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Abstract

We compare two types of universal operators constructed relatively recently by Cabello Sánchez, and the authors. The first operator $\mathbf{\Omega}$ acts on the Gurariĭ space, while the second one $\mathbf{P}_{\mathbb{S}}$ has values in a fixed separable Banach space $\mathbb{S}$. We show that if $\mathbb{S}$ is the Gurariĭ space, then both operators are isometric. We also prove that, for a fixed space $\mathbb{S}$, the operator $\mathbf{P}_{\mathbb{S}}$ is isometrically unique. Finally, we show that $\mathbf{\Omega}$ is generic in the sense of a natural infinite game.

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Contents

1 Universal operators

The purpose of this note is to discuss two constructions of universal operators between separable Banach spaces. We are interested in isometric universality. Namely, an operator U is *universal* if its restrictions to closed subspaces are, up to linear isometries, *all* linear operators of norm not exceeding $\|U\|$. To be more precise, a bounded linear operator $U\colon V \to W$ acting between separable Banach spaces is *universal* if for every linear operator $T\colon X \to Y$ with X, Y separable and $\|T\| \leq \|U\|$, there exist linear isometric embeddings $i\colon X \to V$, $j\colon Y \to W$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\;U\;} & W \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ X & \xrightarrow[\;T\;]{} & Y \end{array}$$

is commutative, that is, $U \circ i = j \circ T$. Such an operator has been relatively recently constructed by the authors [5]. Another recent work [2], due to Cabello Sánchez and the present authors, contains in particular a construction of a linear operator that is universal in a different sense. Namely, let us say that a bounded linear operator $U\colon V \to W$ is *left-universal* (for operators into W) if for every linear operator $T\colon X \to W$ with X separable and $\|T\| \leq \|U\|$ there exists a linear isometric embedding $i\colon X \to V$ for which the diagram

$$\begin{array}{ccc} V & \xrightarrow{\;U\;} & W \\ {\scriptstyle i}\uparrow & \nearrow{\scriptstyle T} & \\ X & & \end{array}$$

is commutative, that is, $U \circ i = T$. Clearly, if W is isometrically universal in the class of all separable Banach spaces then a left-universal operator with values into W is universal. The left-universal operator U constructed in [2] had been later essentially used (with a suitable space W) for finding an isometrically universal graded Fréchet space [1]. There exist other concepts of universality in operator theory, see the introduction of [5] for more details and references.

Let us note the following simple facts related to universal operators.

Proposition 1.1. *Let $U\colon V \to W$ be a bounded linear operator acting between separable Banach spaces.*

(1) *If U is universal then both V and W are isometrically universal among the class of separable Banach spaces.*

(2) *Assume U is left-universal. Then $\ker U$ is isometrically universal among the class of separable Banach spaces. Furthermore, U is right-invertible, that is, there exists an isometric embedding $e\colon W \to V$ such that $U \circ e = \mathrm{id}_W$.*

(3) *Assume U is (left-)universal. Then λU is (left-)universal for every $\lambda > 0$.*

Proof. (1) Fix a separable Banach space X. Taking the zero operator $T\colon X \to 0$, we see that V contains an isometric copy of X. Taking the identity id_X, we see that W contains an isometric copy of X.

(2) The same argument as above, using the zero operators, shows that $\ker U$ is isometrically universal. Taking the identity id_W, we obtain the required isometric embedding $e\colon W \to V$.

(3) Assume U is universal, fix $\lambda > 0$ and fix $T\colon X \to Y$ with $\|T\| \leq \lambda\|U\|$. Then $\|\lambda^{-1}T\| \leq \|U\|$, therefore there are isometric embeddings $i\colon X \to V$, $j\colon Y \to W$ such that $U \circ i = j \circ (\lambda^{-1}T)$. Finally, $(\lambda U) \circ i = j \circ T$. If U is left-universal, the argument is the same, the only difference is that $j = \mathrm{id}_W$. □

By (3) above, we may restrict attention to non-expansive operators. It turns out that there is an easy way of constructing left-universal operators, once we have in hand an isometrically universal space. The argument below was pointed out to us by Przemysław Wojtaszczyk.

Example 1.2. Let V be an isometrically universal Banach space and let W be an arbitrary Banach space. Consider $V \oplus W$ with the maximum norm and let

$$\pi\colon V \oplus W \to W$$

be the canonical projection. Given a non-expansive operator $T\colon X \to W$ with X separable, choose an isometric embedding $e\colon X \to V$ and define $j\colon X \to V \oplus W$ by $j(x) = (e(x), T(x))$. Then j is an isometric embedding and $\pi \circ j = T$, showing that π is left-universal. Of course, if additionally W is isometrically universal, then π is a universal operator.

Perhaps the most well known universal Banach space is $\mathscr{C}([0,1])$, the space of all continuous (real or complex) valued functions on the unit interval, endowed with the maximum norm. In view of the example above, there exists a universal operator from $\mathscr{C}([0,1]) \oplus \mathscr{C}([0,1])$ onto $\mathscr{C}([0,1])$. This leads to (at least potentially) many other universal operators, namely:

Proposition 1.3. *Let V, W be isometrically universal separable Banach spaces. Then there exists a universal operator from V into W.*

Proof. Fix a universal operator $\pi\colon E \to F$ (for instance, $E = \mathscr{C}([0,1]) \oplus \mathscr{C}([0,1])$ and $F = \mathscr{C}([0,1])$) and fix a linear isometric embedding $e\colon E \to V$. Using the amalgamation property for linear operators, we find a separable Banach space V', a linear isometric embedding $e'\colon F \to V'$, and a non-expansive linear operator $\Omega\colon V \to V'$ for which the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ \Omega\ } & V' \\ {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle e'} \\ E & \xrightarrow[\ \pi\ ]{} & F \end{array}$$

is commutative. As W is isometrically universal, we may additionally assume that $V' = W$, replacing Ω by $i \circ \Omega$ and e' by $i \circ e'$, where i is a fixed isometric embedding of V' into W. It is evident that now Ω is a universal operator, because of the universality of π. □

As a consequence, there exists a universal operator on $\mathscr{C}([0,1])$. We do not know whether there exists a left-universal operator on $\mathscr{C}([0,1])$. The situation changes when replacing $[0,1]$ with the Cantor set $2^{\mathbb{N}}$. The space $\mathscr{C}(2^{\mathbb{N}})$ is linearly isomorphic (but not isometric) to $\mathscr{C}([0,1])$ and it is isometrically universal, too. Furthermore, $\mathscr{C}(2^{\mathbb{N}}) \oplus \mathscr{C}(2^{\mathbb{N}})$ with the maximum norm is linearly isometric to $\mathscr{C}(2^{\mathbb{N}})$, because the disjoint sum of two copies of the Cantor set is homeomorphic to the Cantor set. Thus, Example 1.2 provides a left-universal operator on $\mathscr{C}(2^{\mathbb{N}})$.

Another, not so well known, universal Banach space is the *Gurariĭ space*. This is the unique, up to a linear isometry, separable Banach space $\mathbb{G}$ satisfying the following condition:

(G) For every $\varepsilon > 0$, for every finite-dimensional spaces $X_0 \subseteq X$, for every linear isometric embedding $f_0 \colon X_0 \to \mathbb{G}$ there exists a linear ε-isometric embedding $f \colon X \to \mathbb{G}$ such that $f \restriction X_0 = f_0$.

By an *ε-isometric embedding* (briefly: *ε-embedding*) we mean a linear operator f satisfying

$$(1-\varepsilon)\|x\| \leq \|f(x)\| \leq (1+\varepsilon)\|x\|$$

for every x in the domain of f. The space $\mathbb{G}$ was constructed by Gurariĭ [6]; its uniqueness was proved by Lusky [9].

The universal operator constructed in [5] has a special property that actually makes it unique, up to linear isometries. Below we quote the precise result.

Theorem 1.4 ([5]). *There exists a non-expansive linear operator $\mathbf{\Omega} \colon \mathbb{G} \to \mathbb{G}$ with the following property:*

(G) *Given $\varepsilon > 0$, given a non-expansive operator $T \colon X \to Y$ between finite-dimensional spaces, given $X_0 \subseteq X$, $Y_0 \subseteq Y$ and isometric embeddings $i \colon X_0 \to U$, $j \colon Y_0 \to V$ such that $\mathbf{\Omega} \circ i = j \circ (T \restriction X_0)$, there exist ε-embeddings $i' \colon X \to U$, $j' \colon Y \to V$ satisfying*

$$\|i' \restriction X_0 - i\| \leq \varepsilon, \quad \|j' \restriction Y_0 - j\| \leq \varepsilon, \qquad \text{and} \qquad \|\mathbf{\Omega} \circ i' - j' \circ T\| \leq \varepsilon.$$

Furthermore, $\mathbf{\Omega}$ is a universal operator and property (G) specifies it uniquely, up to a linear isometry.

According to [5], we shall call condition (G) the *Gurariĭ property*. What makes this operator of particular interest is perhaps its *almost homogeneity*:

Theorem 1.5 ([5]). *Given finite-dimensional subspaces X_0, X_1, Y_0, Y_1 of $\mathbb{G}$, given linear isometries $i\colon X_0 \to X_1$, $j\colon Y_0 \to Y_1$ such that $\mathbf{\Omega} \circ i = j \circ \mathbf{\Omega}$, for every $\varepsilon > 0$ there exist bijective linear isometries $I\colon \mathbb{G} \to \mathbb{G}$, $J\colon \mathbb{G} \to \mathbb{G}$ satisfying $\mathbf{\Omega} \circ I = J \circ \mathbf{\Omega}$ and $\|I \restriction X_0 - i\| < \varepsilon$, $\|J \restriction Y_0 - j\| < \varepsilon$.*

We now describe the left-universal operators constructed in [2]. Fix a separable Banach space $\mathbb{S}$.

Theorem 1.6 ([2, Section 6]). *There exists a non-expansive linear operator $\mathbf{P}_{\mathbb{S}}\colon V_{\mathbb{S}} \to \mathbb{S}$ with $V_{\mathbb{S}}$ a separable Banach space, satisfying the following condition:*

(‡) *For every finite-dimensional spaces $X_0 \subseteq X$, for every non-expansive linear operator $T\colon X \to \mathbb{S}$, for every linear isometric embedding $e\colon X_0 \to V_{\mathbb{S}}$ such that $\mathbf{P}_{\mathbb{S}} \circ e = T \restriction X_0$, for every $\varepsilon > 0$ there exists an ε-embedding $f\colon X \to V_{\mathbb{S}}$ satisfying*

$$\|f \restriction X_0 - e\| \leq \varepsilon \qquad \text{and} \qquad \|\mathbf{P}_{\mathbb{S}} \circ f - T\| \leq \varepsilon.$$

Furthermore, $\mathbf{P}_{\mathbb{S}}$ is left-universal for operators into $\mathbb{S}$.

We shall say that an operator P has the *left-Gurariĭ* property if it satisfies (‡) in place of $\mathbf{P}_{\mathbb{S}}$. Of course, unlike the Gurariĭ property, the left-Gurariĭ property involves a parameter $\mathbb{S}$, namely, the common range of the operators.

Actually, the projection $\mathbf{P}_{\mathbb{S}}$ was constructed in [2] in case where $\mathbb{S}$ had some additional property, needed only for determining the domain of $\mathbf{P}_{\mathbb{S}}$. Moreover, [2] deals with p-Banach spaces, where $p \in (0, 1]$, however $p = 1$ gives exactly the result stated above. Operators $\mathbf{P}_{\mathbb{S}}$ have the following property which can be called *almost left-homogeneity*.

Theorem 1.7. *Given finite-dimensional subspaces X_0, X_1 of $V_{\mathbb{S}}$, a linear isometry $h\colon X_0 \to X_1$ such that $\mathbf{P}_{\mathbb{S}} \circ h = \mathbf{P}_{\mathbb{S}} \restriction X_0$, for every $\varepsilon > 0$ there exists a bijective linear isometry $H\colon V_{\mathbb{S}} \to \mathbb{S}$ satisfying $\mathbf{P}_{\mathbb{S}} \circ H = \mathbf{P}_{\mathbb{S}}$ and $\|H \restriction X_0 - h\| < \varepsilon$.*

In this note we present a proof that condition (‡) determines $\mathbf{P}_{\mathbb{S}}$ uniquely, up to linear isometries. The arguments will also provide a proof of Theorem 1.7. Furthermore, we show that $\mathbf{\Omega} = \mathbf{P}_{\mathbb{G}}$ and that $\mathbf{\Omega}$ is a generic operator in the space of all non-expansive operators on the Gurariĭ space into itself, in the sense of a natural variant of the Banach-Mazur game.

2 Properties of $\mathbf{\Omega}$ and $\mathbf{P}_{\mathbb{S}}$

Let us recall the following easy fact concerning finite-dimensional normed spaces (cf. [4, Thm. 2.7] or [1, Claim 2.3]). It actually says that the strong operator topology is equivalent to the norm topology in the space of linear operators with a fixed finite-dimensional domain.

Lemma 2.1. *Let A be a vector basis of a finite-dimensional normed space E. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every Banach space X, for every linear operator $f\colon E \to X$ the following implication holds:*

$$\max_{a \in A} \|f(a)\| \le \delta \implies \|f\| \le \varepsilon.$$

Proof. Fix $M > 0$ satisfying the following condition:

(*) $\max_{a \in A} |\lambda_a| \le M$ whenever $x = \sum_{a \in A} \lambda_a a$ and $\|x\| \le 1$.

Such M clearly exists, because of compactness of the unit ball of E. Now, given $\varepsilon > 0$, let $\delta = \varepsilon / (M \cdot |A|)$. Suppose $\max_{a \in A} \|f(a)\| \le \delta$. Then, given $x = \sum_{a \in A} \lambda_a a$ with $\|x\| \le 1$, we have

$$\|f(x)\| \le \sum_{a \in A} |\lambda_a| \cdot \|f(a)\| \le |A| \cdot M \cdot \delta = \varepsilon.$$

We conclude that $\|f\| \le \varepsilon$. □

The following result, in case $\mathbb{S} = \mathbb{G}$ can be found in [1].

Theorem 2.2. *Let $P\colon V \to \mathbb{S}$ be a linear operator. The following conditions are equivalent.*

(a) *P has the left-Gurariĭ property (‡).*

(b) *For every finite-dimensional spaces $X_0 \subseteq X$, for every non-expansive linear operator $T\colon X \to \mathbb{S}$, for every linear isometric embedding $e\colon X_0 \to V$ such that $P \circ e = T \restriction X_0$, for every $\varepsilon > 0$ there exists an ε-embedding $f\colon X \to V$ satisfying*

$$f \restriction X_0 = e \qquad \text{and} \qquad P \circ f = T.$$

Proof. Obviously, (b) is stronger than (‡).

Fix $\varepsilon > 0$ and fix a vector basis A of X such that $A_0 = X_0 \cap A$ is a basis of X_0. We may assume that $\|a\| = 1$ for every $a \in A$. Fix $\delta > 0$ and apply the left-Gurariĭ property for δ instead of ε. We obtain a δ-embedding $f\colon X \to V$ such that $\|f \restriction X_0 - e\| \le \delta$ and $\|P \circ f - T\| \le \delta$. Define $f'\colon X \to V$ by the conditions $f'(a) = e(a)$ for $a \in A_0$ and $f'(a) = f(a)$ for $a \in A \setminus A_0$. Note that $\|f'(a) - f(a)\| \le \delta$ for every $a \in A$. Thus, if δ is small enough, then by Lemma 2.1, we can obtain that f' is an ε-embedding. Furthermore, $\|P \circ f' - P \circ f\| \le \varepsilon$ (recall that δ depends on ε and the norm of X only), therefore $\|P \circ f' - T\| \le \varepsilon + \delta$.

The arguments above show that for every $\varepsilon > 0$ there exists an ε-embedding $f'\colon X \to V$ extending e and satisfying $\|P \circ f' - T\| \le \varepsilon$.

Let us apply this property for δ instead of ε, where δ is taken from Lemma 2.1. We obtain a δ-embedding $f\colon X \to V$ extending e and satisfying $\|P \circ f - T\| \le \delta$.

Given $a \in A \setminus A_0$, the vector

$$w_a = P(f(a)) - T(a)$$

has norm $\leq \delta$. Define $f' \colon X \to V$ by the conditions $f' \restriction X_0 = e$ and

$$f'(a) = f(a) - w_a$$

for $a \in A \setminus A_0$. Lemma 2.1 implies that f' is an ε-embedding, because $\|f'(a) - f(a)\| = \|w_a\| \leq \delta$ for $a \in A \setminus A_0$. Finally, given $a \in A \setminus A_0$, we have

$$Pf'(a) = Pf(a) - w_a = T(a)$$

and the same obviously holds for $a \in A_0$. Thus $P \circ f' = T$. $\square$

The proof of the next result is just a suitable adaptation of the arguments above, therefore we skip it.

Proposition 2.3. *Let $\Omega \colon V \to W$ be a linear operator. The following conditions are equivalent.*

(a) *Ω has the Gurariĭ property* (G).

(b) *Given $\varepsilon > 0$, given a non-expansive operator $T \colon X \to Y$ between finite-dimensional spaces, given $X_0 \subseteq X$, $Y_0 \subseteq Y$ and isometric embeddings $i_0 \colon X_0 \to V$, $j_0 \colon Y_0 \to W$ such that $\Omega \circ i_0 = j_0 \circ (T \restriction X_0)$, there exist ε-embeddings $i \colon X \to V$, $j \colon Y \to W$ satisfying*

$$i \restriction X_0 = i_0, \quad j \restriction Y_0 = j_0, \qquad \text{and} \qquad \Omega \circ i = j \circ T.$$

The last result of this section is the key step towards identifying $\mathbf{\Omega}$ with $\mathbf{P}_{\mathbb{G}}$.

Theorem 2.4. *The operator $\mathbf{\Omega}$ has the left-Gurariĭ property (i.e., it satisfies condition (‡) of Theorem 1.6 with $\mathbb{S} = \mathbb{G}$). In particular, it is left-universal.*

Proof. Fix a non-expansive linear operator $T \colon X \to \mathbb{G}$ with X finite-dimensional, and fix an isometric embedding $e \colon X_0 \to \mathbb{G}$, where X_0 is a linear subspace of X and $T \restriction X_0 = \mathbf{\Omega} \circ e$. Let $Y_0 = Y = T[X] \subseteq \mathbb{G}$ and consider T as an operator from X to Y. Applying the Gurariĭ property with $i = e$ and j the inclusion $Y_0 \subseteq \mathbb{G}$, we obtain an ε-embedding $e' \colon X \to \mathbb{G}$ which is ε-close to e and satisfies $\|\mathbf{\Omega} \circ e' - T\| \leq \varepsilon$. This is precisely condition (‡) from Theorem 1.6. $\square$

In order to conclude that $\mathbf{\Omega} = \mathbf{P}_{\mathbb{G}}$, it remains to show that (‡) determines the operator uniquely. This is done in the next section.

3 Uniqueness of $\mathbf{P}_{\mathbb{S}}$

Before proving that the left-Gurariĭ property determines the operator uniquely, we quote the following crucial lemma from [3].

Lemma 3.1. *Let $\varepsilon > 0$ and let $f\colon E \to F$ be an ε-embedding, where E, F are Banach spaces. Let $\pi\colon E \to \mathbb{S}$, $\varrho\colon F \to \mathbb{S}$ be non-expansive linear operators such that $\|\varrho \circ f - \pi\| \le \varepsilon$. Then there exists a norm on $Z = X \oplus Y$ such that the canonical embeddings $i\colon X \to Z$, $j\colon Y \to Z$ are isometric, $\|j \circ f - i\| \le \varepsilon$ and the operator $t\colon Z \to \mathbb{S}$ defined by $t(x, y) = \pi(x) + \varrho(y)$ is non-expansive.*

Note that the operator t satisfies $t \circ i = \pi$ and $t \circ j = \varrho$. Actually, the norm mentioned in the lemma above does not depend on the operators π, ϱ. It is defined by the following formula:

$$(*) \qquad \|(x, y)\| = \inf\Big\{\|x - w\|_X + \|y - f(w)\|_Y + \varepsilon\|w\|_X \colon w \in X\Big\},$$

where $\|\cdot\|_X$, $\|\cdot\|_Y$ denote the norm of X and Y, respectively. An easy exercise shows that $(*)$ is the required norm, proving Lemma 3.1.

Theorem 3.2. *Let $\mathbb{S}$ be a separable Banach space and let $\pi\colon E \to \mathbb{S}$, $\pi'\colon E' \to \mathbb{S}$ be non-expansive linear operators, both with the left-Gurariĭ property. If E, E' are separable Banach spaces, then there exists a linear isometry $i\colon E \to E'$ such that $\pi = \pi' \circ i$. In particular, π and π' are linearly isometric to $\mathbf{P}_{\mathbb{S}}$.*

Proof. It suffices to prove the following

Claim 3.3. *Let $E_0 \subseteq E$ be a finite-dimensional space, $0 < \varepsilon < 1$, let $i_0\colon E_0 \to E'$ be an ε-embedding such that $\pi' \circ i_0 = \pi \restriction E_0$. Then for every $v \in E$, $v' \in E'$, for every $\eta > 0$ there exists an η-embedding $i_1\colon E_1 \to E'$ with E_1 finite-dimensional and the following conditions are satisfied:*

1. *$v \in E_1$ and $\operatorname{dist}(v', i_1[E_1]) < \eta$;*
2. *$\|i_0 - i_1 \restriction E_0\| < \varepsilon + \eta$ and $\pi' \circ i_1 = \pi$.*

Using Claim 3.3 together with the separability of E and E', we can construct a sequence $i_n\colon E_n \to E'$ of linear operators such that i_n is a 2^{-n}-embedding, $\bigcup_{n\in\omega} E_n$ is dense in E and $\bigcup_{n\in\omega} i_n[E_n]$ is dense in E' and

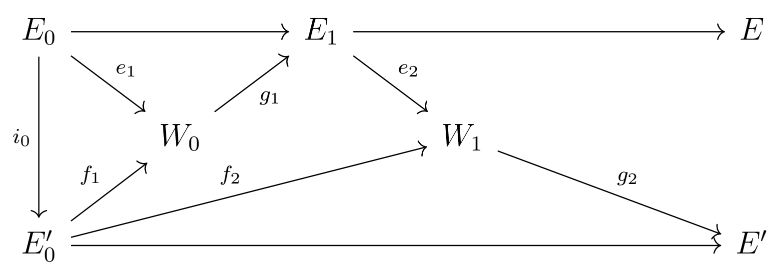
$$\|i_n - i_{n+1} \restriction E_n\| \le 2^{-n} + 2^{-n-1} \qquad \text{and} \qquad \pi' \circ i_{n+1} = \pi$$

for every $n \in \omega$. It is evident that $\{i_n\}_{n\in\omega}$ converges pointwise to a linear isometry whose completion i is the required bijection from E onto E' satisfying $\pi' \circ i = \pi$. Thus, it remains to prove Claim 3.3.

This will be carried out by making two applications of Lemma 3.1.

Fix $0 < \delta < 1$, more precise estimations for δ will be given later. Let $E'_0 \subseteq E'$ be a finite-dimensional space containing v' and such that $i_0[E_0] \subseteq E'_0$. Applying Lemma 3.1, we obtain linear isometric embeddings $e_1 \colon E_0 \to W_0$, $f_1 \colon E'_0 \to W_0$ and a non-expansive operator $t_0 \colon W_0 \to \mathbb{S}$ such that $t_0 \circ e_1 = \pi \restriction E_0$, $t_0 \circ f_1 = \pi' \restriction E'_0$, and $\|e_1 - f_1 \circ i_0\| \leq \varepsilon$. Knowing that π has the left-Gurariĭ property, by Theorem 2.2 applied to the isometric embedding e_1, we obtain a δ-embedding $g_1 \colon W_0 \to E$ such that $g_1 \circ e_1$ is identity on E_0 and $\pi \circ g_1 = t_0$.

Now note that $g_1 \circ f_1$ is a δ-embedding of E'_0 into a finite-dimensional subspace E_1 of E. Without loss of generality, we may assume that $v \in E_1$. Applying Lemma 3.1 again to $g_1 \circ f_1$, we obtain linear isometric embeddings $e_2 \colon E_1 \to W_1$, $f_2 \colon E'_0 \to W_1$ and a non-expansive linear operator $t_1 \colon W_1 \to \mathbb{S}$ such that $t_1 \circ e_2 = \pi \restriction E_1$, $t_1 \circ f_2 = \pi' \restriction E'_0$, and $\|e_2 \circ g_1 \circ f_1 - f_2\| \leq \delta$. Knowing that π' has the left-Gurariĭ property and using Theorem 1.6 for the isometric embedding f_2, we obtain a δ-embedding $g_2 \colon W_1 \to E'$ such that $g_2 \circ f_2$ is identity on E'_0 and $\pi' \circ g_2 = t_1$. The configuration is described in the following diagram, where the horizontal arrows are inclusions, the triangle $E_0 E'_0 W_0$ is ε-commutative, and the triangle $E'_0 E_1 W_1$ is δ-commutative.



It remains to check that $i_1 := g_2 \circ e_2$ is the required δ-embedding.

First, recall that $v \in E_1$, $v' \in E'_0$ and $v' = g_2(f_2(v'))$. Thus, using the fact that $\|g_2\| \leq 1 + \delta$, we get

$$\begin{aligned} \|i_1 g_1 f_1(v') - v'\| &= \|g_2 e_2 g_1 f_1(v') - g_2 f_2(v')\| \\ &\leq (1+\delta)\|e_2 g_1 f_1(v') - f_2(v')\| \\ &\leq (1+\delta)\delta\|v'\|. \end{aligned}$$

Now if $(1+\delta)\delta\|v'\| < \eta$, then we conclude that $\operatorname{dist}(v', i_1[E_1]) < \eta$, therefore condition (1) is satisfied.

Given $x \in E_1$, note that

$$\pi' i_1(x) = \pi' g_2 e_2(x) = t_1 e_2(x) = \pi(x).$$

Here we have used the fact that $\pi' \circ g_2 = t_1$ and $t_\circ e_2 = \pi \restriction E_1$.

Furthermore, given $x \in E_0$, we have

$$\begin{aligned} \|i_1(x) - i_0(x)\| = \|g_2 e_2(x) - i_0(x)\| &= \|g_2 e_2 g_1 e_1(x) - g_2 f_2 i_0(x)\| \\ &\leq (1+\delta)\|e_2 g_1 e_1(x) - f_2 i_0(x)\|, \end{aligned}$$

because $\|g_2\| \le 1+\delta$. On the other hand,

$$\begin{aligned}\|e_2 g_1 e_1(x) - f_2 i_0(x)\| &\le \|e_2 g_1 e_1(x) - e_2 g_1 f_1 i_0(x)\| + \|e_2 g_1 f_1 i_0(x) - f_2 i_0(x)\| \\ &= \|g_1 e_1(x) - g_1 f_1 i_0(x)\| + \|e_2 g_1 f_1 i_0(x) - f_2 i_0(x)\| \\ &\le (1+\delta)\|e_1(x) - f_1 i_0(x)\| + \delta\|i_0(x)\| \\ &\le (1+\delta)\varepsilon\|x\| + \delta(1+\varepsilon)\|x\| \le (\varepsilon + 3\delta)\|x\|.\end{aligned}$$

Here we have used the following facts: e_2 is an isometric embedding, g_1 is a δ-embedding, i_0 is an ε-embedding, $\|e_2 g_1 f_1 - f_2\| \le \delta$, $\|e_1 - f_1 i_0\| \le \varepsilon$ and $\varepsilon < 1$.

Finally, $\|i_1(x) - i_0(x)\| \le (1+\delta)(\varepsilon + 3\delta)\|x\| \le (\varepsilon + 7\delta)\|x\|$. Summarizing, if $(1+\delta)\delta\|v'\| < \eta$ and $7\delta < \eta$ then conditions (1), (2) are satisfied. This completes the proof. □

Note that if $\mathbb{S}$ is the trivial space, the proof above reduces to the well known uniqueness of the Gurariĭ space, shown by this way in [8]. Furthermore, the arguments above can be applied to $\pi = \pi' = \mathbf{P}_{\mathbb{S}}$ and $i_0 = h$, thus proving Theorem 1.7. Theorems 2.4 and 3.2 yield the following result, announced before.

Corollary 3.4. $\mathbf{\Omega} = \mathbf{P}_{\mathbb{G}}$.

In particular, $V_{\mathbb{G}} = \mathbb{G}$. It has been shown in [2] that $V_{\mathbb{S}} = \mathbb{G}$ as long as $\mathbb{S}$ is a (separable) *Lindenstrauss space*, namely, an isometric L_1 predual or (equivalently) a locally almost 1-injective space. Instead of going into details, let us just say that Lindenstrauss spaces are those (separable) Banach spaces that are linearly isometric to a 1-complemented subspace of the Gurariĭ space. The non-trivial direction was proved by Wojtaszczyk [10]. Thus, since $\mathbf{P}_{\mathbb{S}}$ is a projection, if $V_{\mathbb{S}}$ is linearly isometric to $\mathbb{G}$ then $\mathbb{S}$ is necessarily a Lindenstrauss space.

4 Generic operators

Inspired by the result of [7], let us consider the following infinite game for two players *Eve* and *Adam*. Namely, Eve starts by choosing a non-expansive linear operator $T_0\colon E_0 \to F_0$, where E_0, F_0 are finite-dimensional normed spaces. Adam responds by a non-expansive linear operator $T_1\colon E_1 \to F_1$, such that $E_1 \supseteq E_0$, $F_1 \supseteq F_0$ are again finite-dimensional and T_1 extends T_0. Eve responds by a further non-expansive linear extension $T_2\colon E_2 \to F_2$, and so on. So at each stage of the game we have a linear operator between finite-dimensional normed spaces. After infinitely many steps we obtain a chain of non-expansive operators $\{T_n\colon E_n \to F_n\}_{n\in\omega}$. Let $T_\infty\colon E_\infty \to F_\infty$ denote the completion of its union, namely, E_∞ is the completion of $\{E_n\}_{n\in\omega}$, F_∞ is the completion of $\{F_n\}_{n\in\omega}$ and $T_\infty \restriction E_n = T_n$ for every $n \in \omega$. So far, we cannot say who wins the game.

Let us say that a (necessarily non-expansive) linear operator $U\colon X \to Y$ is *generic* if Adam has a strategy making the operator T_∞ isometric to U. Recall that operators U, V are *isometric* if there are bijective linear isometries i, j such that $U \circ j = i \circ V$.

Theorem 4.1. *The operator* $\mathbf{\Omega}$ *is generic.*

Proof. Let us fix a non-expansive linear operator $U : \mathbb{G} \to \mathbb{G}$ between separable Banach spaces satisfying (G). Adam's strategy can be described as follows.

Fix a countable set $\{v_n : a_n \to b_n\}_{n\in\mathbb{N}}$ linearly dense in $U : \mathbb{G} \to \mathbb{G}$. Let $T_0\colon E_0 \to F_0$ be the first move of Eve. Adam finds isometric embeddings $i_0 : E_0 \to \mathbb{G}$, $j_0 : F_0 \to \mathbb{G}$ and finite-dimensional spaces $E_0 \subset E_1$, $F_0 \subset F_1$ together with isometric embeddings $i_1 : E_1 \to \mathbb{G}$, $j_1 : F_1 \to \mathbb{G}$ and non-expansive linear operators $T_1\colon E_1 \to F_1$ such that T_1 extends T_0, $a_0 \in i_1[E_1]$, $b_0 \in j_1[F_1]$.

Suppose now that $n = 2k > 0$ and $T_n : E_n \to F_n$ was the last move of Eve. We assume that linear isometric embeddings $i_{n-1} : E_{n-1} \to \mathbb{G}$, $j_{n-1} : F_{n-1} \to \mathbb{G}$ have already been fixed. Using (G) from Theorem 1.4 we choose linear isometric embeddings $i_n : E_n \to \mathbb{G}$, $j_n : F_n \to \mathbb{G}$ such that $i_n \restriction E_{n-1}$ is 2^{-k}-close to i_{n-1}, $j_n \restriction F_{n-1}$ is 2^{-k}-close to j_{n-1} and $U \circ i_n$ is 2^{-k}-close to $j_n \circ T_n$.

Let $\{T_n : E_n \to F_n\}_{n\in\mathbb{N}}$ be the chain of non-expansive operators between finite-dimensional normed spaces resulting from a fixed play, when Adam was using his strategy. In particular, Adam has recorded sequences $\{T_n : E_n \to F_n\}_{n\in\mathbb{N}}$, $\{i_n : E_n \to \mathbb{G}\}_{n\in\mathbb{N}}$, $\{j_n : F_n \to \mathbb{G}\}_{n\in\mathbb{N}}$ of linear isometric embeddings such that $i_{2n+1} \restriction E_{2n-1}$ is 2^{-n}-close to i_{2n-1} and $j_{2n+1} \restriction F_{2n-1}$ is 2^{-n}-close to j_{2n-1} for each $n \in \mathbb{N}$.

Let $T_\infty\colon E_\infty \to F_\infty$ denote the completion of those unions, namely, E_∞ is the completion of $\{E_n\}_{n\in\omega}$, F_∞ is the completion of $\{F_n\}_{n\in\omega}$ and $T_\infty \restriction E_n = T_n$ for every $n \in \omega$. The assumptions that $i_{2n+1}[E_{2n+1}]$ contains all the vectors $a_0, \dots, a_n$ and $j_{2n+1}[F_{2n+1}]$ contains all the vectors $b_0, \dots, b_n$ ensures that both $i_\infty[E_\infty]$, $j_\infty[F_\infty]$ are dense in $\mathbb{G}$, where $i_\infty\colon E_\infty \to \mathbb{G}$, $j_\infty\colon F_\infty \to \mathbb{G}$ are pointwise limits of $\{i_n\}_{n\in\mathbb{N}}$ and $\{j_n\}_{n\in\mathbb{N}}$, respectively. More precisely, $i_\infty \restriction E_k$ is the pointwise limit of $\{i_n \restriction E_k\}_{n\geq k}$ and $j_\infty \restriction F_k$ is the pointwise limit of $\{j_n \restriction F_k\}_{n\geq k}$ for every $k \in n \in \mathbb{N}$. In particular, both i_∞ and j_∞ are surjective linear isometries.

Finally, $U \circ i_\infty = j_\infty \circ T_\infty$, because $U \circ i_{2k}$ is 2^{-k}-close to $j_{2k} \circ T_{2k}$ for every $k \in \mathbb{N}$. This completes the proof.

□

Question 4.2. Is $\mathbf{\Omega}$ generic in the space of all non-expansive operators on the Gurariĭ space? Being "generic" means of course that the set

$$\{i \circ \mathbf{\Omega} \circ j \colon i, j \text{ bijective linear isometries of } \mathbb{G}\}$$

is residual in the space of all non-expansive operators on $\mathbb{G}$. Here, it is natural to consider the pointwise convergence (i.e., strong operator) topology.

One could also consider a "parametrized" variant of the game above, where the two players build a chain of non-expansive operators from finite-dimensional normed spaces into a fixed Banach space $\mathbb{S}$. If $\mathbb{S}$ is separable then similar arguments as in the proof of Theorem 4.1 show that the second player has a strategy leading to $\mathbf{P}_\mathbb{S}$. Thus, a variant of Question 4.2 makes sense: Is it true that isometric copies of $\mathbf{P}_\mathbb{S}$ form a residual set in a suitable space of operators?

After concluding that $\mathbf{\Omega} = \mathbf{P}_{\mathbb{G}}$, it seems that the "parametrized" construction of universal projections is better in the sense that it "captures" both the Gurariĭ space $\mathbb{G}$ (when the range is the trivial space $\{0\}$) and the universal operator $\mathbf{\Omega}$ (when the range equals $\mathbb{G}$), but also other examples, including projections from the Gurariĭ space onto any separable Lindenstrauss space (see [10] and [2]).

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