(\frac{1}{v(x)}\right)\right]^{\frac{q}{p}} u(x)\,dx\right)^{\frac{1}{q}} < \infty. \tag{4.3}$$

This was perhaps one of the motivations to obtain several equivalent conditions for the inequality (1.2). Let us mention that the inequality (4.2) has also independently been studied in [4], [10], [13].

Our aim, in this section, is to study the following bilinear type inequality involving the geometric mean operator T:

$$\left(\int_0^\infty (Tf)^q(x)\,(Tg)^q(x)u(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^{p_1}(x)v_1(x)\,dx\right)^{\frac{1}{p_1}} \times \left(\int_0^\infty f^{p_2}(x)v_2(x)\,dx\right)^{\frac{1}{p_2}}. \tag{4.4}$$

We prove the following:

Theorem 4.1. *Let* $1 < \max(p_1, p_2) < q < \infty$. *Then the inequality* (4.4) *holds for all* $f, g \in \mathfrak{M}^+$ *if and only if*

$$\tilde{\mathcal{B}} := \sup_{t>0} t^{-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \left(\int_0^t \left[T\left(\frac{1}{v_1(x)}\right)\right]^{\frac{q}{p_1}} \left[T\left(\frac{1}{v_2(x)}\right)\right]^{\frac{q}{p_2}} u(x)\,dx\right)^{\frac{1}{q}} < \infty.$$

Proof. By using (4.3) from Theorem C, we get

$$\begin{aligned} C &= \sup_{g\in\mathfrak{M}^+} \sup_{f\in\mathfrak{M}^+} \frac{\left(\int_0^\infty (Gf)^q(x)\,(Gg)^q(x)u(x)\,dx\right)^{\frac{1}{q}}}{\|f\|_{p_1,v_1}\|g\|_{p_2,v_2}} \\ &\approx \sup_{g\in\mathfrak{M}^+} \|g\|^{-1}_{p_2,v_2} \sup_{t_1>0} t_1^{-\frac{1}{p_1}} \left(\int_0^{t_1} \left[G\left(\frac{1}{v_1(x)}\right)\right]^{\frac{q}{p_1}} (Gg)^q(x)u(x)\,dx\right)^{\frac{1}{q}} \end{aligned}$$

$$
\begin{aligned}
&= \sup_{t_1>0} t_1^{-\frac{1}{p_1}} \sup_{g\in\mathfrak{M}^+} \|g\|_{p_2,v_2}^{-1} \left(\int_0^{t_1} \left[G\left(\frac{1}{v_1(x)}\right)\right]^{\frac{q}{p_1}} (Gg)^q(x)u(x)\,dx\right)^{\frac{1}{q}} \\
&\approx \sup_{t_1>0} t_1^{-\frac{1}{p_1}} \sup_{0<t_2<t_1} t_2^{-\frac{1}{p_2}} \left(\int_0^{t_2} \left[G\left(\frac{1}{v_1(x)}\right)\right]^{\frac{q}{p_1}} \left[G\left(\frac{1}{v_2(x)}\right)\right]^{\frac{q}{p_2}} u(x)\,dx\right)^{\frac{1}{q}} \\
&= \sup_{t>0} t^{-\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} \left(\int_0^{t} \left[G\left(\frac{1}{v_1(x)}\right)\right]^{\frac{q}{p_1}} \left[G\left(\frac{1}{v_2(x)}\right)\right]^{\frac{q}{p_2}} u(x)\,dx\right)^{\frac{1}{q}} \\
&= \tilde{\mathcal{B}}.
\end{aligned}
$$

□

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