On the Role of Reference Maps in *hp*-FEM

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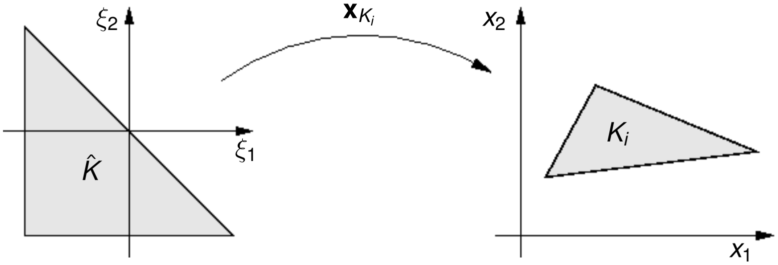
PANM 13, Prague, Czech Republic, May 28 - 31, 2006

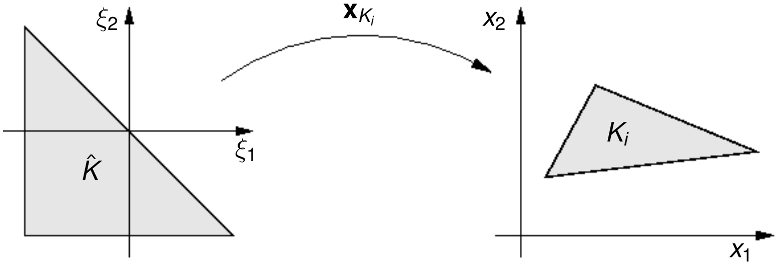
Outline

Affine concept: Reference domain & reference maps



- Define shape functions on $\hat{K}$

Affine concept: Reference domain & reference maps



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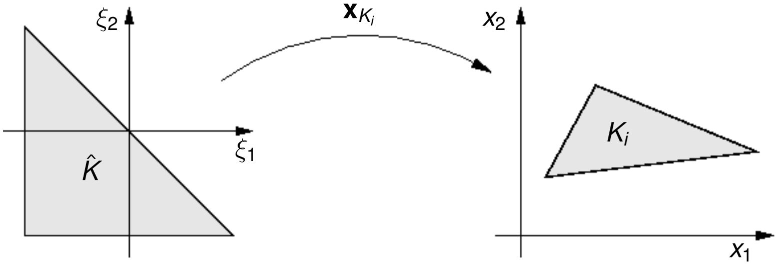
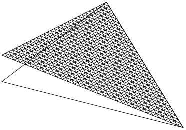
- Define connectivity data

Affine concept: Reference domain & reference maps



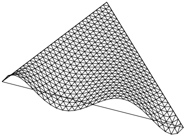
- Define shape functions on $\hat{K}$
- Define connectivity data
- Move weak formulation from K_i to $\hat{K}$

Hierarchic shape functions



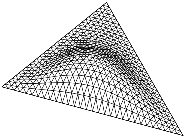
- Vertex functions

Hierarchic shape functions



- Vertex functions
- Edge functions

Hierarchic shape functions



- Vertex functions
- Edge functions
- Bubble functions

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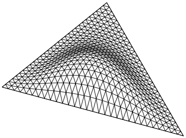

- Vertex functions
- Edge functions
- Bubble functions

Number of shape functions

- Vertex: 3
- Edge: $(p-1)$ on each edge
- Bubble: $(p-1)(p-2)/2$

Bubble functions

Monomial-based (Babuška et al, mid-1970s)

$$\varphi^b_{n_1,n_2} = \lambda_1 (\lambda_2)^{n_1} (\lambda_3)^{n_2}, \quad 1 \le n_1, n_2, \quad n_1 + n_2 \le p - 1$$

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Lobatto-based (Ainsworth, around 2000)

$$\varphi^b_{n_1,n_2,t} = \lambda_1\lambda_2\lambda_3\phi_{n_1-1}(\lambda_3 - \lambda_2)\phi_{n_2-1}(\lambda_1 - \lambda_3)$$

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Jacobi-based (Beuchler, 2005)

L-shape domain problem

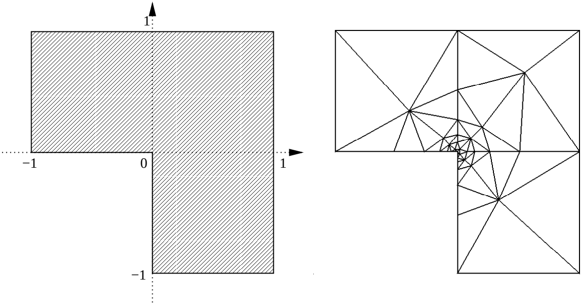


Figure: The L-shape domain and its partition.

L-shape domain problem

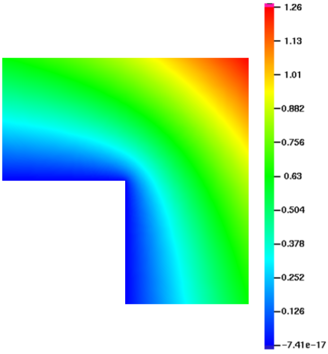
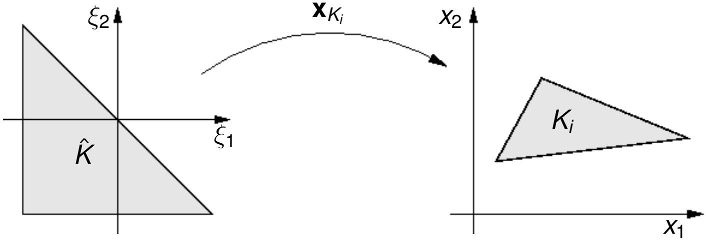


Figure: Exact solution *u* (left) and the norm of its gradient (right).

Problem #1



Several ways to map the central vertex of $\hat{K}$:

- vertex w. *lowest index ("random")* in K_i
- vertex w. *minimum angle* in K_i
- vertex w. *medium angle* of K_i
- vertex w. *maximum angle* of K_i

Effect on Lobatto shape functions

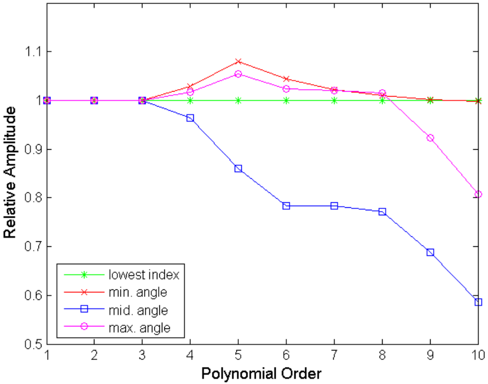
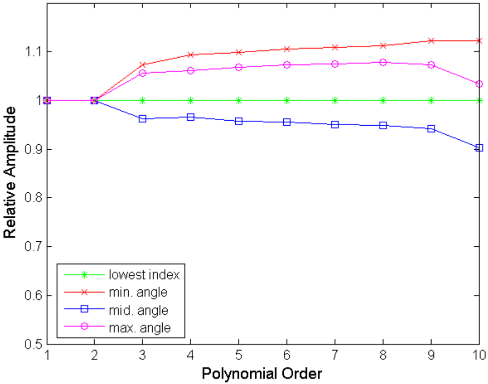


Figure: Condition number of stiffness matrix, $p = 1, 2, \ldots, 10$.

Effect of Jacobi shape functions



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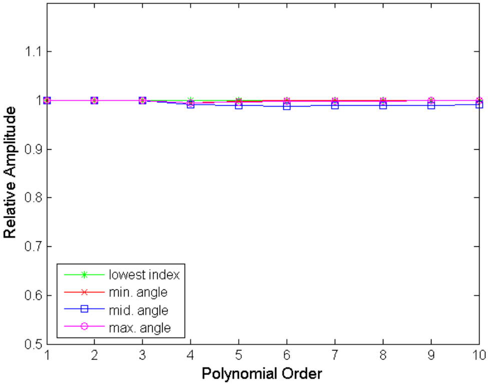
Effect on generalized eigenfunctions



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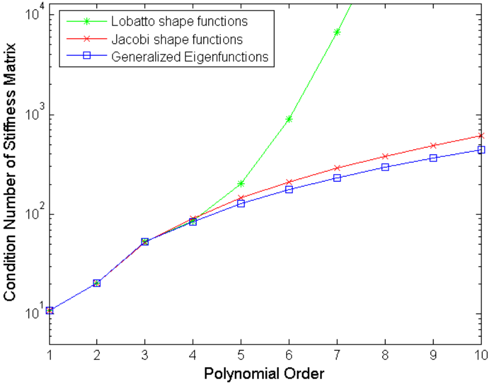
Conditioning comparison



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- $\mathcal{B}_p = \{g_1, g_2, \ldots, g_{(p-1)(p-2)/2}\}$ – arbitrary basis of W

$$\psi_k = \sum_{j=1}^{(p-1)(p-2)/2} y_{jk} g_j$$

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- $SY = MY\Lambda$
- Solution: LAPACK, Matlab

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- Symmetric linear elliptic problem → inner product

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- Transformation to reference domain:

$$\int_{K_i} \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} = \int_{\hat{K}} |J_{K_i}| \, \nabla \hat{u} \cdot \underbrace{\left(\frac{D\mathbf{x}_{K_i}}{D\xi} \right)^{-1} \left(\frac{D\mathbf{x}_{K_i}}{D\xi} \right)^{-T}}_{\text{symmetric positive definite}} \nabla \hat{v} \, \mathrm{d}\boldsymbol{\xi}$$

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- Energetic inner product cannot be fully exploited!

Abandon the affine concept?

- Orthogonalize bubbles on each K_i under

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- Excellent parallel preconditioner
- Note: ON basis on K_i not unique

How to choose ON bubbles on elements?

Generalized eigenfunctions in elements

- BB block diagonal in both stiffness and mass matrices

Mass matrix

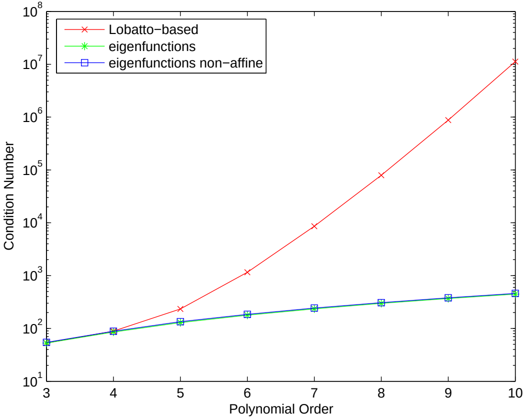
VV	VE	VB
VE	EE	EB
VB	EB	D

Stiffness matrix

VV	VE	VB
VE	EE	BE
VB	EB	I

- Needs further study

Affine vs. Non-Affine (Laplace operator)



Curl-Curl operator

- Maxwell's equations

$$\begin{aligned} \nabla \times \left(\mu_r^{-1} \nabla \times \boldsymbol{E}\right) - \kappa^2 \epsilon_r \boldsymbol{E} = \boldsymbol{F} \quad & \text{in } \Omega, \\ \boldsymbol{E} \cdot \tau = 0 \quad & \text{on } \Gamma_P, \\ \mu_r^{-1} \nabla \times \boldsymbol{E} - i\kappa\lambda \boldsymbol{E} \cdot \tau = \boldsymbol{g} \cdot \tau \quad & \text{on } \Gamma_I. \end{aligned}$$

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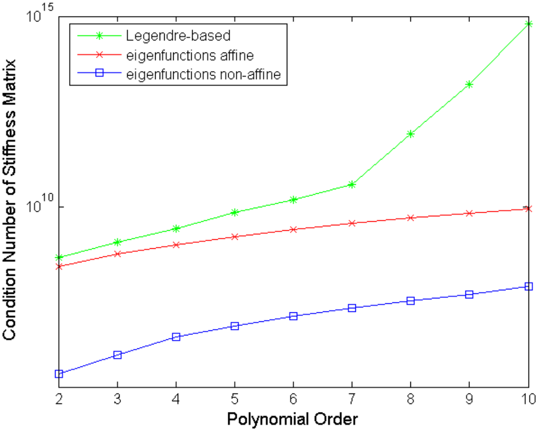
- Use generalized eigenfunctions:
  - outstanding conditioning properties
  - minimal dependence on reference maps

Affine vs. Non-Affine (Maxwell's equations)



Conclusion and Outlook

- Choice of ref. maps influences discrete problem
  - $\Rightarrow$ use generalized eigenfunctions
- Ref. maps incompatible with energetic inner product
  - $\Rightarrow$ leave affine concept
  - $\Rightarrow$ may not be needed for elliptic problems
  - $\Rightarrow$ more serious for Maxwell's equations

Outlook

- Stokes, linear convection-diffusion, Navier-Stokes, etc.
- goal: monolithic *hp*-FEM for coupled problems

The End

Thank You!