

On the complexity of finding falsifying assignments for Herbrand disjunctions

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Abstract

Suppose that Φ is a consistent sentence. Then there is no Herbrand proof of $\neg\Phi$, which means that any Herbrand disjunction made from the prenex form of $\neg\Phi$ is falsifiable. We show that the problem of finding such a falsifying assignment is hard in the following sense. For every total polynomial search problem R , there exists a consistent Φ such that finding solutions to R can be reduced to finding a falsifying assignment to an Herbrand disjunction made from $\neg\Phi$. It has been conjectured that there are no complete total polynomial search problems. If this conjecture is true, then for every consistent sentence Φ , there exists a consistency sentence Ψ , such that the search problem associated with Ψ cannot be reduced to the search problem associated with Φ .

1 Introduction

Let $\Phi := \forall x_1 \dots \forall x_k \phi(x_1, \dots, x_k)$ be a universal sentence (where ϕ is an open formula). According to Herbrand's theorem, Φ is inconsistent if and only if, for some terms τ_{ij} , the disjunction $\bigvee_{i=1}^n \neg\phi(\tau_{i1}, \dots, \tau_{ik})$ is a propositional tautology. Thus if Φ is consistent, every conjunction $\bigwedge_{i=1}^n \phi(\tau_{i1}, \dots, \tau_{ik})$ is a satisfiable proposition. In this paper we study the computational problem of finding satisfying assignments for such conjunctions assuming that Φ is consistent. We call this problem the *Herbrand consistency search* for Φ . This problem can be viewed from three different perspectives:

1. We ask how difficult it is to verify that Φ is consistent; more precisely, how difficult it is to verify that a given disjunction is not a Herbrand proof of $\neg\Phi$. This is somewhat similar to the well-known problem about finitistic consistency statements, where we ask how difficult it is to find a proof that there is no proof of $\neg\Phi$ of length n , see [7]. The two problems are, however, of essentially different nature. For one thing, we consider all

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proofs of length n when we talk about finitistic consistency. For another, transforming the usual proofs (Hilbert style, sequent calculus with cuts, etc.) into Herbrand proofs results in nonelementary blowup of size.

2. A model of a consistent sentence can be built on the Herbrand universe (the set of all terms in the language of Φ). To this end we have to decide the truth of all atomic formulas so that the resulting structure is a model of Φ . Herbrand consistency search can be then viewed as the problem of deciding the truth values of atomic formulas in order to obtain a *partial model* of Φ .

3. It is important to fully understand the complexity of special cases of the **NP**-complete problem SAT (satisfiability of CNF formulas). Each consistent universal sentence whose matrix is a CNF gives us a natural class of CNFs in the way described above. For some sentences, we can show that the problem is solvable in polynomial time. For some other sentences, we believe that this is not the case, but it is unlikely that we can prove that they are **NP**-hard problems, because the CNFs are all satisfiable. Instead, we can argue that for strong sentences their Herbrand consistency search problems are not solvable in polynomial time, because these problems capture the complexity of all total polynomial search problems, as we explain below.

A *polynomial search problem* is given by a binary relation $R(x, y)$ decidable in polynomial time and a polynomial bound on the length of y in terms of the length of x . The task is, for a given x , to find y such that the relation holds true and the polynomial bound is satisfied, if there is any such y . A *total polynomial search problem* is a polynomial search problem that has a solution for every x . While there are polynomial search problems that are **NP**-hard, it seems unlikely that we could prove **NP**-hardness of a total polynomial search problem. The additional condition of totality prevents us to use any known techniques for showing **NP**-hardness, which suggests that it may actually be impossible.

There are naturally defined reductions of one total polynomial search problem to another. This enables us to study classes of these problems closed under reductions and a number of important classes have been defined [6]. These classes are useful for classification of specific search problems. In proof complexity such classes are used to characterize certain sentences provable in fragments of Bounded Arithmetic (see, e.g., [4, 9]). The structure of the quasiorder of polynomial reducibility has not been much studied, except for specific classes of problems. One can easily show that for every finite set of total polynomial search problems there is another one to which all are reducible. Since the condition of totality is not syntactical, we are not able to prove that there is a greatest element in this quasiorder, i.e., that there is a complete total polynomial search problem. We conjecture that there is none.

In this paper we prove that every total polynomial search problem is reducible to the Herbrand consistency search for some consistent sentence Φ . This means that Herbrand consistency search problems can have arbitrary high complexity in the hierarchy of total polynomial search problems. We also prove that polynomial reducibility reflects the strength of consistent sentences in the sense that if a universal sentence Ψ logically follows from Φ , then the Herbrand consistency search for Ψ is reducible to Herbrand consistency search for Φ . This is in line of our project to find connections between provability and computational complexity,

see Section 6.4 of [8]. We will also define Herbrand consistency search for general sentences in prenex form, but the relation between provability and reducibility of the corresponding Herbrand consistency search problems is not clear for sentences that are not universal.

The conjecture that there are no complete total polynomial search problems can be partially justified by showing an oracle with respect to which it holds true. Since we have not found this result in the literature, we present it in the last section.

2 Preliminaries

We will consider first order logic without equality, but constants and function symbols will play an important role. Let $\Sigma := \exists x_1, \dots, x_k \sigma(x_1, \dots, x_k)$ be an existential sentence (where x_1, \dots, x_k are all variables in σ). *Herbrand's Theorem* states that Σ is provable (logically valid) if and only if there exist terms τ_{ij} , $i = 1, \dots, n$, for some n , $j = 1, \dots, k$ such that

$$\bigvee_{i=1}^n \sigma(\tau_{i1}, \dots, \tau_{ik})$$

is a propositional tautology (see, e.g., [3, 2]). We will study the dual version of this statement: a universal sentence $\Phi := \forall x_1, \dots, x_k \phi(x_1, \dots, x_k)$ is consistent if and only if for all families of terms τ_{ij} , $i = 1, \dots, n$, $j = 1, \dots, k$,

$$\bigwedge_{i=1}^n \phi(\tau_{i1}, \dots, \tau_{ik})$$

is satisfiable as a propositional formula (i.e., we can assign truth values to the atomic formulas so that the truth value of the conjunction is truth). A general sentence in a prenex form can be transformed into a universal sentence by skolemization, which we denote by

$$Sk(\forall \bar{x} \exists \bar{y} \forall \bar{z} \exists \bar{u} \dots \phi(\bar{x}, \bar{y}, \bar{z}, \bar{u} \dots)) := \forall \bar{x} \forall \bar{z} \dots \phi(\bar{x}, \bar{f}(\bar{x}), \bar{z}, \bar{g}(\bar{x}, \bar{z}) \dots),$$

where we use bars to denote strings of symbols and f, g, \dots are new function symbols. If Φ is $\bigwedge_i \Phi_i$, where the sentences Φ_i are in prenex form, then we define $Sk(\Phi)$ to be $\bigwedge_i Sk(\Phi_i)$ (where each term uses different function symbols). Clearly, $Sk(\Phi) \vdash \Phi$, but the opposite is not true in general. For the sake of simplicity, we will only define Herbrand consistency search for conjunctions of prenex sentences, although Herbrand's theorem has been proved for general sentences.

Definition 1 *Let Φ be a consistent sentence which is a conjunction of sentences in prenex form. Let $\phi(x_1, \dots, x_k)$ be the matrix (the quantifier-free part) of the skolemization of Φ . Then $HCS(\Phi)$, the Herbrand Consistency Search for Φ , is the following total polynomial search problem:*

- given terms τ_{ij} in the language of ϕ , $i = 1, \dots, n$, $j = 1, \dots, k$, find a truth assignment to the atomic subformulas occurring in $\phi(\tau_{i1}, \dots, \tau_{ik})$, for $i = 1, \dots, n$, that makes $\bigwedge_{i=1}^n \phi(\tau_{i1}, \dots, \tau_{ik})$ true.

Example 1. Consider an axiomatization of the theory of dense linear orderings. Using a Skolem function $f(x, y)$, we can present it as a universal theory with the axioms (stated without the universal quantifiers)

$$\begin{aligned} 0 < 1, \\ \neg x < x, \\ x < y \vee x = y \vee y < x, \\ x < y \wedge y < z \rightarrow x < z, \\ x < f(x, y) \wedge f(x, y) < y, \end{aligned}$$

plus the identity and equality axioms. Let $\phi(x, y, z)$ be the conjunction of these axioms. Given terms $\tau_{i,j}$, $i = 1, \dots, m$, $j = 1, 2, 3$, we can easily (certainly in polynomial time) find truth assignments to the atomic formulas $\tau_{i,j} = \tau_{i',j'}$ and $\tau_{i,j} < \tau_{i',j'}$ such that the conjunction $\bigwedge_i \phi(\tau_{i,1}, \tau_{i,2}, \tau_{i,3})$ becomes true. To find such an assignment we need only to find an interpretation of the terms in a finite linear ordering and then to assign the truth values according to this interpretation. To find such an interpretation, we start by ordering the variables of the terms $\tau_{i,j}$ in an arbitrary way. Then we gradually extend the ordering to more complex subterms of the terms $\tau_{i,j}$. Specifically, having an interpretation of terms τ and σ and a non-interpreted term $f(\tau, \sigma)$, we place $f(\tau, \sigma)$ on an arbitrary position strictly between τ and σ .

Example 2. Let Φ be a prenex form sentence axiomatizing a fragment of Peano Arithmetic. Consider a skolemization of Φ . If Φ is sufficiently complex, some Skolem functions may be difficult to compute, or they even may be non-computable. Then finding interpretation in a finite part of the natural numbers may also be difficult. Note, however, that this does not imply that finding a satisfying truth assignment must be difficult. In particular, finding such an assignment is always doable in nondeterministic polynomial time whatever the complexity of the Skolem functions is.

Definition 2 *A total polynomial search problem is defined by a binary relation $R(x, y)$ computable in polynomial time and a polynomial p such that for every x there exists y such that $|y| \leq p(|x|)$ and $R(x, y)$. The task is, for a given x , to find a y satisfying the two conditions above.*

Here we use $|x|$ to denote the length of x , i.e., the number of bits in an encoding of x . In the following definition we will omit polynomial bounds on ys and assume that they are implicit in R and S .

Definition 3 *Let R and S be total polynomial search problems. We say that R is polynomially reducible to S if R can be solved in polynomial time using an oracle that gives solutions to S . We say that R is many-one polynomially reducible to S , if it is polynomially reducible using one query to the oracle for S .*

Clearly, both relations are reflexive and transitive. Note that if $\mathbf{P}=\mathbf{NP}$, then every search problem is reducible to every other one. Hence we can only prove non-reducibility assuming some conjectures in computational complexity.

3 Main result

Theorem 3.1 *For every total polynomial search problem R , there exist a consistent universal sentence Φ such that the problem R is many-one polynomially reducible to $HCS(\Phi)$.*

Proof. Given a total polynomial search problem R , the sentence Φ will express that R is total. This can, certainly, be done in various ways, but it does not automatically guarantee that we can reduce R to $HCS(\Phi)$. Therefore we have to describe the formalization in more detail.

We start with a brief high-level overview of the proof. We will take a Turing machine M that decides in polynomial time the relation R and express that for a given x there exists y and an accepting computation of M on the inputs x and y . Thus the first step is to define terms that will represent an input word x . Then we need to ensure that the bits of x are encoded into the truth values of some atomic formulas. To this end we use an elementary theory of the successor function S and use terms (numerals) $S^i(0)$ as indices of a one dimensional array. Specifically, we use atomic formulas $P(x, S^i(0))$ to determine the bits of x ($P(x, S^i(0))$ false means $x_i = 0$, $P(x, S^i(0))$ true means $x_i = 1$). A computation of M can be represented by a two-dimensional array with entries in a finite alphabet. The elements of the alphabet can be encoded by bit strings of length d for some constant d . So we represent the computation by d ternary relations $Q_k(z, s, t)$. The second part of the input y will be implicitly encoded in the array. Given a term τ representing an input word x , the term $F(\tau)$, where F is a function symbol, will denote the object representing the computation. Thus the bits of the array corresponding to $F(\tau)$ will be defined by the truth values of $Q_k(F(\tau), S^i(0), S^j(0))$. The matrix of Φ will be a conjunction of several formulas which we can view as axioms of a simple theory describing computations of M . One of the axioms says that M accepts, so the implicitly encoded y must be such that $R(x, y)$ holds true. It will not be hard to see that we need only a polynomial number of term instances of the axioms in order to guarantee that the truth values encode a computation on the input word correctly. In fact these term instances can easily be defined from the input word. The implicitly encoded y can also be easily read from the truth values, thus the construction gives a many-one polynomial reduction.

Now we describe the formalization in more detail, but since it is fairly routine, we leave some parts to the reader.

Let a total polynomial search problem be given by a relation R computable in polynomial time. So we assume that for every x there exists a y such that $R(x, y)$ and the length of y is bounded by a polynomial in the length of x . Let M be a (deterministic) Turing machine that in polynomial time decides the relation $R(x, y)$. We will also assume that M has a certain form that will make the formalization easier. Specifically, we will assume the following properties of M .

1. For given $x, y \in \{0, 1\}^*$, M always stops after $p(|x|)$ steps, where p is some polynomial, provided that the input word x is coded appropriately (see below). This means that it reaches one of the two final states, one of which is the accepting state and the other is the rejecting state.

2. The tape of M is infinite in one direction. The squares of the tape will be indexed by $0, 1, 2, \dots$. We will view squares as having d registers indexed $1, \dots, d$; every register contains 0 or 1. The contents of a square encode the symbol on the tape, the presence/non-presence of the head and the state of the machine.
3. Registers 1 and 2 will be used to encode x . The content of registers 1 are the bits of x and registers 2 determine the end of the word x (the first 1 in register 2 is in the first square after the end of x). The input word y will be coded by registers 3 and 4 in the same way. An occurrence of 1 in register 5 marks the position of the head of the machine.
4. Initially all registers with numbers greater than 5 contain zeros. Register 5 contains only one 1 and this is in the square 0.
5. Register 6 will be used to determine that M has stopped and rejected; i.e., if 1 occurs in any of the registers 6, then the machine rejects.
6. The machine starts by looking for the mark that determines the end of x . After that it looks for the mark that determines the end of y . If it does not find it in the given polynomial limit, it will stop and reject. If the mark is all right, the machine computes the relation $R(x, y)$, i.e., it will stop and accept iff the relation holds true.

Our sentence Φ will use relation symbols $=, P(x, t), Q_i(z, s, t)$, for $i = 1, \dots, d$, constants $0, \Lambda$, and function symbols $S(x), f_0(x), f_1(x), \ell(x), F(x)$. The sentence will be a universal closure of formulas that we present in a form of a finite number of axioms.

First we need

1. the axioms of identity and the axiom of equality for S

$$s = t \rightarrow S(s) = S(t).$$

We do not postulate the axioms of equality for other function and relation symbols, since we only need them to derive the inequalities in Lemma 3.2. Note that these axioms can be stated using three variables, say, r, s, t . The symbol S represents the successor function, so we postulate the usual axioms

2. $0 \neq S(t), s \neq t \rightarrow S(s) \neq S(t)$.

We leave the proof of the following easy fact to the reader.

Lemma 3.2 *The propositions $S^i(0) \neq S^j(0)$ for all $i, j \leq n, i \neq j$ are derivable using propositional logic from the term instances of axioms 1. and 2. for all terms of the form $S^k(0), k \leq n$.*

Next we need some axioms in order to be able to write down terms that represent input words x . The intended interpretation of the predicate P is: the i -th bit of x is 0 if $P(x, S^i(0))$ is false, and 1 otherwise. The constant Λ represents the empty word and $\ell(x)$ represents the length of a binary word x . Therefore our first axiom is

3. $\ell(\Lambda) = 0$.

The functions f_0 and f_1 add bits 0 and 1 at the end of the word.

4. $\ell(f_0(x)) = S(\ell(x)) \wedge$
 $\neg P(f_0(x), \ell(x)) \wedge$
 $(s \neq \ell(x) \rightarrow (P(f_0(x), s) \equiv P(x, s))).$

5. $\ell(f_1(x)) = S(\ell(x)) \wedge$
 $P(f_1(x), \ell(x)) \wedge$
 $(s \neq \ell(x) \rightarrow (P(f_1(x), s) \equiv P(x, s))).$

Thus given a word $w = (w_0, \dots, w_{n-1}) \in \{0, 1\}^n$, the term $f_{w_{n-1}} \dots f_{w_1} f_{w_0}(\Lambda)$ represents it in our theory. We need also to show that this fact has a propositional proof using a small number of instances of the axioms.

Lemma 3.3 *Let $\tau = f_{w_{n-1}} \dots f_{w_1} f_{w_0}(\Lambda)$. The propositions*

$$(\neg)^{w_0} P(\tau, 0), (\neg)^{w_1} P(\tau, S(0)), \dots, (\neg)^{w_{n-1}} P(\tau, S^{n-1}(0)),$$

and

$$\ell(\tau) \neq 0, \ell(\tau) \neq S(0), \dots, \ell(\tau) \neq S^{n-1}(0), \ell(\tau) = S^n(0),$$

are derivable using propositional logic from term instances of axioms 1.-5. for terms $S^k(0)$, $k = 0, \dots, n$, and $\Lambda, f_{w_0}(\Lambda), \dots, f_{w_{n-1}} \dots f_{w_1} f_{w_0}(\Lambda)$. (We denote by $(\neg)^0$ the empty symbol, and $(\neg)^1$ stands for \neg .)

Proof. By induction construct such proofs for all subterms of τ . The induction step is done using Lemma 3.2 and axioms 4. and 5. ■

We represent a computation of the machine by a two dimensional array where each entry has d registers, each register containing one bit. The first index is time, the second is a position on the tape. The content of the k -th register in time s and position t is determined by a predicate $Q_k(z, s, t)$. The variable z stands for the entire array. The sentence Φ will express the fact that, for every x , there exists y such that M accepts the input (x, y) . We do not need to mention y explicitly, because it is encoded in the array z . We use skolemization to eliminate the existential quantifier, thus the array will be represented by $F(x)$.

The initial configuration of the machine is formalized by the following axioms.

6. $Q_1(F(x), 0, t) \equiv P(x, t), Q_2(F(x), 0, t) \equiv \ell(x) = t$.

(Clearly, the predicate $P(x, t)$ is dispensable and can be replaced by $Q_1(0, F(x), t)$, but it would complicate the presentation above.) The second input is encoded in the same way using Q_3 and Q_4 , but we do not need any axioms about it.

7. $Q_5(F(x), 0, 0), \neg Q_5(F(x), 0, S(t)),$

8. $\neg Q_i(F(x), 0, t)$, for $i = 6, \dots, d$.

The transition function is formalized by axioms of the form:

9. $Q_i(F(x), S(s), 0) \equiv \rho_i$,
 $Q_i(F(x), S(s), S(t)) \equiv \psi_i$,
for $i = 1, \dots, d$,

where ρ_i and ψ_i are propositions composed from atomic formulas of the form $Q_j(F(x), s, 0)$, $Q_j(F(x), s, S(0))$, respectively, $Q_j(F(x), s, t)$, $Q_j(F(x), s, S(t))$, $Q_j(F(x), s, SS(t))$ for $j = 1, \dots, d$.

Finally, we postulate that the machine never rejects the input:

10. $\neg Q_6(F(x), s, t)$.

Let $\phi(x, r, s, t)$ be the conjunction of the axioms 1.-10., and let Φ be $\forall x \forall r \forall s \forall t \phi(x, r, s, t)$.

To show that Φ satisfies the theorem, we have first to show that Φ is consistent. To this end we take a function γ such that $R(x, \gamma(x))$ is true for all x . We interpret the predicate symbols $P(x, t)$, $Q_i(z, s, t)$, for $i = 1, \dots, d$, constants $0, \Lambda$, and function symbols $S(x)$, $f_0(x)$, $f_1(x)$, $\ell(x)$ as explained above. The function symbol $F(x)$ represents the function that maps a given string x to the array encoding the computation of the machine M on the input $(x, \gamma(x))$. Hence Φ is consistent.

Second, we have to construct a reduction from the search problem to finding truth assignments of the term instances of ϕ . The reduction is defined as follows. Let $w \in \{0, 1\}^n$ be given. Let τ_l denote the term $f_{w_l} \dots f_{w_1} f_{w_0}(\Lambda)$ for $l = 0, \dots, n - 1$. Finding a solution u such that $R(w, u)$ will be reduced to finding a truth assignment to the atomic formulas of

$$\bigwedge_{i,j,k=0}^{p(n)} \bigwedge_{l=0}^{n-1} \phi(F(\tau_l), S^i(0), S^j(0), S^k(0))$$

that makes this formula true. We will denote this formula by Ψ_w . Note that we need the second conjunction to run over all numbers $l = 0, \dots, n - 1$, because we need to derive formulas from Lemmas 3.2 and 3.3, but the instances of the axioms 6.-10. for $x = \tau_l$, $l < n - 1$ will not be used.

It is clear that Ψ_w can be constructed in polynomial time, so we only need to show that from any satisfying assignment A , we can construct some u in polynomial time such that $R(w, u)$. We start by observing that according to Lemma 3.2, $A(P(\tau_{n-1}, S^i(0))) = \top$ iff $w_i = 1$ for $i = 0, \dots, n - 1$. Similarly for the atomic formulas $\ell(\tau_{n-1}) = S^i(0)$ for $i = 0, \dots, n$, so the truth values for these formulas represent w . By axioms 6., w is also correctly represented by the truth values of $Q_1(F(\tau_{n-1}), 0, S^i(0))$ and $Q_2(F(\tau_{n-1}), 0, S^i(0))$. The axioms 7.-9. then ensure that the truth values of $Q_k(F(\tau_{n-1}), S^i(0), S^j(0))$, for $i, j = 0, \dots, p(n)$, $k = 1, \dots, d$ represent a computation of the Turing machine M on w and some u , where u is coded by the truth values of $Q_3(F(\tau_{n-1}), 0, S^i(0))$ and $Q_4(F(\tau_{n-1}), 0, S^i(0))$. Since the machine must

stop within the limit $p(n)$ and the instances of the axiom 10. ensure that it does not reject, the computation must be accepting. Hence the string u is such that $R(w, u)$. We just note that u can easily be constructed from the truth assignment A . ■

4 Reductions among HCS problems

In order to show connection between provability of $\Phi \rightarrow \Psi$ and polynomial reducibility of $HCS(\Psi)$ to $HCS(\Phi)$, we prove that provability implies polynomial reducibility if Ψ is universal.

Proposition 4.1 *Let Φ be a consistent sentence in a prenex form. Let Ψ be a universal sentence such that $\Phi \vdash \Psi$. Then $HCS(\Psi)$ is polynomially reducible to $HCS(\Phi)$.*

Proof. Note that $Sk(\Phi) \vdash \Phi$ and $HCS(Sk(\Phi))$ is the same as $HCS(\Phi)$. Hence we can w.l.o.g. assume that Φ is universal.

Let Φ and Ψ be $\forall x_1 \dots x_k \phi(x_1, \dots, x_k)$ and $\forall y_1 \dots y_l \psi(y_1, \dots, y_l)$ and assume that $\Phi \vdash \Psi$. Then we have

$$\vdash \forall y_1 \dots y_l \exists x_1 \dots x_k (\phi(x_1, \dots, x_k) \rightarrow \psi(y_1, \dots, y_l)).$$

The herbrandization of this sentence is

$$\vdash \exists x_1 \dots x_k (\phi(x_1, \dots, x_k) \rightarrow \psi(c_1, \dots, c_l)),$$

where c_1, \dots, c_l are new constants. According to Herbrand's theorem, there exist terms τ_{ij} such that

$$\bigvee_i (\phi(\tau_{i1}, \dots, \tau_{ik}) \rightarrow \psi(c_1, \dots, c_l)) \quad (1)$$

is a propositional tautology.

Let $\sigma_{p1}, \dots, \sigma_{pl}$, $p = 1, \dots, n$, be terms in the language of ψ . We substitute these terms into (1) for c_1, \dots, c_l . The resulting formulas propositionally imply the following formula

$$\bigwedge_p \bigwedge_i \phi(\tau_{pi1}^*, \dots, \tau_{pik}^*) \rightarrow \bigwedge_p \psi(\sigma_{p1}, \dots, \sigma_{pl}),$$

where $\tau_{pij}^* := \tau_{ij}[\sigma_{p1}/c_1, \dots, \sigma_{pl}/c_l]$. Thus in order to satisfy $\bigwedge_p \psi(\sigma_{p1}, \dots, \sigma_{pl})$, it suffices to satisfy $\bigwedge_p \bigwedge_i \phi(\tau_{pi1}^*, \dots, \tau_{pik}^*)$. In general, the latter formula is not an instance of $HCS(\Phi)$ because Ψ may use other function symbols. However, note that the role of terms is only to determine which atomic formulas are same and which are different. Hence to get an instance of $HCS(\Phi)$ that is essentially the same propositional formula, it suffices to replace the maximal terms that are not in the language of Φ by variables (the same variables for the same terms, of course). ■

We will prove a similar result for existential sentences. Note that $HCS(\exists y_1, \dots, y_m \alpha(y_1, \dots, y_m))$ is trivial, because the skolemization of the sentence does not contain any variables, hence it is a finite problem. Thus one has to state the result in a slightly different way.

Lemma 4.2 *Let Φ be a consistent universal sentence, let $\alpha(y_1, \dots, y_m)$ be an open formula with m free variables and let c_1, \dots, c_m be constants not occurring in Φ and α . Then $HCS(\Phi \wedge \forall \bar{y}(\alpha(\bar{y}) \rightarrow \alpha(\bar{c})))$ is polynomially reducible to $HCS(\Phi)$.*

Proof. Let Φ be $\forall x_1, \dots, x_n \phi(x_1, \dots, x_k)$. Let an instance of $HCS(\Phi \wedge \forall \bar{y}(\alpha(\bar{y}) \rightarrow \alpha(\bar{c})))$ be given; i.e., we want to find a satisfying assignment for

$$\bigwedge_{i=1}^n (\phi(\tau_{i1}, \dots, \tau_{ik}) \wedge (\alpha(\sigma_{i1}, \dots, \sigma_{im}) \rightarrow \alpha(c_1, \dots, c_m))) \quad (2)$$

for given terms τ_{ij} , σ_{il} . We suppose that we have an oracle for $HCS(\Phi)$. Denote by $F := \bigwedge_{i=1}^n \phi(\tau_{i1}, \dots, \tau_{ik})$. Let F_i denote F where we substitute $c_1 \mapsto \sigma_{i1}, \dots, c_m \mapsto \sigma_{im}$ for $i = 1, \dots, n$. Now we apply our oracle to $F \wedge \bigwedge_{i=1}^n F_i$. The terms in this formula may contain constants c_i , which are not in the language of Φ , but we can interpret them as variables to satisfy the formal definition of $HCS(\Phi)$. Let A be a truth assignment for the atomic formulas of $F \wedge \bigwedge_{i=1}^n F_i$ that makes the formula true. Extend A to an arbitrary assignment that gives truth values also to those atomic formulas of $\alpha(c_1, \dots, c_m)$ and $\alpha(\sigma_{i1}, \dots, \sigma_{im})$, $i = 1, \dots, n$, for which A is not defined (e.g., let them be all false). Now we consider two cases.

1. The assignment A' satisfies the formula (2). Then we are done.
2. Assume it does not. Then, for some i , it satisfies $\alpha(\sigma_{i1}, \dots, \sigma_{im})$. We define a truth assignment A'' for the formula (2) using the part of A' that assigns values to F_i and $\sigma_{i1}, \dots, \sigma_{im}$. Specifically, given $\beta(c_1, \dots, c_m)$, an atomic subformula of (2), or an atomic subformula $\alpha(c_1, \dots, c_m)$, we assign to it the values that A' gives to $\beta(\sigma_{i1}, \dots, \sigma_{im})$. Thus A'' satisfies $\bigwedge_{i=1}^n \phi(\tau_{i1}, \dots, \tau_{ik})$, because A' satisfies F_i , and it also satisfies $\bigwedge_{i=1}^n (\alpha(\sigma_{i1}, \dots, \sigma_{im}) \rightarrow \alpha(c_1, \dots, c_m))$, because A' satisfies $\alpha(\sigma_{i1}, \dots, \sigma_{im})$. Thus A'' satisfies (2). ■

Proposition 4.3 *Let Φ be a consistent sentences in a prenex form $\forall x_1, \dots, x_k \phi(x_1, \dots, x_k)$ and let α be an open formula with m variables. Suppose that $\Phi \vdash \exists y_1, \dots, y_m \alpha(y_1, \dots, y_m)$. Then $HCS(\exists \bar{y} \forall \bar{x} (\phi(\bar{x}) \wedge \alpha(\bar{y})))$ is polynomially reducible to $HCS(\Phi)$.*

Proof. The skolemization of the sentence $\exists \bar{y} \forall \bar{x} (\phi(\bar{x}) \wedge \alpha(\bar{y}))$ is the universal sentence

$$\Psi := \forall x_1, \dots, x_k (\phi(x_1, \dots, x_k) \wedge \alpha(c_1, \dots, c_m))$$

where c_1, \dots, c_m are new constants. This sentence is provable from $\Phi \wedge \forall \bar{y} (\alpha(\bar{y}) \rightarrow \alpha(\bar{c}))$, because Φ proves $\exists \bar{y} \alpha(\bar{y})$. Thus, according to Proposition 4.1, $HCS(\Psi)$ is polynomially reducible to $HCS(\Phi \wedge \forall \bar{y} (\alpha(\bar{y}) \rightarrow \alpha(\bar{c})))$. This problem in turn is reducible to Φ by Lemma 4.2. The polynomial reducibility of $HCS(\Psi)$, hence also of $HCS(\exists \bar{y} \forall \bar{x} (\phi(\bar{x}) \wedge \alpha(\bar{y})))$, follows by transitivity of reducibility. ■

5 Provably total search problems in T and $HCS(T)$

Since Herbrand consistency search is defined for sentences, we will only consider finitely axiomatized theories. If T contains a sufficiently strong fragment of arithmetic, or set theory, and T is sound, we can formalize polynomial time computations in T . Then we can associate with T the class of all search problems that are provably total in T . These are problems that can be defined by a formula ρ such that $T \vdash \forall x \exists y \rho(x, y)$. In order to avoid trivialization, we have to restrict the formulas ρ to a class of formulas that define polynomial time relations in a natural way. Some theories have symbols for every polynomial time computable relation, e.g., Cook's PV [5]. We can also use formulas that define **NP** relations, e.g., Buss's Σ_1^b in bounded arithmetic T_2 [1]. Then the problem of characterizing provably total search problems is, essentially, equivalent to the problem of characterizing provable sentences that are universal closures of Σ_1^b formulas. Since $HCS(T)$ is a polynomial search problem associated with T , a natural question arises, whether or not $HCS(T)$ is in the class of polynomial search problems provably total in T . (Here we assume that T is axiomatized by sentences in prenex form.) We state this question in a slightly more general way.

Problem 1 *Let T be a finitely axiomatized theory, sufficiently strong to be able to formalize polynomial time computations. Is there a polynomial search problem R that is provably total in T and such that $HCS(T)$ is polynomially reducible to R ?*

We note that if T is sufficiently strong, then T does not prove the totality for the natural formalization of $HCS(T)$. Indeed, if T proves Herbrand's theorem, then T can prove that natural provability (in Hilbert-style calculi, sequent calculi with cuts, etc.) is equivalent to provability in the sense of Herbrand. Hence the sentence expressing the totality of $HCS(T)$ is equivalent to the formal consistency of T . Thus by Gödel's incompleteness theorem, it is not provable.

A natural approach to solving the problem positively is to try to express $HCS(T)$ in the following way:

$$\rho(x, y) \vee \sigma(z),$$

where ρ is the natural formalization of $HCS(T)$ and $\sigma(z)$ expresses that z is a proof of contradiction from T . The task of this search problem is either to find a satisfying assignment for term instances of the matrix ϕ of T as required by $HCS(T)$, or a proof of contradiction. If T is consistent, then this formula is equivalent to $\rho(x, y)$, hence defines $HCS(T)$. If, moreover, T is sufficiently strong, then it does prove $\forall x \exists y, z (\rho(x, y) \vee \sigma(z))$, but this does not suffice. We need y and z to be polynomially bounded:

$$\forall x \exists y, z (|y|, |z| \leq p(|x|) \wedge (\rho(x, y) \vee \sigma(z))),$$

for some polynomial p . The problem is only to bound $|z|$, since $|y|$ is polynomially bounded according to the definition of search problems. Thus we need (to be able to prove from T) that if for some x , $\rho(x, y)$ is unsatisfiable, then there exists z , a proof of contradiction from T , of at most polynomial length. If $\rho(x, y)$ is unsatisfiable, we know how to construct a

contradiction—we have an unsatisfiable propositional formula, hence we can derive a contradiction in propositional calculus. However, we do not know if such a proof can have polynomial length. Thus we do not see how to use this approach to solve the problem and we tend to conjecture that the answer is negative.

6 A relativization

We conjecture that there is no complete total polynomial search problem. In order to support this conjecture, we will construct an oracle relative to which there is no complete polynomial search problem. We will only prove the proposition for many-one reductions, but the same argument can surely be used for general reductions.

Proposition 6.1 *There exists an oracle R such that relative to R , there is no complete total polynomial search problem with respect to many-one polynomial reductions.*

Proof. We start by observing that the condition that R has a many-one polynomial reduction to S can be equivalently defined as follows: there exist polynomial time computable functions $f(x)$ and $g(x, y)$ such that for all x and y

$$S(f(x), y) \rightarrow R(x, g(x, y))$$

holds true.

The oracle that we construct will be represented by a ternary relation $R(p, x, y)$ on binary strings. We will view p as a parameter that specifies a binary relation $R_p(x, y)$ that may be a total polynomial search problem. We will construct R so that the condition

$$R_p(x, y) \rightarrow |y| \leq |x|$$

is satisfied for all x and y . Let ρ and f, g be definitions of a binary relation and two functions by means of polynomial time oracle Turing machines. Given an oracle R , we denote by ρ_R and f_R, g_R the corresponding relation and functions. We will assume that the conditions $\rho(x, y) \rightarrow |y| \leq |x|$ and $|g(x, y)| \leq |x|$ are ensured by the definition of ρ . We need to construct R so that the following holds true for every ρ :

1. either ρ_R is not total, i.e.,

$$\exists x \forall y \neg \rho_R(x, y), \tag{3}$$

2. or for some p , R_p is total, but not reducible to ρ_R , i.e., for every f, g ,

$$\forall x \exists y R_p(x, y) \quad \wedge \tag{4}$$

$$\exists x \exists y (\rho_R(f_R(x), y) \wedge \neg R_p(x, g_R(x, y))). \tag{5}$$

Our procedure will have two loops—outer and inner. In the outer loop we go over all definitions ρ ; in the inner one we go over all pairs of definitions of f, g . In the process we will define R gradually for more and more triples p, x, y . At each stage R is defined only for a finite number of parameters p . At the beginning of the i th outer loop we take p_i such that no value of $R_{p_i}(x, y)$ has been fixed so far and gradually define $R_{p_i}(x, y)$. At each stage of this loop R_{p_i} will be defined only for a finite number of pairs x, y .

The outer loop serves us to diagonalize over definitions ρ , which means that at the end of round i the conditions 1. and 2. above will be satisfied for the i th ρ . The partial definition of R will be denoted by R^i . Similarly, in the inner loop we diagonalize over functions f, g . Let R^{ij} denote the j th step of the inner loop inside of the loop i . Then we will get that either 1. holds true and the loop stops, or 2. holds true for the j th pair f, g .

Suppose we are in the outer loop i . At the beginning of each inner loop j we assume the following properties of $R^{i(j-1)}$, the oracle so far defined. For every x for which some value of $R_{p_i}(x, y)$ has been fixed, there exists some y , such that $R_{p_i}(x, y')$ (and $|y'| \leq |x|$). Let n_{ij} be a sufficiently large number and let x_{ij} , $|x_{ij}| = n_{ij}$ be a string such that the following conditions are satisfied.

- $R_{p_i}(x_{ij}, y)$ has not been fixed for any y and,
- for every y' , $|y'| \leq |f(x_{ij})|$, the Turing machines of ρ, f and g cannot query all strings y of length n_{ij} when used on the inputs $x_{ij}, y', f(x_{ij})$ (because of the polynomial bounds on the computations of ρ, f and g).

(The string x_{ij} can be the string of n_{ij} zeros.) First we extend the oracle so that for every x , $|x| < n_{ij}$, there is some y , $|y| \leq |x|$, such that (x, y) is in R_p . This is possible, because we assume that for the strings x used in previous stages this has already been ensured. We now consider two cases.

Case 1: The currently defined oracle can be extended so that condition (5) is satisfied for x_{ij} and some y , ($|y| \leq |x_{ij}|$). In this case we fix the minimum number of values of R that are needed to ensure this condition. Then it is still possible to add pairs (x, y) to R_p to ensure $\exists y R_p(x, y)$ for all strings x so far used.

Case 2: The opposite is true. This means that for every extension of the so far specified oracle R , and every y , $|y| \leq |x_{ij}|$, the implication

$$\rho_R(f_R(x_{ij}), y) \rightarrow R_{p_i}(x_{ij}, g_R(x_{ij}, y))$$

is satisfied. In particular, the implication will be satisfied if we fix R so that for all z , $|z| \leq |x_{ij}|$, $\neg R_{p_i}(x_{ij}, z)$. It follows that $\neg \rho_R(f_R(x_{ij}), y)$ for all y , $|y| \leq |f_R(x_{ij})|$. Hence (3) holds true for all further extension of so far defined R . ■

7 Conclusions

We still do not fully understand the relation between provability and polynomial reducibility of the corresponding Herbrand consistency search problems. The most important problem

is:

Problem 2 *Let Φ and Ψ be consistent sentences in prenex forms. Suppose that $\Phi \vdash \Psi$. Is then $HCS(\Psi)$ polynomially reducible to $HCS(\Phi)$?*

We have only been able to solve the problem in two special cases: for universal sentences Ψ and for existential sentences Ψ . If the answer is negative, then the concept of Herbrand consistency is not well-behaved. In such a case it would be better to compare the provability of $\Phi \rightarrow \Psi$ with the reducibility of $HCS(\Psi)$ to $HCS(\alpha)$ for all prenex sentences α derivable from Φ .

Another interesting open problem is whether or not polynomial reducibility of $HCS(\Psi)$ to $HCS(\Phi)$ implies $\Phi \vdash \Psi$ at least in some special cases.

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References

- [1] S. R. Buss: Bounded Arithmetic. Bibliopolis, Naples, 1986.
- [2] S. R. Buss. On Herbrand's Theorem. In Logic and Computational Complexity, Lecture Notes in Computer Science 960, 1995, Springer-Verlag, pp. 195-209.
- [3] S. R. Buss: An Introduction to Proof Theory, in Handbook of Proof Theory, edited by S. R. Buss. Elsevier, Amsterdam, 1998, pp. 1-78.
- [4] S.R. Buss and J. Krajíček. An application of Boolean complexity to separation problems in bounded arithmetic, Proc. of the London Math. Soc. 69(3):1-21, 1994.
- [5] S. A. Cook: Feasibly Constructive Proofs and the Propositional Calculus. Proc. 7th Annual Symp. on Theory of Computing 1975, pp. 83-97
- [6] D. S. Johnson, C. Papadimitriou, and M. Yannakakis: How easy is local search? Journal of Computer System Science, 37(1): 79-100, 1988
- [7] P. Pudlák: On the length of proofs of finitistic consistency statements in first order theories. In: Logic Colloquium 84, North Holland P.C., 1986, pp. 165-196.
- [8] P. Pudlák: Logical Foundations of Mathematics and Computational Complexity. Springer, 2013.
- [9] A. Skelley and N. Thapen: The provably total search problems of bounded arithmetic. Proceedings of the London Mathematical Society, Vol 103(1):106-138, 2011.