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Abstract

We consider the compressible Navier-Stokes system describing the motion of a viscous fluid confined to a straight layer $\Omega_\delta = (0, \delta) \times \mathbb{R}^2$. We show that the weak solutions in the 3D domain converge strongly to the solution of the 2D incompressible Navier-Stokes equations (Euler equations) when the Mach number ε tends to zero as well as $\delta \to 0$ (and the viscosity goes to zero).

Key words: compressible Navier-Stokes system, dimension reduction, low Mach number limit, vanishing viscosity.

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1 Introduction and main results

The paper is devoted to the problem of the limit passage from three-dimensional to two-dimensional geometry, and from compressible and viscous to incompressible viscous or inviscid fluid.

In the infinite slab geometry

$$\Omega_\delta = (0,\delta)\times\mathbb{R}^2, \quad \delta>0,$$

we consider the following compressible Navier-Stokes system describing the motion of a barotropic fluid,

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x\left(\varrho_\varepsilon \mathbf{u}_\varepsilon\right) = 0, \tag{1.1}$$

$$\partial_t\left(\varrho_\varepsilon \mathbf{u}_\varepsilon\right) + \mathrm{div}_x\left(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\right) + \frac{1}{\varepsilon^2}\nabla_x p\left(\varrho_\varepsilon\right) = \mu \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon), \tag{1.2}$$

where μ is the shear viscosity and we assume the bulk viscosity to be zero, $\varepsilon>0$ is the Mach number and

$$\mathbb{S}(\nabla_x\mathbf{u}) = \left(\nabla_x\mathbf{u} + \nabla_x'\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right),\ p\left(\varrho\right) = A\varrho^\gamma,\ A>0,\ \gamma>\frac{3}{2}. \tag{1.3}$$

The system is supplemented with the initial conditions

$$\mathbf{u}_\varepsilon\left(0,x\right) = \mathbf{u}_{0,\varepsilon}\left(x\right),\ \varrho_\varepsilon\left(0,x\right) = \varrho_{0,\varepsilon} = 1 + \varepsilon\varrho_{0,\varepsilon}^{(1)}, \tag{1.4}$$

the complete slip boundary conditions

$$\mathbf{u}_\varepsilon\cdot\mathbf{n}|_{\partial\Omega_\delta} = 0,\ [\mathbb{S}(\nabla_x\mathbf{u})\mathbf{n}]_{\tan}|_{\partial\Omega_\delta} = 0, \tag{1.5}$$

and the far field conditions for the velocity and density,

$$\mathbf{u}_\varepsilon \to 0, \quad \varrho_\varepsilon \to 1 \ \text{ as } \ |x|\to\infty. \tag{1.6}$$

Let $x_h = (x_1,x_2)$ and for a function defined in Ω_δ, denote the average in the x_3 variable as

$$\overline{f}(x_h) = \overline{f}^\delta(x_h) = \frac{1}{\delta}\int_0^\delta f(x_h,x_3)dx_3.$$

We assume the thickness δ of the domain Ω_δ depends on ε such that $\delta = \delta(\varepsilon)\to 0$ as $\varepsilon\to 0$. If $(\overline{\varrho_{0,\varepsilon}}, \overline{\mathbf{u}_{0,\varepsilon}}) \to (1,\mathbf{u}_0)$ in a certain sense, then the formal limits of $(\overline{\varrho_\varepsilon}, \overline{\mathbf{u}_\varepsilon})$-the average of the solution $(\varrho_\varepsilon,\mathbf{u}_\varepsilon)$ to the initial-boundary value problems (1.1)-(1.6)-are the incompressible Navier-Stokes equations in $\mathbb{R}^2$, namely

$$\partial_t\mathbf{v} + \left(\mathbf{v}\cdot\nabla_h\right)\mathbf{v} + \nabla_h\pi - \mu\Delta\mathbf{v} = 0,\ \mathrm{div}_h\mathbf{v} = 0 \tag{1.7}$$

supplemented with the initial value

$$\mathbf{v}_0\left(x_h\right) = \mathbf{H}\left(\mathbf{u}_{0,h}\right)\left(x_h\right) \in L^2(\mathbb{R}^2;\mathbb{R}^2), \tag{1.8}$$

see Theorem 1.4 below. Note that here we use notation $\mathbf{u}_h = (u_1, u_2)$ for a vector field $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$, $\mathbf{v} = (v_1, v_2)$ always represents a vector field in $\mathbb{R}^2$ and

$$\nabla_h = (\partial_{x_1}, \partial_{x_2}), \quad \mathrm{div}_h = \nabla_h \cdot, \ \Delta_h = \nabla_h \cdot \nabla_h = \partial_{x_1 x_1} + \partial_{x_2 x_2},$$

while $\mathbf{H} = \mathrm{Id} - \nabla_h \Delta_h^{-1} \mathrm{div}_h$ is the Helmholtz projection to solenoidal vector fields in $\mathbb{R}^2$.

Finally, in addition to $\delta = \delta(\varepsilon) \to 0$, if we assume $\mu = \mu(\varepsilon) \to 0$ as $\varepsilon \to 0$, we obtain the following Euler equations in the plane $\mathbb{R}^2$.

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla_h)\, \mathbf{v} + \nabla_h \pi = 0, \ \mathrm{div}_h \mathbf{v} = 0.$$

The goal of this paper is to rigorously justify these two multiple limit passages. We recall that in [18, 20] P. L. Lions and N. Masmoudi initiated the study of incompressible (and inviscid) limit of global weak solutions to the compressible Navier-Stokes equations. See also more recent works [1, 3, 6, 7, 8], among others, on analysis of multi-scale singular limit of compressible viscous fluids. G. Raugel and G. R. Sell have first studied the thin domain problem to the incompressible fluids, see [12, 21]. We also note that in a recent paper [10], the authors considered the incompressible inviscid limit on expanding domains. As in most cases of the singular limit problems in fluid dynamics, the main difficulties occur due to poor a priori bounds on the weak solutions as well as the high oscillation of acoustic waves due to ill-prepared data. Our approach is a combination of regularization and Strichartz estimates appeared in the context of singular limit problems in the whole space, see [2, 22], among others.

1.1 Weak solution to the compressible system

Following Maltese and Novotný [19] or Ducomet et al. [4] we define the weak solutions to the compressible Navier-Stokes system.

Definition 1.1. We say that $(\varrho, \mathbf{u})$ is a weak solution to the compressible Navier-Stokes system if

• the functions $(\varrho, \mathbf{u})$ belongs to the class

$$\varrho - 1 \in L^\infty\left([0,T]; L^\gamma(\Omega) + L^2(\Omega)\right), \quad \varrho \geq 0 \ \text{ a.a. in } \ (0,T) \times \Omega, \tag{1.9}$$

$$\mathbf{u} \in L^2\left(0,T; W^{1,2}_{\mathbf{n}}(\Omega; \mathbb{R}^3)\right), \quad \varrho \mathbf{u} \in L^\infty\left(0,T; L^2(\Omega) + L^{\frac{2\gamma}{\gamma+1}}(\Omega)\right). \tag{1.10}$$

• $\varrho - 1 \in C_{\mathrm{weak}}\left([0,T]; L^\gamma(\Omega) + L^2(\Omega)\right)$, and the continuity equation is satisfied in the weak sense,

$$\int_\Omega \varrho\varphi(\tau,\cdot)\,dx - \int_\Omega \varrho_0\varphi(0,\cdot)\,dx = \int_0^\tau \int_\Omega \varrho\left(\partial_t\varphi + \mathbf{u}\cdot\nabla_x\varphi\right) dxdt \tag{1.11}$$

for all $\tau \in [0,T]$ and any test function $\varphi \in C_c^\infty\left([0,T] \times \overline{\Omega}\right)$.

• $\varrho\mathbf{u} \in C_{\text{weak}}\left([0,T]; L^2(\Omega) + L^{\frac{2\gamma}{\gamma+1}}(\Omega)\right)$, and the momentum equation is satisfied in the weak sense,

$$\int_\Omega \varrho\mathbf{u}\cdot\phi(\tau,\cdot)\,dx - \int_\Omega \varrho_0\mathbf{u}_0\cdot\phi(0,\cdot)\,dx$$

$$= \int_0^\tau\int_\Omega \left(\varrho\mathbf{u}\cdot\partial_t\phi + \varrho\mathbf{u}\otimes\mathbf{u}:\nabla_x\phi + p(\varrho)\,\mathrm{div}_x\phi\right)dxdt - \mu\int_0^\tau\int_\Omega \mathbb{S}(\nabla\mathbf{u}):\nabla_x\phi dxdt \tag{1.12}$$

for all $\tau \in [0,T]$ and any test function $\phi \in C_c^\infty\left([0,T]\times\overline{\Omega};\mathbb{R}^3\right)$.

• the energy inequality

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{E(\varrho,1)}{\varepsilon^2}\right](\tau)\,dx + \mu\int_0^\tau\int_\Omega \mathbb{S}(\nabla\mathbf{u}):\nabla\mathbf{u}dxdt$$

$$\le \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{E(\varrho_0,1)}{\varepsilon^2}\right]dx \tag{1.13}$$

holds for a.e. $\tau \in [0,T]$, where

$$E(\varrho,1) = H(\varrho) - H'(1)(\varrho-1) - H(1),$$

with

$$H(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}dz.$$

1.2 Main results

To state our result, we first introduce the following classical result to the target system-the initial value problem to two dimensional Navier-Stokes equations (1.7), see [16] for example.

Theorem 1.2. *Given $\mathbf{v}_0 \in L^2(\mathbb{R}^2)$, $\mathrm{div}_h\mathbf{v}_0 = 0$ in the sense of distribution, there exists a unique weak solution*

$$\mathbf{v} \in C([0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^2)) \cap L^2_{loc}(0,\infty; W^{1,2}(\mathbb{R}^2;\mathbb{R}^2)),\ \mathbf{v}(0,\cdot) = \mathbf{v}_0$$

to (1.7) such that for any $\phi(t,x_h) \in C_c^\infty([0,T]\times\mathbb{R}^2;\mathbb{R}^2)$, $\mathrm{div}_h\phi = 0$,

$$\int_{\mathbb{R}^2} \mathbf{v}\cdot\phi(\tau,x_h)dx_h - \int_{\mathbb{R}^2}\mathbf{v}_0(x_h)\cdot\phi(0,x_h)dx_h$$

$$= \int_0^\tau\int_{\mathbb{R}^2} \mathbf{v}\cdot\partial_t\phi + \mathbf{v}\cdot\nabla_h\mathbf{v}\cdot\phi - \nabla_h\mathbf{v}:\nabla_h\phi dx_h dt. \tag{1.14}$$

for any $\tau \in [0,T]$.

Remark 1.3. In fact we only need the definition of weak solution to (1.7)-(1.8) and its uniqueness, from which we have the strong convergence of the whole sequence $\overline{\mathbf{u}_\varepsilon}$.

The first result of the present paper is the following theorem on the incompressible and thin domain limit. We assume $\delta \to 0$ as $\varepsilon \to 0$ while the viscosity $\mu > 0$ is fixed.

Theorem 1.4. *Let $\varrho_\varepsilon, \mathbf{u}_\varepsilon$ be the weak solution to the compressible Navier-Stokes system (1.1)-(1.6) with the initial data*

$$\overline{|\mathbf{u}_{0,\varepsilon}|^2} \text{ bounded in } L^1(\mathbb{R}^2),\ \overline{\left|\varrho_{0,\varepsilon}^{(1)}\right|^2} \text{ bounded in } L^1 \cap L^\infty(\mathbb{R}^2) \tag{1.15}$$

uniformly for $\varepsilon \in (0,1)$ such that

$$\overline{\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon}} \to \mathbf{u}_0 = (\mathbf{u}_{0,h}, 0) \in L^2(\mathbb{R}^2; \mathbb{R}^3) \tag{1.16}$$

as $\varepsilon \to 0$. Then

$$\overline{\varrho_\varepsilon} \to 1 \text{ in } L^\infty(0,T; L^2 + L^\gamma(\mathbb{R}^2)),\ \overline{\mathbf{u}_\varepsilon} \to (\mathbf{v}, 0) \text{ in } L^2(0,T; L^2_{loc}(\mathbb{R}^2)) \tag{1.17}$$

for any $T > 0$, where $\mathbf{v}$ is the unique weak solution to the initial value problem (1.7)-(1.8).

We also consider the inviscid incompressible limit, meaning the viscosity $\mu = \mu(\varepsilon) \to 0$ as $\varepsilon \to 0$. To this end, let us recall the following classical result, see [16] for example.

Theorem 1.5. *Given $\mathbf{v}_0 \in W^{3,2}(\mathbb{R}^2)$, $\operatorname{div}_h \mathbf{v}_0 = 0$, there exists a unique solution*

$$\mathbf{v} \in C^k([0,\infty), W^{3-k,2}(\mathbb{R}^2; \mathbb{R}^2)),\ \pi \in C^k([0,\infty), W^{3-k,2}(\mathbb{R}^2)),\ k = 0,1,2,3$$

to the following initial value problem

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla_h)\mathbf{v} + \nabla_h \pi = 0,\ \operatorname{div}_h \mathbf{v} = 0, \tag{1.18}$$

$$\mathbf{v}(0,x) = \mathbf{v}_0 \tag{1.19}$$

such that for any $T > 0$,

$$\|\mathbf{v}\|_{W^{k,\infty}(0,T;W^{3-k,2}(\mathbb{R}^2;\mathbb{R}^2))} + \|\pi\|_{W^{k,\infty}(0,T;W^{3-k,2}(\mathbb{R}^2))} \le c(T)\|\mathbf{v}_0\|_{W^{3,2}(\mathbb{R}^2)}. \tag{1.20}$$

Our result on incompressible, inviscid and thin domain limit is stated as follows.

Theorem 1.6. *Suppose $\delta, \mu \to 0$ as $\varepsilon \to 0$. Assume there exist $\varrho_0^{(1)} \in L^2(\mathbb{R}^2), \mathbf{u}_0 = (\mathbf{u}_{0,h}, 0) \in L^2(\mathbb{R}^2; \mathbb{R}^3)$ such that*

$$\overline{\left|\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\right|^2},\ \overline{|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0|^2} \to 0 \text{ in } L^1(\mathbb{R}^2) \tag{1.21}$$

and $\mathbf{v}_0 = \mathbf{H}(\mathbf{u}_{0,h}) \in W^{3,2}(\mathbb{R}^2)$, $\nabla_h \Psi_0 = \mathbf{H}^\perp(\mathbf{u}_{0,h}) \in L^2(\mathbb{R}^2; \mathbb{R}^2)$. Let $\mathbf{v}$ be the unique solution to the initial value problem (1.18)-(1.19) and $\varrho_\varepsilon, \mathbf{u}_\varepsilon$ be the weak solution to the compressible Navier-Stokes system (1.1)-(1.6). Then, as $\varepsilon \to 0$,

$$\overline{\varrho_\varepsilon} \to 1 \text{ in } L^\infty(0,T; L^2 + L^\gamma(\mathbb{R}^2)),\ \overline{\sqrt{\varrho_\varepsilon}\mathbf{u}_{\varepsilon,h}} \to \mathbf{v} \text{ in } L^2(0,T; L^2_{loc}(\mathbb{R}^2)) \tag{1.22}$$

for any $T > 0$ and any compact set $K \subset \mathbb{R}^2$.

Remark 1.7. It immediately follows from (1.21) that

$$\overline{\varrho_{0,\varepsilon}^{(1)}} \to \varrho_0^{(1)} \text{ in } L^2(\mathbb{R}^2),\ \overline{\mathbf{u}_{0,\varepsilon}} \to \mathbf{u}_0 \text{ in } L^2(\mathbb{R}^2;\mathbb{R}^3).$$

Remark 1.8. Comparing with results [19],[5],[4] we are interested in multi-scale singular limit, which means that we study not only reduction of dimension but also low Mach number limit or low Mach number inviscid limit. As a target system we get the weak solution of Navier-Stokes equation or strong solution of Euler equation.

Before the end of this section we introduce some results on regularization that will be used in the following context.

Let $\eta \in (0,1)$ and define $\chi_\eta(z) = \chi(\eta z) \in C_0^\infty(\mathbb{R})$, as

$$\chi(z) = 1,\ |z| \leq 1,\ \chi = 0, |z| \geq 2. \tag{1.23}$$

For a function $f \in L^2(\mathbb{R}^2)$, denote

$$f_\eta = \mathcal{F}^{-1}(\chi_\eta \hat{f}) = \mathcal{F}^{-1}(\chi_\eta) * f,$$

where $\hat{f}$ is the Fourier transform in $\mathbb{R}^2$ and $\mathcal{F}^{-1}$ is its inverse. Then $f_\eta \in C^\infty(\mathbb{R}^2) \cap W^{k,p}(\mathbb{R}^2)$ for any $p \in [1,\infty]$ and $k = 0,1,2,\cdots$. For $f \in L^p(\mathbb{R}^2)$, $p \in [1,\infty]$,

$$\|f_\eta\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)}$$

and

$$f_\eta \to f \text{ in } L^p(\mathbb{R}^2) \text{ as } \eta \to 0,\ p \in [1,\infty).$$

Moreover,

$$\begin{aligned} \|f_\eta\|_{W^{s,p_1}(\mathbb{R}^2)} &\leq c(s,p_1,p_2,\eta)\|f\|_{L^{p_2}(\mathbb{R}^2)}, \\ \|f_\eta\|_{W^{s_1,p_1}(\mathbb{R}^2)} &\leq c(s_1,s_2,p_1,p_2,\eta)\|f_\eta\|_{W^{s_2,p_2}(\mathbb{R}^2)} \end{aligned} \tag{1.24}$$

for any $s, s_1, s_2 \in \mathbb{R}, p_1 \geq p_2 \in [1,\infty]$ and fixed $\eta \in (0,1)$.

2 Uniform bounds

For any function f defined in $(0,T) \times \Omega_\delta$, we introduce the decomposition

$$f = [f]_{ess} + [f]_{res}$$

where

$$[f]_{ess} = \kappa(\varrho) f, \quad [f]_{res} = (1 - \kappa(\varrho)) f,$$

with

$$\kappa(\varrho) \in C_c^\infty(0,\infty), \quad 0 \leq \kappa(\varrho) \leq 1,$$

and $\kappa(\varrho) = 1$ in a neighborhood of 1. The above decomposition is understood in the sense that the *essential* part is the quantity that determines the asymptotic behavior of the system, while the *residual* part will disappear in the limit passage.

We start with the uniform bounds following from the energy inequality (1.13). Dividing both sides of (1.13) by δ and recalling assumption (1.15) added on the initial data, we have the following estimates:

$$\overline{\varrho_\varepsilon \left|\mathbf{u}_\varepsilon\right|^2} \text{ uniformly bounded in } L^\infty\left(0,T;L^1\left(\mathbb{R}^2\right)\right), \tag{2.1}$$

$$\left[\frac{\overline{\varrho_\varepsilon}-1}{\varepsilon}\right]_{ess} \text{ uniformly bounded in } L^\infty\left(0,T;L^2\left(\mathbb{R}^2\right)\right), \tag{2.2}$$

$$\text{ess sup}_{t\in(0,T)}\|\left[\overline{\varrho_\varepsilon}\right]_{res}\|^\gamma_{L^\gamma(\mathbb{R}^2} \leq c\varepsilon^2, \tag{2.3}$$

$$\text{ess sup}_{t\in(0,T)}\|\left[1\right]_{res}\|_{L^1(\mathbb{R}^2)} \leq c\varepsilon^2, \tag{2.4}$$

$$\mu\overline{\nabla_x \mathbf{u}_\varepsilon} \text{ uniformly bounded in } L^2\left(0,T;L^2\left(\mathbb{R}^2;\mathbb{R}^{3\times3}\right)\right). \tag{2.5}$$

As a direct consequence of these bounds,

$$\left[\overline{\varrho_\varepsilon \mathbf{u}_\varepsilon}\right]_{ess} \text{ uniformly bounded in } L^\infty\left(0,T;L^2\left(\mathbb{R}^2;\mathbb{R}^3\right)\right) \tag{2.6}$$

and

$$\left[\overline{\varrho_\varepsilon \mathbf{u}_\varepsilon}\right]_{res} \to 0 \text{ in } L^\infty\left(0,T;L^s\left(\mathbb{R}^2;\mathbb{R}^3\right)\right) \text{ as } \varepsilon\to 0 \text{ for any } s\in\left[1,2\gamma/\left(\gamma+1\right)\right]. \tag{2.7}$$

Also we observe that from (2.2) and (2.3),

$$r_\varepsilon := \frac{\overline{\varrho_\varepsilon}-1}{\varepsilon} \text{ uniformly bounded in } L^\infty(0,T;L^2+L^{\min\{2,\gamma\}}(\mathbb{R}^2)). \tag{2.8}$$

Moreover,

$$\overline{\varrho_\varepsilon} \to 1 \text{ in } L^\infty\left(0,T;L^\gamma\left(\mathbb{R}^2\right)+L^2\left(\mathbb{R}^2\right)\right). \tag{2.9}$$

For fixed μ we have uniform bound of $\overline{\mathbf{u}_\varepsilon}$ in $L^2(0,T;W^{1,2}(\mathbb{R}^2;\mathbb{R}^3))$. To this end we write

$$\int_{\mathbb{R}^2}\overline{\left|\mathbf{u}_\varepsilon\right|^2}dx_h = \int_{\mathbb{R}^2}\overline{\varrho_\varepsilon\left|\mathbf{u}_\varepsilon\right|^2}dx_h + \int_{\mathbb{R}^2}\overline{\left(1-\varrho_\varepsilon\right)\left|\mathbf{u}_\varepsilon\right|^2}dx_h.$$

The first term on the right-hand side is uniformly bounded thanks to (2.1). Let us consider the second one. We write

$$\overline{\varrho_\varepsilon}-1 = \overline{\varrho_\varepsilon^{(1)}} + \overline{\varrho_\varepsilon^{(2)}},$$

with

$$\overline{\varrho_\varepsilon^{(1)}} \to 0 \text{ in } L^\infty\left(0,T;L^\gamma\left(\mathbb{R}^2\right)\right), \tag{2.10}$$

and

$$\overline{\varrho_\varepsilon^{(2)}} \to 0 \text{ in } L^\infty\left(0,T;L^2\left(\mathbb{R}^2\right)\right). \tag{2.11}$$

We have

$$\int_{\mathbb{R}^2}\overline{\left(1-\varrho_\varepsilon\right)\left|\mathbf{u}_\varepsilon\right|^2}dx_h = \frac{1}{\delta}\int_{\Omega_\delta}\left(1-\varrho_\varepsilon\right)\left|\mathbf{u}_\varepsilon\right|^2dx,$$

and

$$\left|\frac{1}{\delta}\int_{\Omega_\delta}(1-\varrho_\varepsilon)\left|\mathbf{u}_\varepsilon\right|^2dx\right| \leq \left(\frac{1}{\delta}\int_{\Omega_\delta}\left|\varrho_\varepsilon^{(1)}\right|^\gamma dx\right)^{1/\gamma}\left(\frac{1}{\delta}\int_{\Omega_\delta}\left|\mathbf{u}_\varepsilon\right|^{2\gamma'}dx\right)^{1/\gamma'}$$

$$+\left(\frac{1}{\delta}\int_{\Omega_\delta}\left|\varrho_\varepsilon^{(2)}\right|^2dx\right)^{1/2}\left(\frac{1}{\delta}\int_{\Omega_\delta}\left|\mathbf{u}_\varepsilon\right|^4dx\right)^{1/2}$$

with $1/\gamma + 1/\gamma' = 1$. From (2.5) and Sobolev embedding, we have $\mathbf{u}_\varepsilon$ uniformly bounded in $L^2\left(0,T;L^6\left(\Omega_\delta;\mathbb{R}^3\right)\right)$. Indeed, for a function f such that $\nabla f \in L^2(\Omega_\delta)$, let $f_\delta(x_1,x_2,x_3) = f(x_1,x_2,\delta x_3), x_3 \in [0,1]$. Applying Sobolev embedding to f_δ in the fixed domain $\mathbb{R}^2\times[0,1]$ we find

$$\frac{1}{\delta}\int_{\Omega_\delta}|f|^6dx = \int_0^1\int_{\mathbb{R}^2}|f_\delta|^6dx \leq C\int_0^1\int_{\mathbb{R}^2}|\nabla f_\delta|^2dx$$

$$= \frac{1}{\delta}\int_{\Omega_\delta}|\nabla_h f|^2+\delta^2|\partial_3 f|^2dx \leq C\frac{1}{\delta}\int_{\Omega_\delta}|\nabla_h f|^2+|\partial_3 f|^2dx \text{ if } \delta\leq 1.$$

Then

$$\left(\frac{1}{\delta}\int_{\Omega_\delta}\left|\mathbf{u}_\varepsilon\right|^{2\gamma'}dx\right)^{1/\gamma'} \leq \left(\frac{1}{\delta}\int_{\Omega_\delta}\left|\mathbf{u}_\varepsilon\right|^2dx\right)^{\frac{3}{2\gamma'}-\frac{1}{2}}\left(\frac{1}{\delta}\int_{\Omega_\delta}\left|\mathbf{u}_\varepsilon\right|^6dx\right)^{\frac{1}{2}-\frac{1}{2\gamma'}},$$

and

$$\left(\frac{1}{\delta}\int_{\Omega_\delta}\left|\mathbf{u}_\varepsilon\right|^4dx\right)^{1/2} \leq \left(\frac{1}{\delta}\int_{\Omega_\delta}\left|\mathbf{u}_\varepsilon\right|^2dx\right)^{1/4}\left(\frac{1}{\delta}\int_{\Omega_\delta}\left|\mathbf{u}_\varepsilon\right|^6dx\right)^{1/4}.$$

From Young's inequality, (2.10) and (2.11), it follows

$$\int_{\mathbb{R}^2}\overline{\left|\mathbf{u}_\varepsilon\right|^2}dx_h \leq C,$$

which gives

$$\overline{\mathbf{u}_\varepsilon} \text{ bounded in } L^2\left(0,T;W^{1,2}\left(\mathbb{R}^2;\mathbb{R}^3\right)\right). \tag{2.12}$$

We emphasize that this uniform bound is only valid for fixed $\mu > 0$.

3 Energy and Strichartz estimates

We consider the following acoustic system in $\mathbb{R}^2$.

$$\varepsilon\partial_t\psi_\varepsilon + \Delta_h\Psi_\varepsilon = 0,\ \varepsilon\partial_t\nabla_h\Psi_\varepsilon + a^2\nabla_h\psi_\varepsilon = 0,\ a^2 = p'(1) > 0, \tag{3.1}$$

supplemented with the initial data

$$\psi_\varepsilon(0,x_h) = \psi_0(x_h) \in W^{m,2}(\mathbb{R}^2),\ \nabla_h\Psi_\varepsilon(0,x_h) = \nabla_h\Psi_0(x_h) \in W^{m,2}(\mathbb{R}^2;\mathbb{R}^2), \tag{3.2}$$

for some $m = 0, 1, 2, \cdots$. The acoustic system conserves energy,

$$\frac{1}{2}\int_{\mathbb{R}^2} |a\psi_\varepsilon(t,x_h)|^2 + |\nabla_h \Psi_\varepsilon(t,x_h)|^2 \, dx_h = \frac{1}{2}\int_{\mathbb{R}^2} |a\psi_0(x_h)|^2 + |\nabla_h \Psi(x_h)|^2 \, dx_h \tag{3.3}$$

for any $t \geq 0$. Also, standard energy estimates give us

$$\|\psi_\varepsilon(t,\cdot)\|_{W^{k,2}(\mathbb{R}^2)} + \|\nabla_h \Psi_\varepsilon(t,\cdot)\|_{W^{k,2}(\mathbb{R}^2)}$$
$$\leq c\left(\|\psi_0\|_{W^{k,2}(\mathbb{R}^2)} + \|\nabla_h \Psi_0\|_{W^{k,2}(\mathbb{R}^2)}\right) \tag{3.4}$$

for $k = 1, 2, \cdots, m$.

The acoustic wave system disperse local energy. We recall the following $L^p - L^q$-estimate as a special case of the well-known Strichartz estimates in $\mathbb{R}^2$, see [11].

$$\|\psi_\varepsilon\|_{L^q(\mathbb{R},L^p(\mathbb{R}^2))} + \|\nabla_h \Psi_\varepsilon\|_{L^q(\mathbb{R},L^p(\mathbb{R}^2))}$$
$$\leq c\varepsilon^{\frac{1}{q}}\left(\|\psi_0\|_{W^{\sigma,2}(\mathbb{R}^2)} + \|\nabla_h \Psi_0\|_{W^{\sigma,2}(\mathbb{R}^2)}\right) \tag{3.5}$$

for any

$$p \in (2,\infty),\ \frac{2}{q} = \frac{1}{2} - \frac{1}{p},\ q \in (4,\infty),\ \sigma = \frac{3}{q} < 1. \tag{3.6}$$

Hence for any $k = 0, 1, \cdots, m-1$,

$$\|\psi_\varepsilon\|_{L^q(\mathbb{R},W^{k,p}(\mathbb{R}^2))} + \|\nabla_h \Psi_\varepsilon\|_{L^q(\mathbb{R},W^{k,p}(\mathbb{R}^2))}$$
$$\leq c\varepsilon^{\frac{1}{q}}\left(\|\psi_0\|_{W^{m,2}(\mathbb{R}^2)} + \|\nabla_h \Psi_0\|_{W^{m,2}(\mathbb{R}^2)}\right). \tag{3.7}$$

Now consider the inhomogeneous case of (3.1),

$$\varepsilon\partial_t \psi_\varepsilon + \Delta_h \Psi_\varepsilon = \varepsilon f_1,\ \varepsilon\partial_t \nabla_h \Psi_\varepsilon + a^2 \nabla_h \psi_\varepsilon = \varepsilon \mathbf{f}_2 \tag{3.8}$$

supplemented with the initial data

$$\psi_\varepsilon(0,x_h) = \psi_0(x_h),\ \nabla_h \Psi_\varepsilon(0,x_h) = \nabla_h \Psi_0(x_h), \tag{3.9}$$

where $f_1, \mathbf{f}_2 \in L^q(0,T;W^{m,2}(\mathbb{R}^2))$ and $\psi_0(x_h), \nabla_h \Psi_0 \in W^{m,2}(\mathbb{R}^2)$. By using Duhamel's principle it is easy to show

$$\|\psi_\varepsilon\|_{L^q(\mathbb{R},W^{k,p}(\mathbb{R}^2))} + \|\nabla_h \Psi_\varepsilon\|_{L^q(\mathbb{R},W^{k,p}(\mathbb{R}^2))}$$
$$\leq c\varepsilon^{\frac{1}{q}}\left(\|\psi_0\|_{W^{m,2}(\mathbb{R}^2)} + \|\nabla_h \Psi_0\|_{W^{m,2}(\mathbb{R}^2)}\right) + c(T)\varepsilon^{\frac{1}{q}}\left(\|f_1\|_{W^{m,2}(\mathbb{R}^2)} + \|\mathbf{f}_2\|_{W^{m,2}(\mathbb{R}^2)}\right) \tag{3.10}$$

for the same k, p, q as above, see [2].

4 Weak to weak limit

This section is devoted to proving Theorem 1.4. Motivated by Lighthill [14], [15], we take average over $(0,\delta)$ in the x_3-variable to the original Navier-Stokes system (1.1)-(1.2) and write the resulting system in the following form in $(0,T)\times\mathbb{R}^2$,

$$\varepsilon\partial_t\left(\frac{\overline{\varrho_\varepsilon}-1}{\varepsilon}\right)+\mathrm{div}_h\left(\overline{\varrho_\varepsilon\mathbf{u}_{\varepsilon h}}\right)=0, \tag{4.1}$$

$$\varepsilon\partial_t\left(\overline{\varrho_\varepsilon\mathbf{u}_{\varepsilon h}}\right)+a^2\nabla_h\left(\frac{\overline{\varrho_\varepsilon}-1}{\varepsilon}\right)$$
$$=\varepsilon\left(\mu\mathrm{div}_h\mathbb{S}(\nabla_h\overline{\mathbf{u}_{\varepsilon h}})-\mathrm{div}_h\overline{\varrho_\varepsilon\mathbf{u}_{\varepsilon h}\otimes\mathbf{u}_{\varepsilon h}}+\frac{1}{\varepsilon^2}\nabla_h\left(\overline{p\left(\varrho_\varepsilon\right)}-a^2\left(\overline{\varrho_\varepsilon}-1\right)-p\left(1\right)\right)\right) \tag{4.2}$$

supplemented with the conditions (1.5) and (1.6), where $a^2=p'(1)$. In fact, the system (4.1) and (4.2) should be understood in the weak sense, namely

$$\int_0^T\int_{\mathbb{R}^2}\varepsilon r_\varepsilon\partial_t\varphi+\overline{\mathbf{m}_{0,\varepsilon}}\cdot\nabla_h\varphi dx_hdt+\varepsilon\int_{\mathbb{R}^2}r_{0,\varepsilon}\varphi(0,x_h)dx_h=0 \tag{4.3}$$

holds for every $\varphi\in C_c^\infty\left([0,T)\times\mathbb{R}^2\right)$, while

$$\int_0^T\int_{\mathbb{R}^2}\varepsilon\overline{\mathbf{m}_\varepsilon}\cdot\partial_t\phi+r_\varepsilon\mathrm{div}_h\phi dx_hdt+\int_{\mathbb{R}^2}\overline{\mathbf{m}_\varepsilon}\cdot\phi(0,x_h)dx_h=\varepsilon\int_0^T\int_{\mathbb{R}^2}\mathbf{f}_\varepsilon:\nabla_h\phi dx_hdt \tag{4.4}$$

for any $\phi\in C_c^\infty\left([0,T)\times\mathbb{R}^2;\mathbb{R}^2\right)$, where

$$r_\varepsilon=\frac{\overline{\varrho_\varepsilon}-1}{\varepsilon},\quad\overline{\mathbf{m}_\varepsilon}=\overline{\varrho_\varepsilon\mathbf{u}_{\varepsilon,h}},\ \mathbf{f}_\varepsilon=\mathbf{f}_\varepsilon^1+\mathbf{f}_\varepsilon^2+\mathbf{f}_\varepsilon^3,$$

$$\mathbf{f}_\varepsilon^1=\overline{\varrho_\varepsilon\mathbf{u}_{\varepsilon,h}\otimes\mathbf{u}_{\varepsilon,h}},\ \mathbf{f}_\varepsilon^2=-\mu\mathbb{S}(\nabla_h\overline{\mathbf{u}_{\varepsilon,h}}),\ \mathbf{f}_\varepsilon^3=\frac{1}{\varepsilon^2}\left(\overline{p\left(\varrho_\varepsilon\right)}-a^2\left(\overline{\varrho_\varepsilon}-1\right)-p\left(1\right)\right)\mathbb{I}_2,$$

such that

$$\mathbf{f}_\varepsilon^2\text{ uniformly bounded in }L^2\left(0,T;L^2\left(\mathbb{R}^2;\mathbb{R}^{2\times2}\right)\right) \tag{4.5}$$

and $\mathbf{f}_\varepsilon^1$, $\mathbf{f}_\varepsilon^3$ uniformly bounded in $L^\infty(0,T;L^1(\mathbb{R}^2;\mathbb{R}^{2\times2}))$ according to the uniform bounds established in (2.1)-(2.5). Hence

$$\mathbf{f}_\varepsilon^1,\ \mathbf{f}_\varepsilon^3\text{ uniformly bounded in }L^\infty(0,T;W^{-s,2}(\mathbb{R}^2;\mathbb{R}^{2\times2})),\ s>1, \tag{4.6}$$

since $L^1(\mathbb{R}^2)$ continuously embedded in $W^{-s,2}(\mathbb{R}^2)$.

The averaged momentum $\overline{\mathbf{m}_\varepsilon}$ can be written in terms of its Helmholtz decomposition, namely

$$\overline{\mathbf{m}_\varepsilon}=\mathbf{H}\left[\overline{\mathbf{m}_\varepsilon}\right]+\mathbf{H}^\perp\left[\overline{\mathbf{m}_\varepsilon}\right],$$

where

$$\mathbf{H}^\perp\left[\overline{\mathbf{m}_\varepsilon}\right]=\nabla_h\Phi_\varepsilon$$

represents the presence of the acoustic waves, with Φ_ε the acoustic potential, while $\mathbf{H}\left[\mathbf{m}_\varepsilon\right]$ the solenoidal part. In the following we will show the compactness of the solenoidal component, while dispersive estimates for the acoustic wave equations will show that $\nabla_h\Phi_\varepsilon$ tends to zero on compact subsets and therefore becomes negligible in the limit $\varepsilon\to0$.

4.1 Compactness of the solenoidal component

As a direct consequence of (2.12), there exists some $\mathbf{V}(t, x_1, x_2) \in \mathbb{R}^3$ such that

$$\overline{\mathbf{u}_\varepsilon} \to \mathbf{V} \text{ weakly in } L^2\left(0, T; W^{1,2}\left(\mathbb{R}^2; \mathbb{R}^3\right)\right). \tag{4.7}$$

From the weak formulation of the continuity equation, it follows

$$\mathrm{div}_x \mathbf{V} = 0 \text{ in } \mathcal{D}',$$

which is equivalent to

$$\mathrm{div}_h \mathbf{v} = 0, \quad \mathbf{v} = \mathbf{V}_h = \mathbf{V}_h\left(t, x_h\right).$$

We remark that in fact the third component of $\mathbf{V}$ is zero according to (2.12) and Poincaré's inequality. In order to show the strong convergence of $\mathbf{H}(\overline{\mathbf{u}_{\varepsilon,h}})$ we first observe that the solenoidal component of the vector field $\overline{\mathbf{m}_\varepsilon}$ is (weakly) compact in time. Indeed, relations (2.6) and (2.7) imply that

$$\overline{\mathbf{m}_\varepsilon} \to \mathbf{v} \text{ weakly-(*) in } L^\infty\left(0, T; \left(L^2 + L^{2\gamma/(\gamma+1)}\right)\left(\mathbb{R}^2; \mathbb{R}^2\right)\right) \tag{4.8}$$

since $\overline{\varrho_\varepsilon} \to 1$. From (4.4) and the bounds (4.6) and (4.5), we have

$$\left[\tau \to \int_{\mathbb{R}^2} \overline{\mathbf{m}_\varepsilon} \cdot \phi dx_h\right] \to \left[\tau \to \int_{\mathbb{R}^2} \mathbf{v} \cdot \phi dx_h\right] \text{ in } C\left[0, T\right] \tag{4.9}$$

for any $\phi(x_h) \in C_c^\infty\left(\mathbb{R}^2; \mathbb{R}^2\right)$, $\mathrm{div}_h \phi = 0$. This compactness in time of $\mathbf{H}(\overline{\mathbf{m}_\varepsilon})$, together with the fact that $\mathbf{H}(\overline{\mathbf{u}_{\varepsilon,h}})$ are uniformly bounded in $L^2(0, T; W^{1,2}(\mathbb{R}^2, \mathbb{R}^2))$, yield

$$\mathbf{H}(\overline{\mathbf{m}_\varepsilon}) \cdot \mathbf{H}(\overline{\mathbf{u}_{\varepsilon,h}}) \to |\mathbf{v}|^2$$

in the sense of distribution according to Lemma 5.1 in [17]. Hence $|\mathbf{H}(\overline{\mathbf{u}_{\varepsilon,h}})|^2 \to |\mathbf{v}|^2$ weakly since

$$\left|\int_0^T \int_{\mathbb{R}^2} (\mathbf{H}(\overline{\mathbf{m}_\varepsilon}) - \mathbf{H}(\overline{\mathbf{u}_{\varepsilon,h}})) \cdot \mathbf{H}(\overline{\mathbf{u}_{\varepsilon,h}})\right| = \left|\int_0^T \int_{\mathbb{R}^2} \mathbf{H}(\overline{(\varrho_\varepsilon - 1)\mathbf{u}_{\varepsilon,h}}) \cdot \mathbf{H}(\overline{\mathbf{u}_{\varepsilon,h}})\right|$$

$$\leq \|\overline{\varrho_\varepsilon} - 1\|_{L_T^\infty(L^2 + L^\gamma(\mathbb{R}^2))} \|\overline{\mathbf{u}_{\varepsilon,h}}\|^2_{L_T^2(L^4 + L^{\frac{2\gamma}{\gamma-1}}(\mathbb{R}^2))} \to 0$$

according to (2.9) and (2.12). We thus conclude by (4.7) that

$$\mathbf{H}(\overline{\mathbf{u}_{\varepsilon,h}}) \to \mathbf{v} \in L^2\left(0, T; L^2\left(\mathbb{R}^2; \mathbb{R}^2\right)\right) \tag{4.10}$$

and

$$\mathbf{H}(\overline{\mathbf{u}_{\varepsilon,h}}) \to \mathbf{v} \in L^2\left(0, T; L^p_{loc}\left(\mathbb{R}^2; \mathbb{R}^2\right)\right) \tag{4.11}$$

for any $p \in [2, \infty)$.

4.2 Compactness of the gradient component

From (4.1)-(4.2) (or its weak formulation (4.3)-(4.4)) we know that $r_\varepsilon = \frac{\overline{\varrho_\varepsilon}-1}{\varepsilon}$ and $\nabla_h \Phi_\varepsilon = \mathbf{H}^\perp(\overline{\varrho_\varepsilon \mathbf{u}_{\varepsilon,h}})$-the gradient part of $\overline{\varrho_\varepsilon \mathbf{u}_{\varepsilon,h}}$, obey the following equations in the sense of distribution.

$$\varepsilon \partial_t r_\varepsilon + \Delta_h \Phi_\varepsilon = 0,\ \varepsilon \partial_t \Phi_\varepsilon + a^2 \nabla_h r_\varepsilon = \varepsilon \mathbf{g}_\varepsilon, \tag{4.12}$$

supplemented with the initial data

$$r_\varepsilon(0,\cdot) = \overline{\varrho_{0,\varepsilon}^{(1)}},\ \nabla_h \Phi_\varepsilon(0,\cdot) = \mathbf{H}^\perp(\overline{\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon,h}}), \tag{4.13}$$

where $\mathbf{g}_\varepsilon = \mathbf{g}_\varepsilon^1 + \mathbf{g}_\varepsilon^2 + \mathbf{g}_\varepsilon^3$ and $\mathbf{g}_\varepsilon^i$ is the corresponding gradient part of $\mathbf{f}_\varepsilon^i$, $i = 1,2,3$ such that

$$\mathbf{g}_\varepsilon^2 \text{ uniformly bounded in } L^2\left(0,T; L^2\left(\mathbb{R}^2; \mathbb{R}^{2\times 2}\right)\right) \tag{4.14}$$

$$\mathbf{g}_\varepsilon^1,\ \mathbf{g}_\varepsilon^3 \text{ uniformly bounded in } L^\infty\left(0,T; W^{-s,2}\left(\mathbb{R}^2; \mathbb{R}^{2\times 2}\right)\right),\ s > 1, \tag{4.15}$$

according to (4.5) and (4.6).

We realize that system (4.12)-(4.13) is nothing but the inhomogeneous acoustic wave system (3.8)-(3.9). In order to apply Strichartz estimates we regularize (4.12)-(4.13) by using the mollifiers χ_η introduced in (1.23) to obtain

$$\varepsilon \partial_t r_{\varepsilon,\eta} + \Delta_h \Phi_{\varepsilon,\eta} = 0,\ \varepsilon \partial_t \Phi_{\varepsilon,\eta} + a^2 \nabla_h r_{\varepsilon,\eta} = \varepsilon \mathbf{g}_{\varepsilon,\eta}, \tag{4.16}$$

with the initial data

$$r_{\varepsilon,\eta}(0,\cdot) = \left(\overline{\varrho_{0,\varepsilon}^{(1)}}\right)_\eta,\ \nabla_h \Phi_{\varepsilon,\eta}(0,\cdot) = \left(\mathbf{H}^\perp(\overline{\varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon,h}})\right)_\eta. \tag{4.17}$$

Now by (1.24) and the Strichartz estimates (3.10) (with $k = 0$ and $p = 4, q = 8$ for example),

$$\begin{aligned} &\|r_{\varepsilon,\eta}\|_{L^q(\mathbb{R}, L^p(\mathbb{R}^2))} + \|\nabla_h \Phi_{\varepsilon,\eta}\|_{L^q_T(L^p(\mathbb{R}^2))} \\ \le\ & c\varepsilon^{\frac{1}{q}}\left(\left\|\overline{r_{\varepsilon}}_{,\eta}(0,\cdot)\right\|_{W^{1,2}(\mathbb{R}^2)} + \|\nabla_h \Phi_{\varepsilon,\eta}(0\cdot)\|_{W^{1,2}(\mathbb{R}^2)}\right) + c(T)\varepsilon^{\frac{1}{q}}\|\mathbf{g}_{\varepsilon,\eta}\|_{W^{1,2}(\mathbb{R}^2)} \\ \le\ & c(\eta)\varepsilon^{\frac{1}{q}} + c(\eta,T)\varepsilon^{\frac{1}{q}},\ \eta \in (0,1) \end{aligned}$$

according to the uniform-in-ε bounds (4.15)-(4.14) on $\mathbf{g}_\varepsilon$ and (1.15) on $\varrho_{0,\varepsilon}^{(1)}$ and $\mathbf{u}_{0,\varepsilon}$. By sending $\varepsilon \to 0$ we find that for any $\eta \in (0,1)$,

$$\nabla_h \Phi_{\varepsilon,\eta} \to 0 \text{ in } L^2(0,T; L^2_{loc}(\mathbb{R}^2)) \tag{4.18}$$

since $p, q > 2$. By using the uniform-in-ε bound of $\nabla_h \Phi_\varepsilon$ in $L^2(0,T; W^{1,2}(\mathbb{R}^2))$, which follows from the corresponding bound (2.12) for $\overline{\mathbf{u}_\varepsilon}$, and (**??**), we have

$$\nabla_h \Phi_\varepsilon - \nabla_h \Phi_{\varepsilon,\eta} \to 0 \text{ in } L^2(0,T; L^2_{loc}(\mathbb{R}^2)) \text{ as } \eta \to 0$$

uniformly for $\varepsilon \in (0,1)$. By writing

$$\nabla_h \Phi_\varepsilon = (\nabla_h \Phi_\varepsilon - \nabla_h \Phi_{\varepsilon,\eta}) + \nabla_h \Phi_{\varepsilon,\eta}$$

and taking $\varepsilon \to 0$ first and then $\eta \to 0$, we finally obtain

$$\nabla_h \Phi_\varepsilon \to 0 \text{ in } L^2(0,T; L^2_{loc}(\mathbb{R}^2)) \text{ as } \varepsilon \to 0 \tag{4.19}$$

and consequently

$$\nabla_h \Phi_\varepsilon \to 0 \text{ in } L^2(0,T; L^p_{loc}(\mathbb{R}^2)) \text{ as } \varepsilon \to 0 \tag{4.20}$$

for any $p \in [2, \infty)$.

4.3 The weak-weak limit passage

The strong convergence (4.20) of $\Phi_\varepsilon = \mathbf{H}^\perp(\overline{\varrho_\varepsilon \mathbf{u}_{\varepsilon,h}})$, together with the uniform bound (2.8) of $r_\varepsilon = \frac{\overline{\varrho_\varepsilon} - 1}{\varepsilon}$ yields

$$\mathbf{H}^\perp(\overline{\mathbf{u}_{\varepsilon,h}}) = \varepsilon \mathbf{H}^\perp\left(r_\varepsilon \mathbf{u}_{\varepsilon,h}\right) + \mathbf{H}^\perp(\overline{\varrho_\varepsilon \mathbf{u}_{\varepsilon,h}}) \to 0 \text{ in } L^2(0,T; L^s_{loc}(\mathbb{R}^2))$$

for $s < \min\{2, \gamma\}$. Hence

$$\mathbf{H}^\perp(\overline{\mathbf{u}_{\varepsilon,h}}) \to 0 \text{ in } L^2(0,T; L^p_{loc}(\mathbb{R}^2))$$

for any $p \in [2, \infty)$ according to (2.12). Together with the strong convergence (4.11) of the solenoidal part we conclude that

$$\overline{\mathbf{u}_{\varepsilon,h}} \to \mathbf{v} \text{ in } L^2(0,T; L^p_{loc}(\mathbb{R}^2)),\ p \in [2, \infty). \tag{4.21}$$

Finally, by applying all these strong convergence in the weak formulation (1.11)-(1.12) (after taking δ-average as in (4.1)-(4.2)), we find

$$\int_{\mathbb{R}^2} \mathbf{v} \cdot \nabla_h \varphi dx = 0$$

for any $\varphi \in C_c^\infty(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} \mathbf{v} \cdot \phi(\tau, x_h) dx_h - \int_{\mathbb{R}^2} \mathbf{v}_0 \cdot \phi(0, x_h) dx_h$$
$$= \int_0^\tau \int_{\mathbb{R}^2} \mathbf{v} \cdot \partial_t \phi + \mathbf{v} \otimes \mathbf{v} : \nabla_h \phi dx_h dt - \int_0^\tau \int_{\mathbb{R}^2} \nabla_h \mathbf{v} : \nabla_h \phi dx_h dt$$

for any $\phi \in C_c^\infty([0,T) \times \mathbb{R}^2)$, $\text{div}\phi = 0$, which are nothing but the weak formulation (1.14) of $\mathbf{v}$-the unique solution to two dimensional Navier-Stokes system (1.7)-(1.8).

5 The relative energy inequality

Motivated by [9], we introduce the relative energy inequality which is satisfied by any weak solution $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ of the Navier-Stokes system (1.1)-(1.6). First, we define a relative energy functional

$$\mathcal{E}\left(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid r, \mathbf{U}\right) = \frac{1}{\delta} \int_{\Omega_\delta} \frac{1}{2} \varrho_\varepsilon \left|\mathbf{u}_\varepsilon - \mathbf{U}\right|^2 + \frac{1}{\varepsilon^2}\left(H(\varrho_\varepsilon) - H'(r)\left(\varrho_\varepsilon - r\right) - H\left(r\right)\right) dx. \tag{5.1}$$

The following relative energy inequality holds.

$$\mathcal{E}\left(\varrho_{\varepsilon},\mathbf{u}_{\varepsilon}\mid r,\mathbf{U}\right)(\tau)+\frac{\mu}{\delta}\int_0^{\tau}\int_{\Omega_\delta}\mathbb{S}\left(\nabla_x\mathbf{u}_{\varepsilon}-\nabla_x\mathbf{U}\right):\left(\nabla_x\mathbf{u}_{\varepsilon}-\nabla_x\mathbf{U}\right)dxdt$$

$$\leq\mathcal{E}\left(\varrho_{\varepsilon},\mathbf{u}_{\varepsilon}\mid r,\mathbf{U}\right)(0)+\frac{1}{\delta}\int_0^{\tau}\mathcal{R}\left(\varrho_{\varepsilon},\mathbf{u}_{\varepsilon}\mid r,\mathbf{U}\right)dt, \tag{5.2}$$

with the remainder term

$$\mathcal{R}\left(\varrho_{\varepsilon},\mathbf{u}_{\varepsilon}\mid r,\mathbf{U}\right)=\int_{\Omega_\delta}\varrho_{\varepsilon}\left(\partial_t\mathbf{U}+\mathbf{u}_{\varepsilon}\cdot\nabla_x\mathbf{U}\right)\cdot\left(\mathbf{U}-\mathbf{u}_{\varepsilon}\right)dx$$

$$+\mu\int_{\Omega_\delta}\mathbb{S}\left(\nabla_x\mathbf{U}\right):\left(\nabla_x\mathbf{U}-\nabla_x\mathbf{u}_{\varepsilon}\right)dx$$

$$+\frac{1}{\varepsilon^2}\int_{\Omega_\delta}\left(\varrho_{\varepsilon}-r\right)\partial_t H'\left(r\right)-p\left(\varrho_{\varepsilon}\right)\mathrm{div}_x\mathbf{U}-\varrho_{\varepsilon}\mathbf{u}_{\varepsilon}\cdot\nabla H'\left(r\right)dx \tag{5.3}$$

for any pair of smooth functions $r,\mathbf{U}$ such that

$$r>0,\quad r-1\in C_c^{\infty}\left([0,T]\times\overline{\Omega_\delta}\right),\quad \mathbf{U}\in C_c^{\infty}\left([0,T]\times\overline{\Omega_\delta};\mathbb{R}^3\right),\quad \mathbf{U}\cdot\mathbf{n}|_{\partial\Omega_\delta}=0. \tag{5.4}$$

Note that the class of test functions $r,\mathbf{U}$ can be extended to a wider ones ensuring all terms appeared in the relative energy inequality make sense.

6 The incompressible inviscid limit

6.1 Test functions

In contrast to Section 4, we consider the acoustic wave equations (3.1)-(3.2) with initial data

$$\psi_0=\varrho_0^{(1)},\ \nabla_h\Psi_0=\mathbf{H}^{\perp}(\mathbf{u}_{0,h}).$$

Let

$$(\psi_{0,\eta},\nabla_h\Psi_{0,\eta}):=((\varrho_0^{(1)})_\eta,\mathbf{H}^{\perp}(\mathbf{u}_{0,h})_\eta).$$

and $\psi_{\varepsilon,\eta},\nabla_h\Psi_{\varepsilon,\eta}$ be the corresponding solution to (3.1). Since the acoustic wave system is linear,

$$\psi_{\varepsilon,\eta}=(\psi_\varepsilon)_\eta,\ \nabla_h\Psi_{\varepsilon,\eta}=(\nabla_h\Psi_\varepsilon)_\eta.$$

Let ε_0 be small enough such that for $\varepsilon\leq\varepsilon_0$, $r_{\varepsilon,\eta}:=1+\varepsilon\psi_{\varepsilon,\eta}>0$. We use the couple

$$[r_{\varepsilon,\eta},\mathbf{U}_{\varepsilon,\eta}],\ \mathbf{U}_{\varepsilon,\eta}=(\mathbf{v}+\nabla_h\Psi_{\varepsilon,\eta},0)$$

as the test function $[r,\mathbf{U}]$ in the relative energy inequality (5.2), where $\mathbf{v}$ the solution to the 2d Euler equations (1.18)-(1.19).

$$\mathcal{E}_{\varepsilon,\eta}\left(\varrho,\mathbf{u}\mid r,\mathbf{U}\right)(\tau)+\frac{\mu}{\delta}\int_0^{\tau}\int_{\Omega_\delta}\mathbb{S}\left(\nabla_x\mathbf{u}-\nabla_x\mathbf{U}\right):\left(\nabla_x\mathbf{u}-\nabla_x\mathbf{U}\right)dxdt$$

$$\leq \mathcal{E}_{\varepsilon,\eta}\left(\varrho, \mathbf{u} \mid r, \mathbf{U}\right)(0) + \frac{1}{\delta}\int_0^\tau \mathcal{R}_{\varepsilon,\eta}\left(\varrho, \mathbf{u} \mid r, \mathbf{U}\right) dt. \tag{6.1}$$

Here to avoid notation complexity we omit the subscript ε of $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ and ε, η of $[r_{\varepsilon,\eta}, \mathbf{U}_{\varepsilon,\eta}]$ unless it is necessary. Also we tacitly admit that, when using addition/dot between a vector $\mathbf{u} \in \mathbb{R}^3$ and another vector $\mathbf{v} \in \mathbb{R}^2$, $\mathbf{v}$ is viewed as a 3d vector such that its third component is zero.

For the initial data we have

$$\mathcal{E}_{\varepsilon,\eta}\left(\varrho, \mathbf{u} \mid r, \mathbf{U}\right)(0) = \frac{1}{\delta}\int_{\Omega_\delta} \frac{1}{2}\varrho_{0,\varepsilon}\left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x$$

$$+\frac{1}{\delta}\int_{\Omega_\delta}\frac{1}{\varepsilon^2}\left[H\left(1+\varepsilon\varrho_{0,\varepsilon}^{(1)}\right) - \varepsilon H'\left(1+\varepsilon\varrho_0^{(1)}\right)\left(\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\right) - H\left(1+\varepsilon\varrho_0^{(1)}\right)\right]\mathrm{d}x, \tag{6.2}$$

where $\mathbf{u}_0 = \mathbf{H}[\mathbf{u}_{0,h}] + \nabla_h\Psi_0$. For the first term on the right hand side of the equality (6.2) we have

$$\frac{1}{\delta}\int_{\Omega_\delta}\frac{1}{2}\varrho_{0,\varepsilon}\left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x = \frac{1}{\delta}\int_{\Omega_\delta}\frac{1}{2}\left|1+\varepsilon\varrho_{0,\varepsilon}^{(1)}\right|\left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x$$

$$\leq \frac{1}{\delta}\int_{\Omega_\delta}\frac{1}{2}\left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x + \frac{1}{\delta}\int_{\Omega_\delta}\frac{1}{2}\left|\varepsilon\varrho_{0,\varepsilon}^{(1)}\right|\left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x$$

$$\leq \frac{1}{\delta}\int_{\Omega_\delta}\frac{1}{2}\left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x + \varepsilon\left\|\overline{\varrho_{0,\varepsilon}^{(1)}}\right\|_{L^\infty(\mathbb{R}^3)}\frac{1}{\delta}\int_{\Omega_\delta}\frac{1}{2}\left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2 \mathrm{d}x$$

$$\leq c\left(1+\varepsilon\right)\left\|\overline{\left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2}\right\|_{L^1(\mathbb{R}^2;\mathbb{R}^3)}. \tag{6.3}$$

For the second term on the right hand side of the equality (6.2), setting $a = 1+\varepsilon\varrho_{0,\varepsilon}^{(1)}$ and $b = 1+\varepsilon\varrho_0^{(1)}$ and observing that

$$H(a) = H(b) + H'(b)(a-b) + \frac{1}{2}H''(\xi)(a-b)^2, \;\; \xi \in (a,b)\,,$$

$$\left|H(a) - H'(b)(a-b) - H(b)\right| \leq c\left|a-b\right|^2,$$

we have

$$\frac{1}{\delta}\int_{\Omega_\delta}\frac{1}{\varepsilon^2}\left[H\left(1+\varepsilon\varrho_{0,\varepsilon}^{(1)}\right) - \varepsilon H'\left(1+\varepsilon\varrho_0^{(1)}\right)\left(\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\right) - H\left(1+\varepsilon\varrho_0^{(1)}\right)\right]\mathrm{d}x$$

$$\leq c\frac{1}{\delta}\int_{\Omega_\delta}\frac{1}{\varepsilon^2}\left(\left|\varepsilon\left(\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\right)\right|^2\right)\mathrm{d}x$$

$$\leq \left\|\overline{\left|\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\right|^2}\right\|_{L^1(\mathbb{R}^2)}. \tag{6.4}$$

Finally, we can conclude

$$[\mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})](0) \leq c[(1+\varepsilon)\left\|\overline{\left|\mathbf{u}_{0,\varepsilon} - \mathbf{u}_0\right|^2}\right\|_{L^1(\mathbb{R}^2;\mathbb{R}^3)} + \left\|\overline{\left|\varrho_{0,\varepsilon}^{(1)} - \varrho_0^{(1)}\right|^2}\right\|_{L^1(\mathbb{R}^2)}].$$

By sending $\varepsilon \to 0$ and then $\eta \to 0$ we find, according to (1.15),

$$\mathcal{E}\left(\varrho_{\varepsilon}, \mathbf{u}_{\varepsilon} \mid r_{\varepsilon}, \mathbf{U}_{\varepsilon}\right)(0) \to 0 \text{ as } \varepsilon \to 0. \tag{6.5}$$

Denote

$$\mathcal{R}_{\varepsilon,\eta}\left(\varrho, \mathbf{u} \mid r, \mathbf{U}\right) = \sum_{j=1}^{3} \mathcal{R}_j.$$

The remaining part of this section is to estimate each $\mathcal{R}_j$ to conclude the proof of Theorem 1.6 by Gronwall's inequality.

In the following we will use notation c, which may change from line to line, to mean a constant depending only on the uniform bound of the given initial data. Notations $c(T), c(\eta, T)$ mean the constants may depending on its components but independent of ε.

6.2 The convective term

We write

$$\begin{aligned}\frac{1}{\delta}\int_0^{\tau} \mathcal{R}_1 dt = {} & \frac{1}{\delta}\int_0^{\tau}\int_{\Omega_\delta} \varrho\left(\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U}\right)\cdot\left(\mathbf{U}-\mathbf{u}\right)dxdt \\ & + \frac{1}{\delta}\int_0^{\tau}\int_{\Omega_\delta} \varrho\left(\mathbf{u}-\mathbf{U}\right)\cdot\nabla\mathbf{U}\cdot\left(\mathbf{U}-\mathbf{u}\right)dxdt. \end{aligned} \tag{6.6}$$

The last term is controlled by

$$\begin{aligned} & \int_0^{\tau} \|\nabla_h \mathbf{v}(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \mathcal{E}_{\varepsilon,\eta}(t)dt + \frac{1}{\delta}\int_0^{\tau}\int_{\Omega_\delta} \varrho\left(\mathbf{u}-\mathbf{U}\right)\cdot\nabla\nabla\Psi\cdot\left(\mathbf{U}-\mathbf{u}\right)dxdt \\ & \leq \int_0^{\tau} c(t)\mathcal{E}_{\varepsilon,\eta}(t)dt - \frac{1}{\delta}\int_0^{\tau}\int_{\Omega_\delta} \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\nabla\Psi dxdt \\ & -\frac{2}{\delta}\int_0^{\tau}\int_{\Omega_\delta} \varrho\left(\mathbf{u}\otimes\mathbf{U}\right) : \nabla\nabla\Psi dxdt + \frac{1}{\delta}\int_0^{\tau}\int_{\Omega_\delta} \varrho\left(\mathbf{U}\otimes\mathbf{U}\right) : \nabla\nabla\Psi dxdt. \end{aligned} \tag{6.7}$$

Applying (1.24) and Sobolev's embedding lemma to $\varrho\mathbf{u}\otimes\mathbf{u}$ term,

$$\begin{aligned} & \left|\frac{1}{\delta}\int_0^{\tau}\int_{\Omega_\delta} \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\nabla\Psi dxdt\right| \leq c(T)\left\|\overline{\varrho|\mathbf{u}|^2}\right\|_{L^\infty_T(L^1(\mathbb{R}^2))}\left\|\nabla^2\Psi\right\|_{L^8_T(L^\infty(\mathbb{R}^2))} \\ & \leq c(\eta,T)\left\|\overline{\varrho|\mathbf{u}|^2}\right\|_{L^\infty_T(L^1(\mathbb{R}^2))}\left\|\nabla^2\Psi\right\|_{L^8_T(W^{1,4}(\mathbb{R}^2))} \leq c(\eta,T)\varepsilon^{\frac{1}{8}} \end{aligned} \tag{6.8}$$

according to the uniform bound (2.1) and Strichart estimate (3.7). Moreover, by using the uniform bound of $\overline{\varrho\mathbf{u}}$ in $L^\infty(0,T;L^2 + L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^2))$,

$$\left|\frac{1}{\delta}\int_0^{\tau}\int_{\Omega_\delta} \varrho\left(\mathbf{u}\otimes\mathbf{U}\right) : \nabla\nabla\Psi dxdt\right|$$

$$\leq c(T)\left\|\overline{\varrho\mathbf{u}}\right\|_{L^\infty_T(L^2+L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^2))}\left\|\mathbf{U}\right\|_{L^\infty_T(L^4+L^{\frac{6\gamma}{2\gamma-3}}(\mathbb{R}^2))}\left\|\nabla^2\Psi\right\|_{L^8_T(L^4(\mathbb{R}^2))+L^6_T(L^6(\mathbb{R}^2))}$$

$$\leq c(T)c(\eta)\left(\varepsilon^{\frac{1}{8}}+\varepsilon^{\frac{1}{6}}\right)\leq c(\eta,T)\varepsilon^{\frac{1}{8}}. \tag{6.9}$$

To handle the last $\mathbf{U}\otimes\mathbf{U}$ term in (6.7), we use the uniform bound (2.8) to obtain

$$\left|\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\left(\mathbf{U}\otimes\mathbf{U}\right):\nabla\nabla\Psi dxdt\right|$$

$$\leq\varepsilon\left|\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\frac{\varrho-1}{\varepsilon}\left(\mathbf{U}\otimes\mathbf{U}\right):\nabla\nabla\Psi dxdt\right|+\left|\int_0^\tau\int_{\mathbb{R}^2}\left(\mathbf{U}\otimes\mathbf{U}\right):\nabla\nabla\Psi dx_hdt\right|$$

$$\leq c(T)\varepsilon+c(\eta,T)\varepsilon^{\frac{1}{8}}\leq c(\eta,T)\varepsilon^{\frac{1}{8}}. \tag{6.10}$$

For the first term on the right side of (6.6),

$$\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\left(\partial_t\mathbf{U}+\mathbf{U}\cdot\nabla_x\mathbf{U}\right)\cdot\left(\mathbf{U}-\mathbf{u}\right)dxdt$$

$$=\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\left(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla_h\mathbf{v}\right)\cdot\left(\mathbf{U}-\mathbf{u}\right)dxdt+\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\partial_t\nabla_h\Psi\cdot\left(\mathbf{U}-\mathbf{u}\right)dxdt$$

$$+\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\nabla_h\Psi\cdot\nabla_h\nabla_h\Psi\cdot\left(\mathbf{U}-\mathbf{u}\right)dxdt$$

$$+\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\left(\mathbf{v}\cdot\nabla_h(\nabla_h\Psi)+\nabla_h\Psi\cdot\nabla_h\mathbf{v}\right)\cdot\left(\mathbf{U}-\mathbf{u}\right)dxdt. \tag{6.11}$$

Since $\mathbf{v}$ is the solution to the Euler equations (1.18), we have

$$\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\left(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla_h\mathbf{v}\right)\cdot\left(\mathbf{U}-\mathbf{u}\right)dxdt=I_1+I_2,$$

where

$$I_1=\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\mathbf{u}\cdot\nabla_h\pi dxdt=\frac{1}{\delta}\int_{\Omega_\delta}\varrho\pi dx\,|_{t=0}^\tau-\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\partial_t\pi dxdt$$

$$=\varepsilon\frac{1}{\delta}\int_{\Omega_\delta}\frac{\varrho-1}{\varepsilon}\pi dx\,|_{t=0}^\tau-\varepsilon\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\frac{\varrho-1}{\varepsilon}\partial_t\pi dxdt\leq c(\eta,T)\varepsilon \tag{6.12}$$

according to (1.20) and (2.2)-(2.4) and

$$|I_2|=\left|\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\mathbf{U}\cdot\nabla_h\pi\mathrm{d}x\mathrm{d}t\right|\leq\left|\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\left(\varrho-1\right)\cdot\mathbf{U}\cdot\nabla_h\pi\mathrm{d}x\mathrm{d}t\right|$$

$$+\left|\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\mathbf{U}\cdot\nabla_h\pi\mathrm{d}x\mathrm{d}t\right|. \tag{6.13}$$

Similarly to the analysis above, for the first term on the right hand side of (6.13), we have

$$\left|\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}(\varrho-1)\cdot\mathbf{U}\cdot\nabla_h\pi\mathrm{d}x\mathrm{d}t\right|\le\varepsilon\left|\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\frac{(\varrho-1)}{\varepsilon}\cdot\mathbf{U}\cdot\nabla_h\pi\mathrm{d}x\mathrm{d}t\right|$$
$$\le c(T)\varepsilon$$

according to (1.20), (2.2)-(2.4) and the energy estimate (3.4). For the second term on the right hand side of (6.13), we have

$$\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\mathbf{U}\cdot\nabla_h\pi\mathrm{d}xdt=\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\mathbf{v}\cdot\nabla_h\pi\mathrm{d}x\mathrm{d}t+\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\nabla_h\Psi\cdot\nabla_h\pi\mathrm{d}xdt. \tag{6.14}$$

Performing integration by parts in the first term on the right-hand side of (6.14), we have

$$\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\mathrm{div}_h\mathbf{v}\cdot\pi\mathrm{d}x\mathrm{d}t=0$$

thanks to incompressibility condition, $\mathrm{div}_h\mathbf{v}=0$. For the second term on the right-hand side of (6.14) using integration by parts and acoustic equation, we have

$$\begin{aligned}\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\nabla_h\Psi\cdot\nabla_h\pi\mathrm{d}x\mathrm{d}t&=-\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\Delta_h\Psi\cdot\pi\mathrm{d}x\mathrm{d}t\\&=\varepsilon\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\partial_t\psi\cdot\pi\mathrm{d}x\mathrm{d}t\\&=\varepsilon\left[\frac{1}{\delta}\int_{\Omega_\delta}\psi\cdot\pi\mathrm{d}x\right]_{t=0}^{t=\tau}-\varepsilon\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\psi\cdot\partial_t\pi\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{6.15}$$

that it goes to zero for $\varepsilon\to0$.

Moreover, by using similar argument as above, the last two terms in (6.11) are of order

$$c(\eta,T)(1+\varepsilon)\|\nabla_h\Psi\|_{L^{\frac{8}{5}}_T(W^{1,4}(\mathbb{R}^2))}\le c(\eta,T)\varepsilon^{\frac{1}{8}}. \tag{6.16}$$

Finally, using $\mathrm{div}\mathbf{v}=0$,

$$\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\partial_t\nabla_h\Psi\cdot(\mathbf{U}-\mathbf{u})\,dxdt=-\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\mathbf{u}\cdot\partial_t\nabla_h\Psi dxdt$$
$$+\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}(\varrho-1)\mathbf{v}\cdot\partial_t\nabla_h\Psi dxdt+\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\partial_t\nabla_h\Psi\cdot\nabla_h\Psi dxdt \tag{6.17}$$

The first term on the right side of (6.17) will be cancelled later by the pressure term while by using the acoustic wave equations (3.1), the second term equals to

$$\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\frac{\varrho-1}{\varepsilon}\varepsilon\partial_t\nabla_h\Psi\cdot\mathbf{v}dxdt=-\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\frac{\varrho-1}{\varepsilon}a^2\nabla_h\psi\cdot\mathbf{v}dxdt$$

$$\leq c(T)\left\|\frac{\overline{\varrho}-1}{\varepsilon}\right\|_{L_T^\infty(L^2+L^{\gamma_2}(\mathbb{R}^2))}\|\mathbf{v}\|_{L_T^\infty(L^4+L^{\frac{4\gamma}{3\gamma-4}}(\mathbb{R}^2))}\|\nabla\psi_h\|_{L_T^8(L^4+L^4(\mathbb{R}^2))}$$

$$\leq c(\eta,T)\varepsilon^{\frac{1}{8}},\ \gamma_2=\min\{2,\gamma\} \tag{6.18}$$

by (2.8). Finally, by using the acoustic equations, $\varepsilon\partial_t\nabla_h\Psi=-a^2\nabla_h\psi$,

$$\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\partial_t\nabla_h\Psi\cdot\nabla_h\Psi dxdt$$

$$=-a^2\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\frac{\varrho-1}{\varepsilon}\nabla_h\psi\cdot\nabla_h\Psi dxdt+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla_h\Psi|^2dx\,|_{t=0}^\tau$$

$$\leq c(\eta,T)\varepsilon^{\frac{1}{8}}+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla_h\Psi|^2dx\,|_{t=0}^\tau\ . \tag{6.19}$$

From (6.6) to (6.19) we find

$$\frac{1}{\delta}\int_0^\tau\mathcal{R}_1dt\leq c(\eta,T)\varepsilon^{\frac{1}{8}}+\int_0^\tau c(t)\mathcal{E}_{\varepsilon,\eta}(t)dt$$

$$+\frac{1}{2}\int_{\mathbb{R}^2}|\nabla_h\Psi|^2dx\,|_{t=0}^\tau-\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\mathbf{u}\cdot\partial_t\nabla_h\Psi dxdt. \tag{6.20}$$

6.3 The dissipative term

We have

$$\frac{1}{\delta}\int_0^\tau\mathcal{R}_2dt=\frac{\mu}{\delta}\int_0^\tau\mathbb{S}(\nabla\mathbf{U}):(\nabla\mathbf{u}-\nabla\mathbf{U})dxdt$$

$$\leq\frac{\mu}{2\delta}\int_0^\tau\mathbb{S}(\nabla\mathbf{u}-\nabla\mathbf{U}):(\nabla\mathbf{u}-\nabla\mathbf{U})dxdt+c\mu\int_0^\tau\int_{\mathbb{R}^2}|\mathrm{div}\mathbb{S}(\nabla\mathbf{U})|^2\,dxdt.$$

Hence the first term can be absorbed by its counterpart on the left side of (6.1) and the second term is dominated by $c(\eta,T)\mu$, which goes to zero as $\varepsilon\to0$ since $\mu=\mu(\varepsilon)\to0$.

6.4 Terms depending on the pressure

Recalling that

$$\frac{1}{\delta}\int_0^\tau\mathcal{R}_3dt=\frac{1}{\varepsilon^2}\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}(\varrho-r)\,\partial_tH'(r)-p(\varrho)\,\mathrm{div}\mathbf{U}-\varrho\mathbf{u}\cdot\nabla H'(r)\,dxdt,$$

where $r=r_{\varepsilon,\eta}=1+\varepsilon\psi_{\varepsilon,\eta}$.

$$\frac{1}{\varepsilon^2}\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\mathbf{u}\cdot\nabla H'(r)\,dx=\frac{1}{\varepsilon}\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\mathbf{u}\cdot\nabla\psi H''(r)dxdt$$

$$=\frac{1}{\delta}\int_0^\tau\int_{\Omega_\delta}\varrho\mathbf{u}\cdot\nabla\psi\frac{H''(1+\varepsilon\psi)-H''(1)}{\varepsilon}dxdt+\frac{1}{\delta}\frac{1}{\varepsilon}\int_0^\tau\int_{\Omega_\delta}a^2\varrho\mathbf{u}\cdot\nabla\psi dxdt$$

since $H''(1) = p'(1) = a^2$. Realizing that

$$\left| \frac{H''(1+\varepsilon\psi) - H''(1)}{\varepsilon} \right| \leq c|\psi|,$$

the first term on the right side is controlled by

$$c(\eta, T) \left\| \overline{\varrho \mathbf{u}} \right\|_{L_T^\infty(L^2 + L^{\frac{2\gamma}{\gamma+1}}(\mathbb{R}^2))} \|\psi\|_{L_T^\infty(L^4+L^\infty(\mathbb{R}^2))} \|\nabla_h \psi\|_{L_T^8(L^4(\mathbb{R}^2)) + L_T^{4\gamma}(L^{\frac{2\gamma}{\gamma-1}}(\mathbb{R}^2))}$$

$$\leq c(\eta, T)\varepsilon^{\min\{\{\frac{1}{8}, \frac{1}{4\gamma}\}} \tag{6.21}$$

By using the acoustic equations,

$$\frac{1}{\delta}\frac{1}{\varepsilon}\int_0^\tau \int_{\Omega_\delta} a^2 \varrho \mathbf{u} \cdot \nabla \psi dx dt = -\frac{1}{\delta}\int_0^\tau \int_{\Omega_\delta} \varrho \mathbf{u} \cdot \partial_t \Psi dx dt,$$

which cancels the same term appeared on the right side of (6.17). Now we write

$$\frac{1}{\varepsilon^2}\frac{1}{\delta}\int_0^\tau \int_{\Omega_\delta} (\varrho - r)\, \partial_t H'(r) - p(\varrho)\, \mathrm{div}\mathbf{U} dx dt$$

$$= \frac{1}{\delta}\int_0^\tau \int_{\Omega_\delta} \frac{(\varrho-1)}{\varepsilon} H''(r) \partial_t \psi dx dt + \int_0^\tau \int_{\mathbb{R}^2} \psi H''(r) \partial_t \psi dx_h dt$$

$$-\frac{1}{\delta}\int_0^\tau \int_{\Omega_\delta} \frac{p(\varrho) - p'(1)(\varrho-1) - p(1)}{\varepsilon^2} \Delta \Psi dx dt - \frac{1}{\delta}\int_0^\tau \int_{\Omega_\delta} p'(1)\frac{\varrho-1}{\varepsilon}\frac{1}{\varepsilon}\Delta_h \Psi dx dt. \tag{6.22}$$

Note that

$$\frac{1}{\delta}\int_0^\tau \int_{\Omega_\delta} \frac{\varrho-1}{\varepsilon} H''(r) \partial_t \psi dx dt = \frac{1}{\delta}\int_0^\tau \int_{\Omega_\delta} \frac{\varrho-1}{\varepsilon} H''(1) \partial_t \psi dx dt$$

$$+\frac{1}{\delta}\int_0^\tau \int_{\Omega_\delta} \frac{\varrho-1}{\varepsilon} \left(H''(r) - H''(1)\right) \partial_t \psi dx dt.$$

We find the first term on the right side is cancelled by the last term in (6.22) while the remaining term equals to

$$-\frac{1}{\delta}\int_0^\tau \int_{\Omega_\delta} \frac{\varrho-1}{\varepsilon}\frac{H''(r) - H''(1)}{\varepsilon} \Delta_h \Psi dx dt$$

$$\leq c(T) \left\| \frac{\overline{\varrho} - 1}{\varepsilon} \right\|_{L_T^\infty(L^2 + L^\gamma(\mathbb{R}^2))} \|\psi\|_{L_T^\infty(L^4 + L^{\frac{4\gamma}{3\gamma-4}}(\mathbb{R}^2))} \|\Delta_h \Psi\|_{L_T^8(L^4+L^4(\mathbb{R}^2))}$$

$$\leq c(\eta, T)\varepsilon^{\frac{1}{8}}. \tag{6.23}$$

Similarly,

$$\int_0^\tau \int_{\mathbb{R}^2} \psi H''(r) \partial_t \psi dx_h dt = \int_0^\tau \int_{\mathbb{R}^2} \psi H''(1) \partial_t \psi dx_h dt$$

$$+\int_0^\tau \int_{\mathbb{R}^2} \psi \left(H''(r) - H''(1)\right) \partial_t \psi dx_h dt$$

$$\leq \frac{1}{2} \int_{\mathbb{R}^2} a^2 |\psi|^2 dx_h |_{t=0}^{\tau} + c(T) \|\psi\|_{L^\infty_T(L^2(\mathbb{R}^2))} \|\psi\|_{L^\infty_T(L^4(\mathbb{R}^2))} \|\Delta_h \Psi\|_{L^{\frac{8}{5}}_T(L^4(\mathbb{R}^2))}$$

$$\leq \frac{1}{2} \int_{\mathbb{R}^2} a^2 |\psi|^2 dx_h |_{t=0}^{\tau} + c(\eta, T)\varepsilon^{\frac{1}{8}}. \tag{6.24}$$

Finally, realizing that $\frac{1}{\delta} \frac{p(\varrho) - p'(1)(\varrho - 1) - p(1)}{\varepsilon^2}$ is uniformly bounded in $L^\infty(0, T; L^1(\Omega))$,

$$\frac{1}{\delta} \int_0^\tau \int_{\Omega_\delta} \frac{p(\varrho) - p'(1)(\varrho - 1) - p(1)}{\varepsilon^2} \Delta_h \Psi dx dt$$

$$\leq c(T) \|\Delta_h \Psi\|_{L^{\frac{8}{5}}_T(L^\infty(\mathbb{R}^2))} \leq c(T) \|\nabla_h \Psi\|_{L^{\frac{8}{5}}_T(W^{2,4}(\mathbb{R}^2))} \leq c(\eta, T)\varepsilon^{\frac{1}{8}}. \tag{6.25}$$

From (6.21) to (6.25) we conclude that

$$\frac{1}{\delta} \int_0^\tau \mathcal{R}_3 dt \leq \frac{1}{2} \int_{\mathbb{R}^2} a^2 |\psi|^2 dx_h |_{t=0}^{\tau} + c(\eta, T)\varepsilon^\alpha, \ \alpha = \min\{\frac{1}{8}, \frac{1}{4\gamma}\} \tag{6.26}$$

6.5 Proof of Theorem 1.6

Using the conservation of energy for acoustic wave system and all estimates in the above three subsections, we find

$$\mathcal{E}_{\varepsilon,\eta}\left(\varrho, \mathbf{u} \mid r, \mathbf{U}\right)(\tau) + \frac{\mu}{\delta} \int_0^\tau \int_{\Omega_\delta} \mathbb{S}\left(\nabla \mathbf{u} - \nabla \mathbf{U}\right) : \left(\nabla \mathbf{u} - \nabla \mathbf{U}\right) dx dt$$

$$\leq c(\eta, T)\varepsilon^\alpha + \int_0^\tau c(t) \mathcal{E}_{\varepsilon,\eta}(t) dt,$$

where $c(t) = \|\nabla_h \mathbf{v}(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \leq c\|\mathbf{v}(t, \cdot)\|_{W^{3,2}(\mathbb{R}^2)}$ according to Sobolev's embedding lemma. By Gronwall's inequality,

$$\mathcal{E}_{\varepsilon,\eta}\left(\varrho, \mathbf{u} \mid r, \mathbf{U}\right)(\tau) + \frac{\mu}{\delta} \int_0^\tau \int_{\Omega_\delta} \mathbb{S}\left(\nabla \mathbf{u} - \nabla \mathbf{U}\right) : \left(\nabla \mathbf{u} - \nabla \mathbf{U}\right) dx dt$$

$$\leq c(\eta, T)\varepsilon^\alpha + c(T)\mathcal{E}_{\varepsilon,\eta}(0), \ \text{ a.e. } \tau \in (0, T), \tag{6.27}$$

where $c(T) = \exp \int_0^T \|\nabla_h \mathbf{v}(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} dt$. Sending $\varepsilon \to 0$ and then $\eta \to 0$ we find

$$\lim_{\eta \to 0} \lim_{\varepsilon \to 0} \mathcal{E}\left(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid r_{\varepsilon,\eta}, \mathbf{U}_{\varepsilon,\eta}\right)(\tau) = 0 \text{ uniformly in } \tau \in (0, T),$$

as well as

$$\lim_{\varepsilon \to 0} \mathcal{E}\left(\varrho_\varepsilon, \mathbf{u}_\varepsilon \mid r_\varepsilon, \mathbf{U}_\varepsilon\right)(\tau) = 0 \text{ uniformly in } \tau \in (0, T),$$

where $r_\varepsilon = 1 + \psi_\varepsilon$, $\mathbf{U}_\varepsilon = (\mathbf{v} + \nabla_h \Psi_\varepsilon, 0)$. We thus conclude the proof of Theorem 1.6 by realizing that $\nabla_h \Psi_{\varepsilon,\eta} \to 0$ in $L^q(0, T; L^p(\mathbb{R}^2))$ as $\varepsilon \to 0$ for any $p >$

$2, q > 4$ according to the Strichartz estimate (3.7). Indeed, for any compact set $K \subset \mathbb{R}^2$,

$$\left\| \overline{\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon} - \mathbf{v} \right\|_{L^2_T(L^2(K))} \leq \left\| \overline{\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon} - \mathbf{U}_{\varepsilon,\eta} \right\|_{L^2_T(L^2(\mathbb{R}^2))} + c(T,K) \left\| \nabla_h \Psi_{\varepsilon,\eta} \right\|_{L^q_T(L^p(K))},$$

which vanishes as $\varepsilon \to 0$ and then $\eta \to 0$. Finally we remark that if one assumes that the initial data $\nabla_h \Psi_0 \in W^{3,2}(\mathbb{R}^2)$, then the regularization procedure can be omitted.

7 Conclusion

We derive as a target system a weak solution of incompressible Navier-Stokes equation and the strong solution of incompressible Euler equation. What remains open is to derive-using the singular limit-the strong solution of incompressible Navier-Stokes in case of ill-prepared data. The case of getting the strong solution of incompressible case for well prepared data can be seen as corollary of "inviscid" case. Another very interesting problem is to prove reduction of dimension from weak solution of compressible 3D barotropic case to weak solution of 2D barotropic case.

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