A computational comparison of methods diminishing spurious oscillations in finite element solutions of convection–diffusion equations

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joint work with

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$$-\varepsilon\,\Delta u + \mathbf{b}\cdot\nabla u = f \quad \text{in } \Omega,$$

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u_{bh} ... function defined on Ω approximating u_b on Γ^D

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$$u_h \in u_{bh} + V_h\,,$$

$$\varepsilon\,(\nabla_h u_h, \nabla_h v_h) + (\mathbf{b}\cdot\nabla_h u_h, v_h) = (f, v_h) + (g, v_h)_{\Gamma^N} \quad \forall\, v_h \in V_h$$

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$$R_h(u) = -\varepsilon\,\Delta_h u + \mathbf{b}\cdot\nabla_h u - f$$

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Christie *et al*. (1976), Brooks and Hughes (1982), Mizukami (1985), Hughes *et al*. (1986), Tezduyar and Park (1986), Johnson (1990, 1992), Codina *et al*. (1992), Franca *et al*. (1992), Hughes (1995), Stynes and Tobiska (1995), Almeida and Silva (1997), Brezzi *et al*. (1997, 1998), Codina (1998), Hughes *et al*. (1998), Oñate (1998), Ramage (1998,1999), Fischer *et al*. (1999), Shih and Elman (1999), Brezzi *et al*. (2000), Tezduyar and Osawa (2000), Harari *et al*. (2001), Elman and Ramage (2002), Akin and Tezduyar (2004), Galeão *et al*. (2004)

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optimal choice of τ in general not known

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– published results do not allow to draw a clear conclusion concerning their advantages and drawbacks

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Tezduyar, Park, CMAME (1986)

Galeão, do Carmo, CMAME (1988)

Johnson, CMAME (1990)

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Almeida, Silva (1997)

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Johnson, Schatz, Wahlbin, Math. Comput. (1987)

Codina, CMAME (1993)

Knopp, Lube, Rapin, CMAME (2002)

Burman, Ern, CMAME (2002)

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$$\widetilde{\varepsilon} = \frac{1}{2}\max\left\{0, C - \frac{2\,\varepsilon\,|\nabla_h u_h|}{|R_h(u_h)|\,\mathrm{diam}(K)}\right\}\mathrm{diam}(K)\,\frac{|R_h(u_h)|}{|\nabla_h u_h|}$$

modified method of Burman, Ern (2002)

$$\widetilde{\varepsilon} = \frac{\tau\,|\mathbf{b}|^2\,|R_h(u_h)|}{|\mathbf{b}|\,|\nabla_h u_h| + |R_h(u_h)|}$$

Edge stabilization methods

$$\sum_{K \in \mathscr{T}_h} \int_{\partial K} \Psi_K(u_h)\, \mathrm{sign}(\mathbf{t}_{\partial K} \cdot \nabla(u_h|_K))\, \mathbf{t}_{\partial K} \cdot \nabla(v_h|_K)\, \mathrm{d}\sigma$$

Edge stabilization methods

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Burman, Hansbo, CMAME (2004)

Burman, Ern, Math. Comput. (2005)

Edge stabilization methods

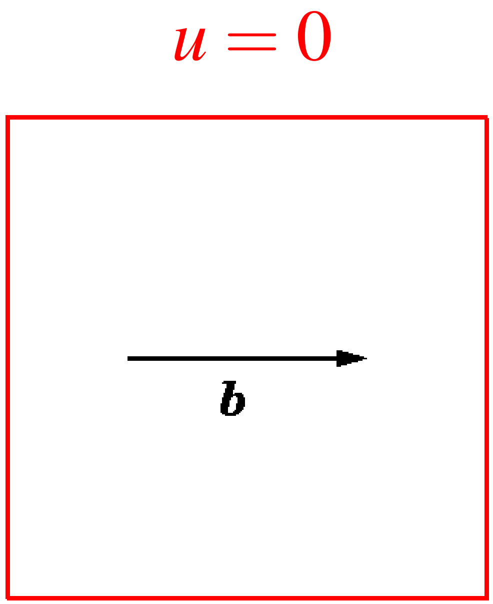
$$\sum_{K \in \mathscr{T}_h} \int_{\partial K} \Psi_K(u_h)\, \mathrm{sign}(\mathbf{t}_{\partial K} \cdot \nabla(u_h|_K))\, \mathbf{t}_{\partial K} \cdot \nabla(v_h|_K)\, \mathrm{d}\sigma$$

Burman, Hansbo, CMAME (2004)

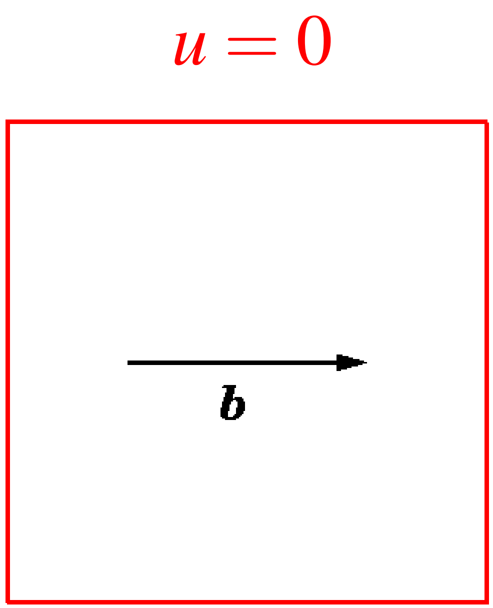
Burman, Ern, Math. Comput. (2005)

discrete maximum principle for P_1^c finite elements

Example 1 (convection with a source term)



Example 1 (convection with a source term)

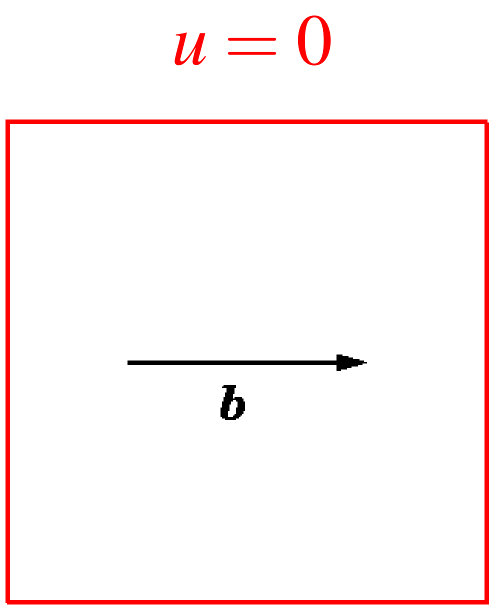


$$\varepsilon = 10^{-8}$$

$$|\mathbf{b}| = 1$$

$$f = 1$$

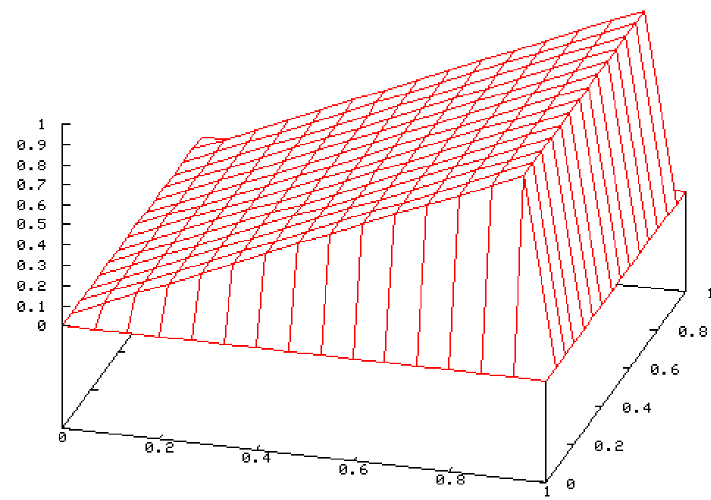
Example 1 (convection with a source term)



$$\varepsilon = 10^{-8}$$

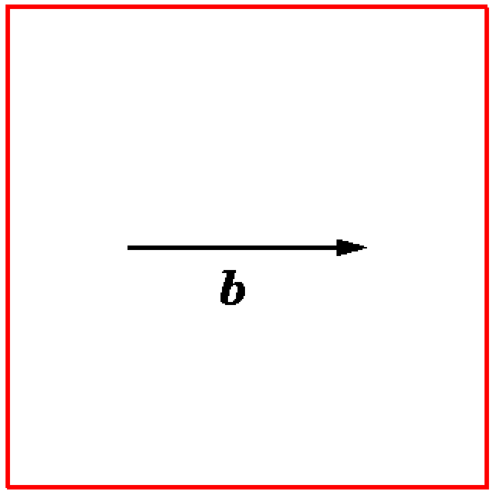
$$|\mathbf{b}| = 1$$

$$f = 1$$



Example 1 (convection with a source term)

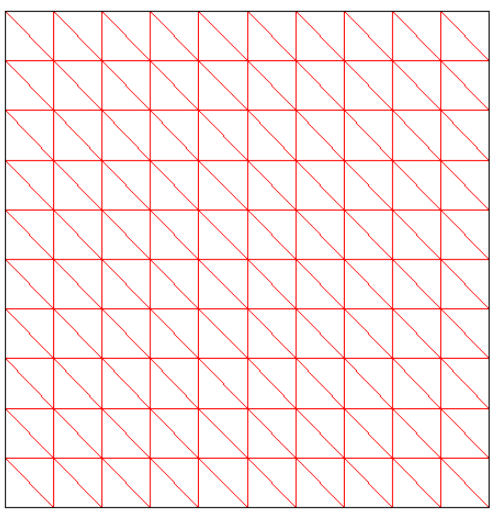
$u = 0$



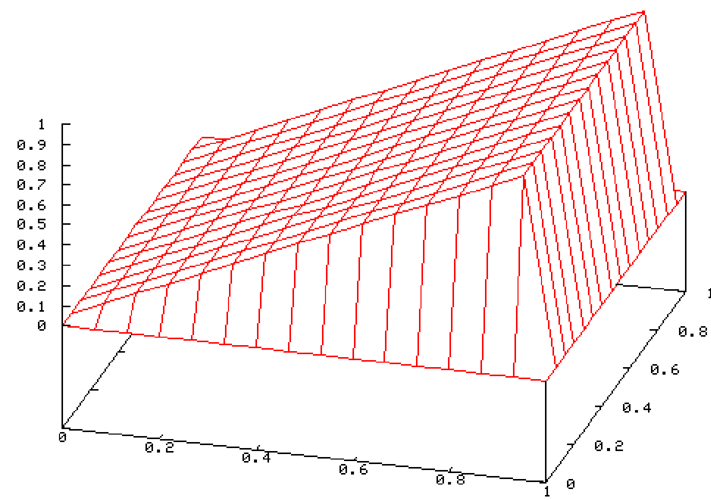
$$\varepsilon = 10^{-8}$$

$$|\mathbf{b}| = 1$$

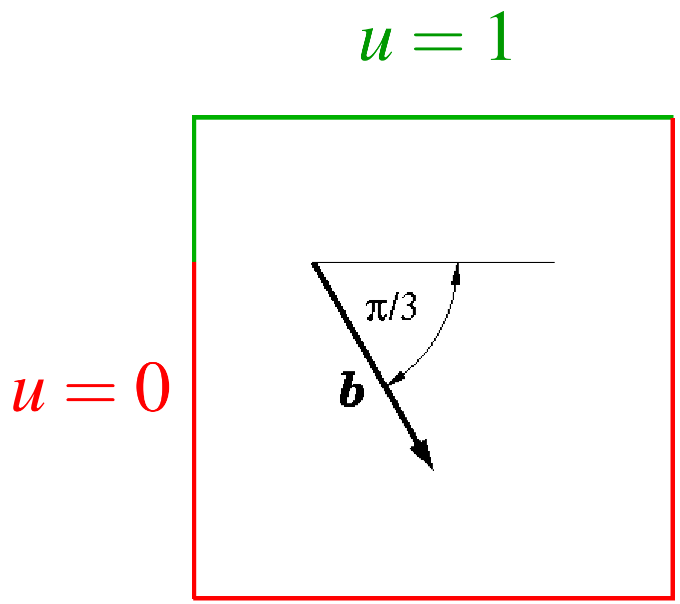
$$f = 1$$



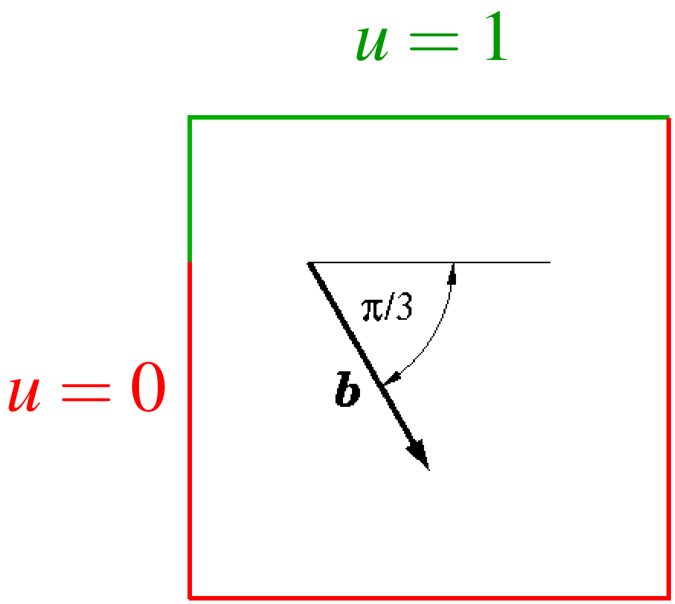
65×65 points



Example 2 (convection skew to the mesh)



Example 2 (convection skew to the mesh)

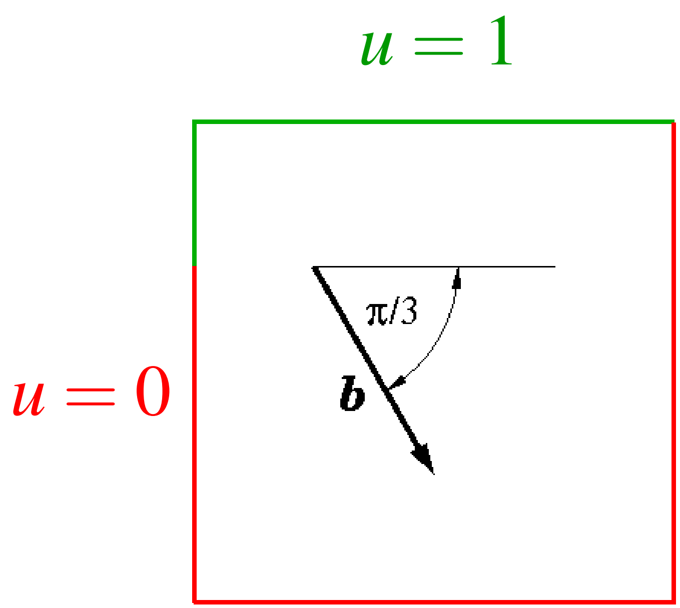


$\varepsilon = 10^{-8}$

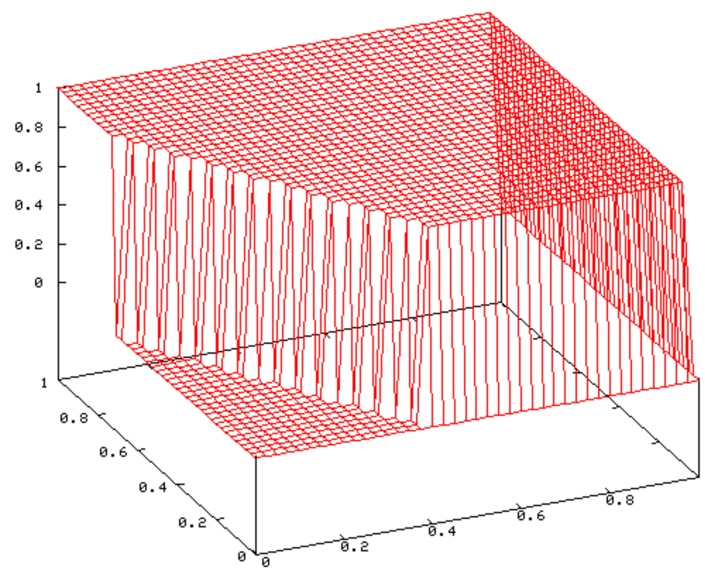
$|\mathbf{b}| = 1$

$f = 0$

Example 2 (convection skew to the mesh)



$$\varepsilon = 10^{-8}$$
$$|\mathbf{b}| = 1$$
$$f = 0$$



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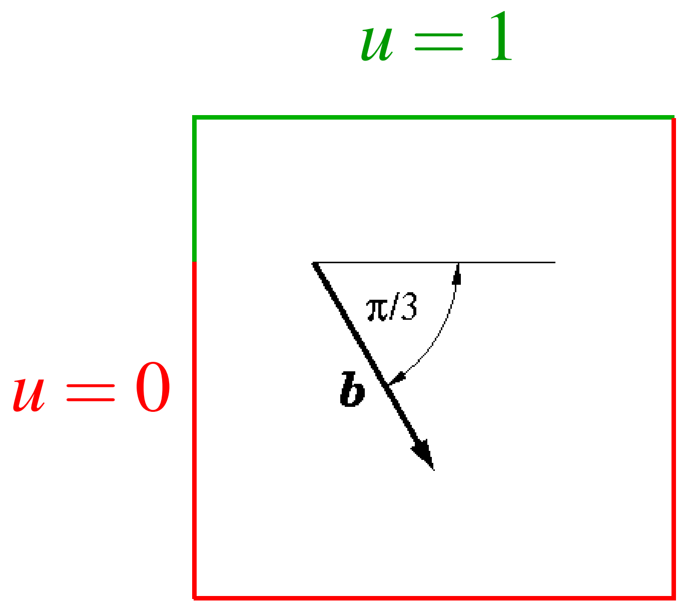
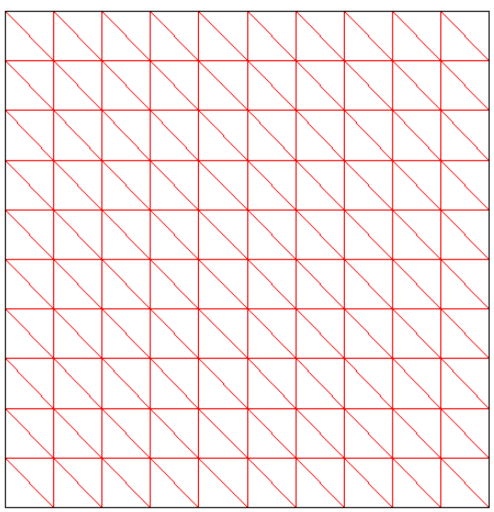
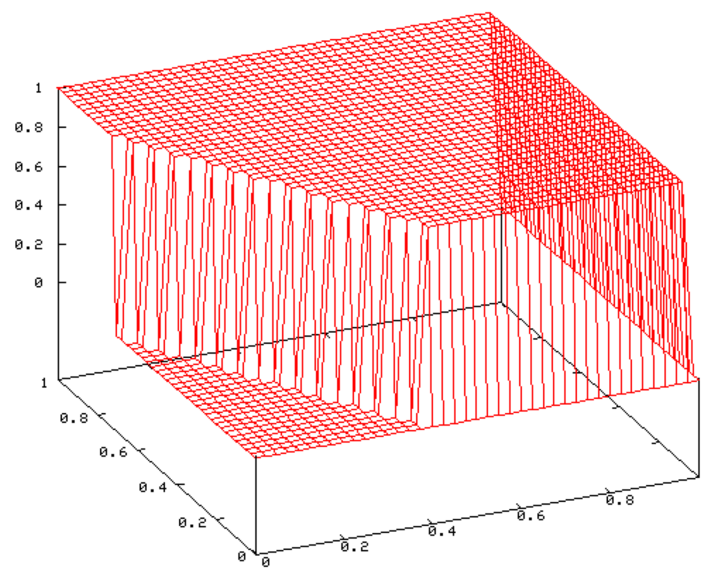


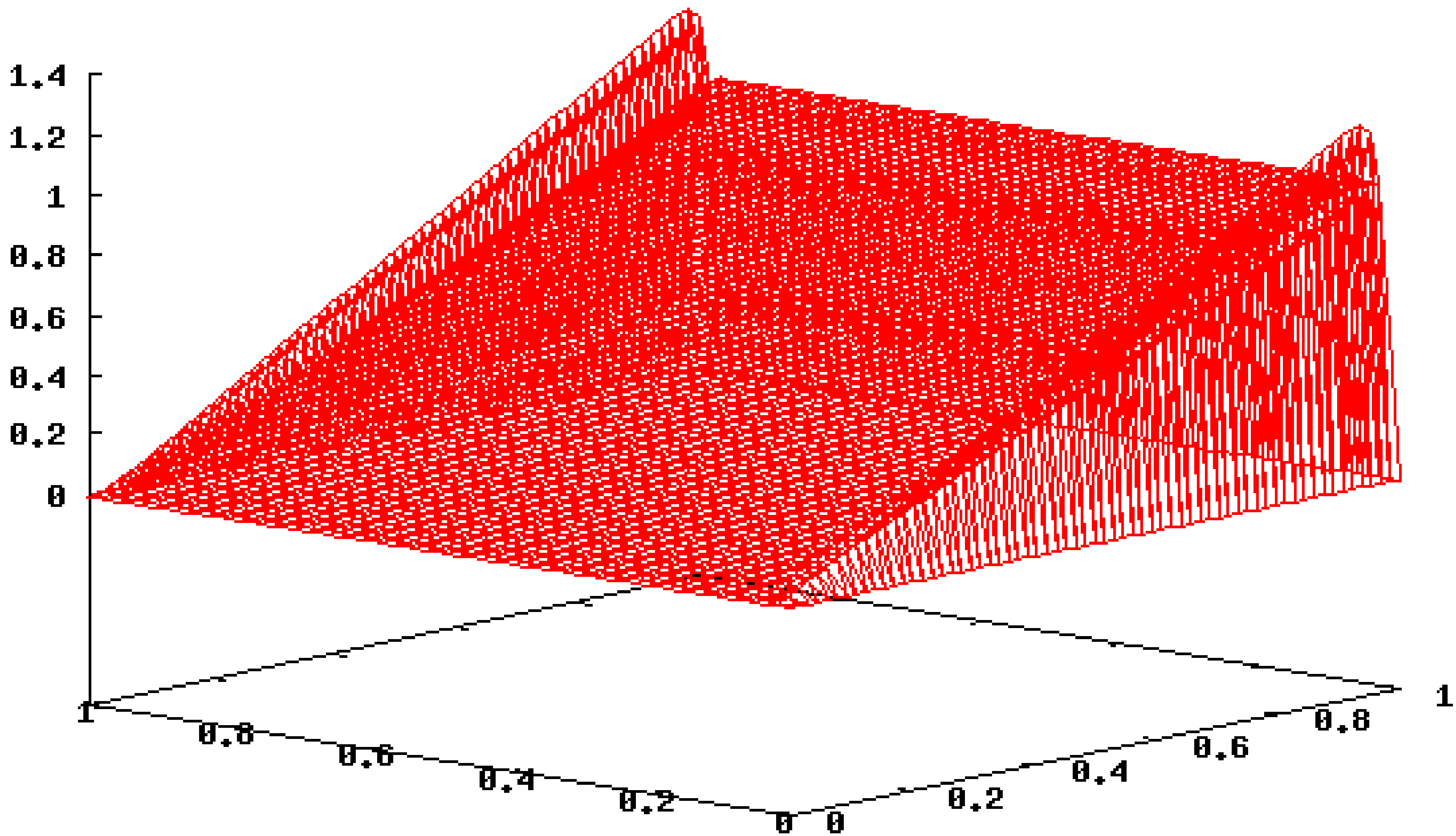
$$\varepsilon = 10^{-8}$$
$$|\mathbf{b}| = 1$$
$$f = 0$$



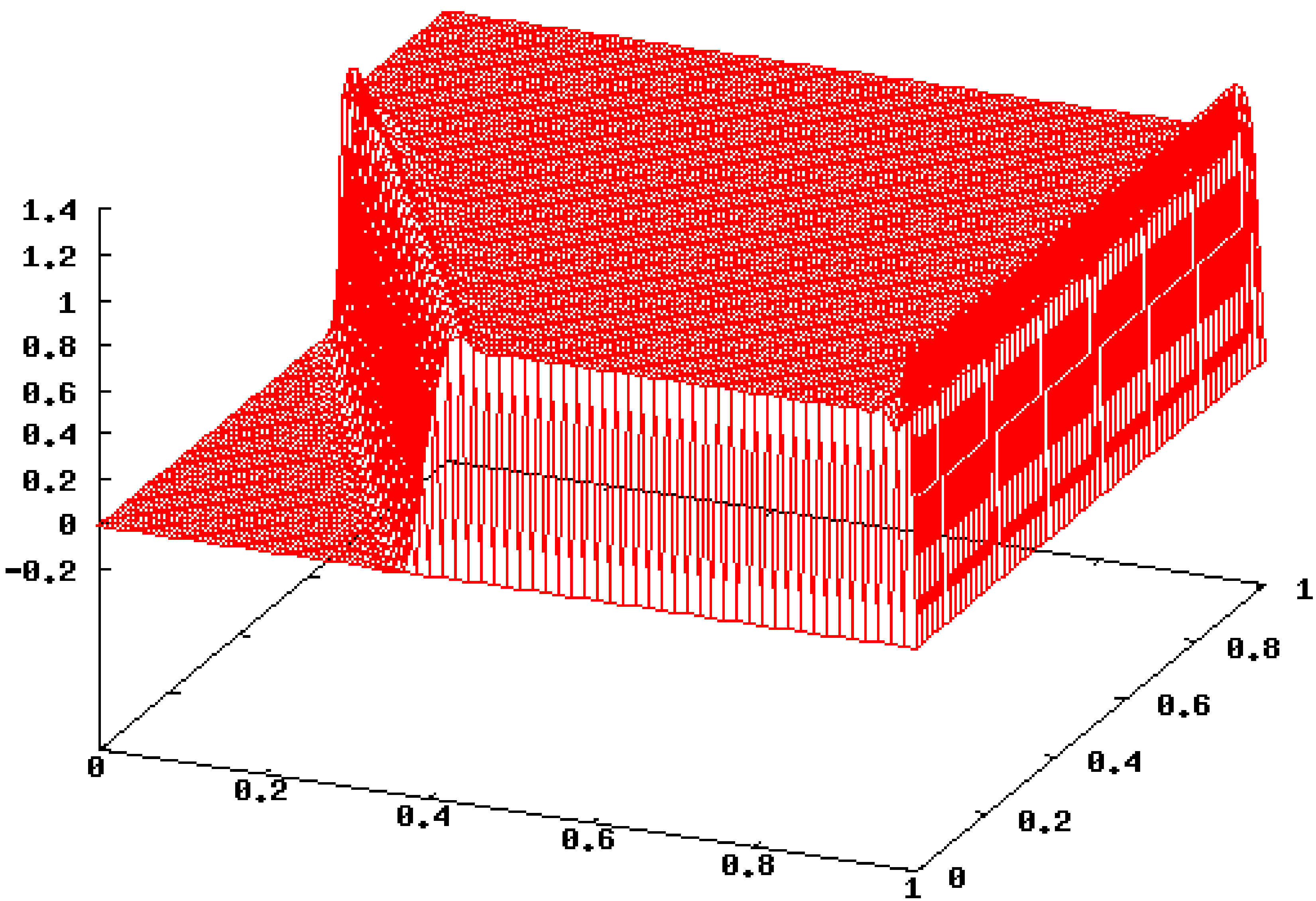
65×65 points

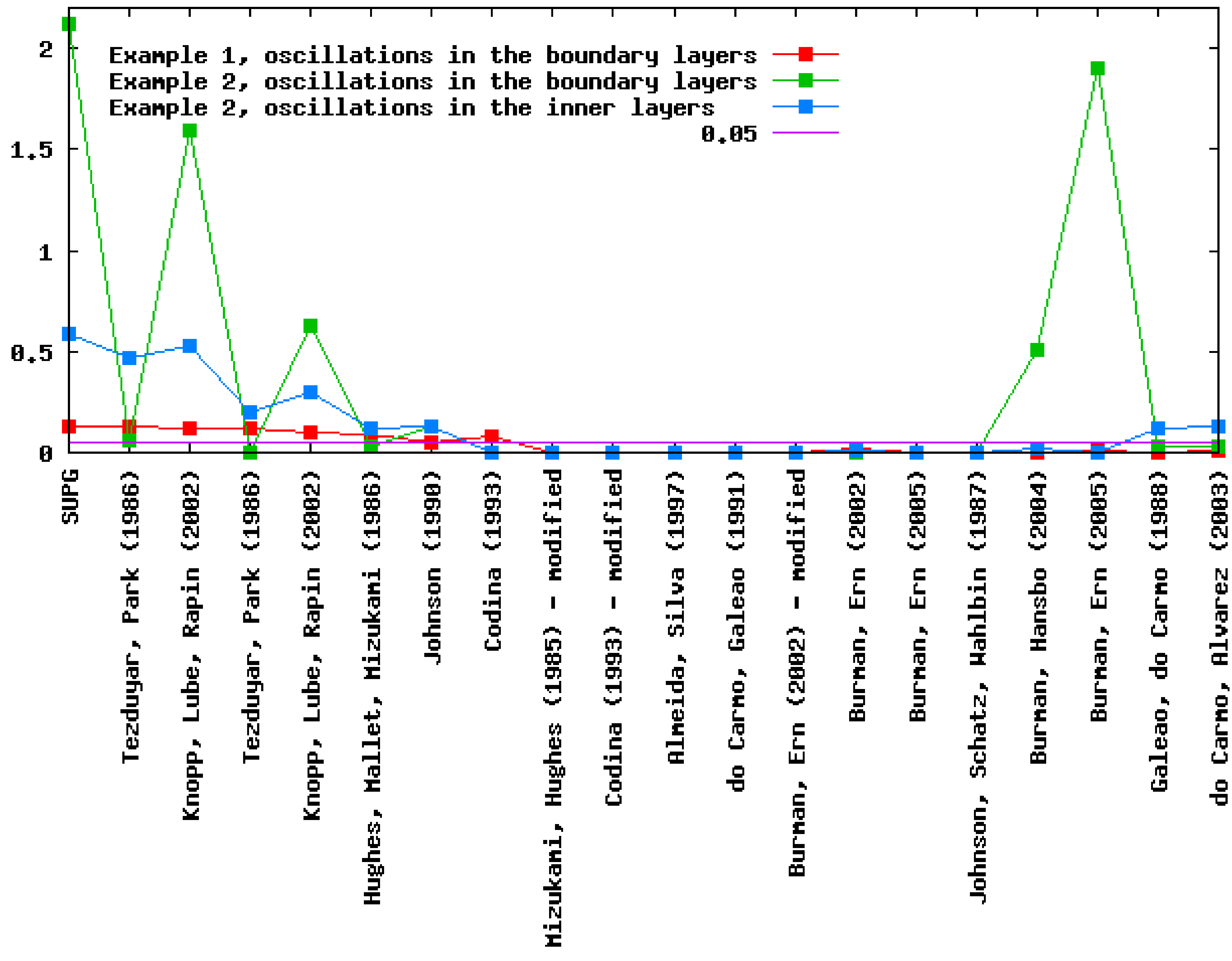


Example 1: SUPG solution

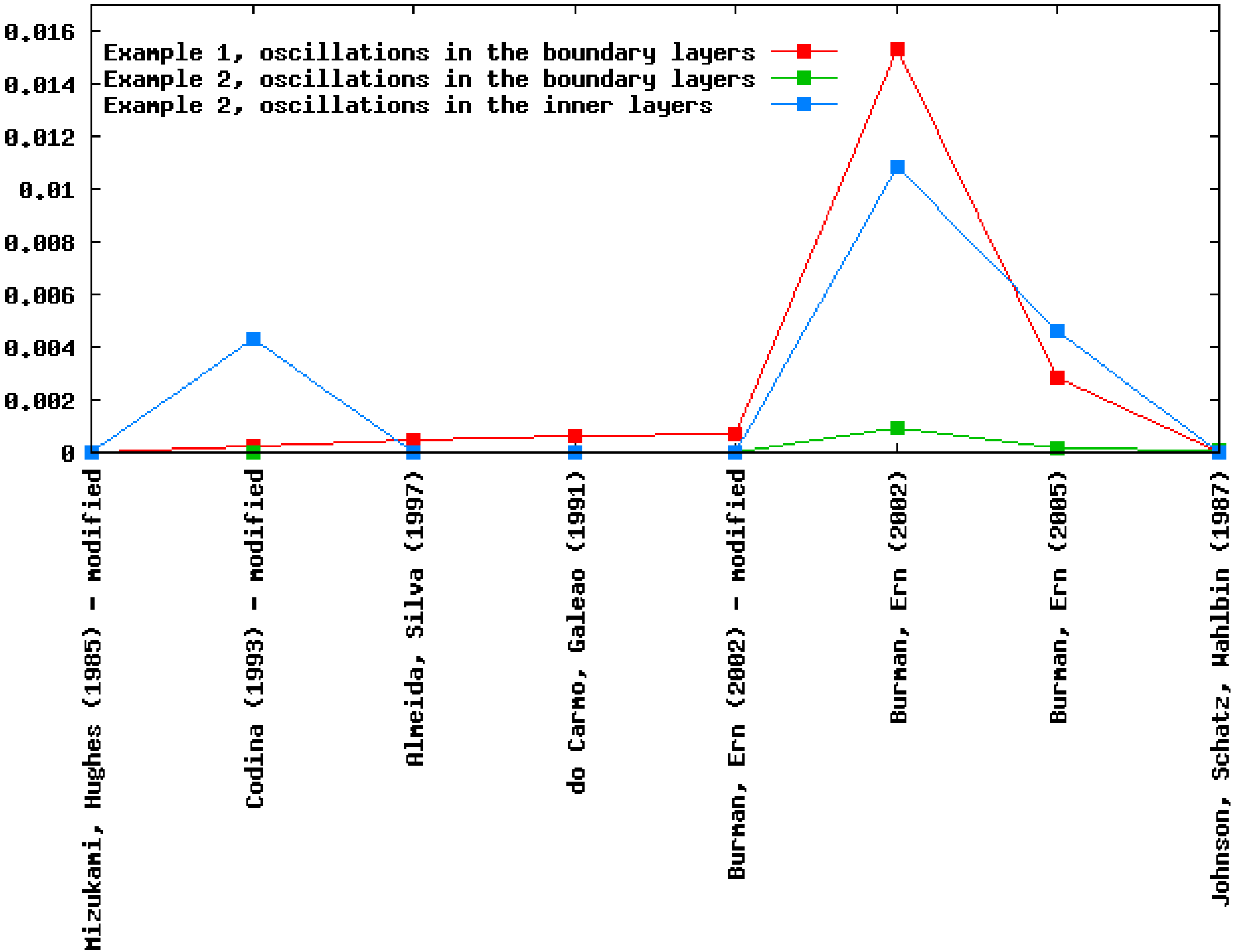


Example 2: SUPG solution

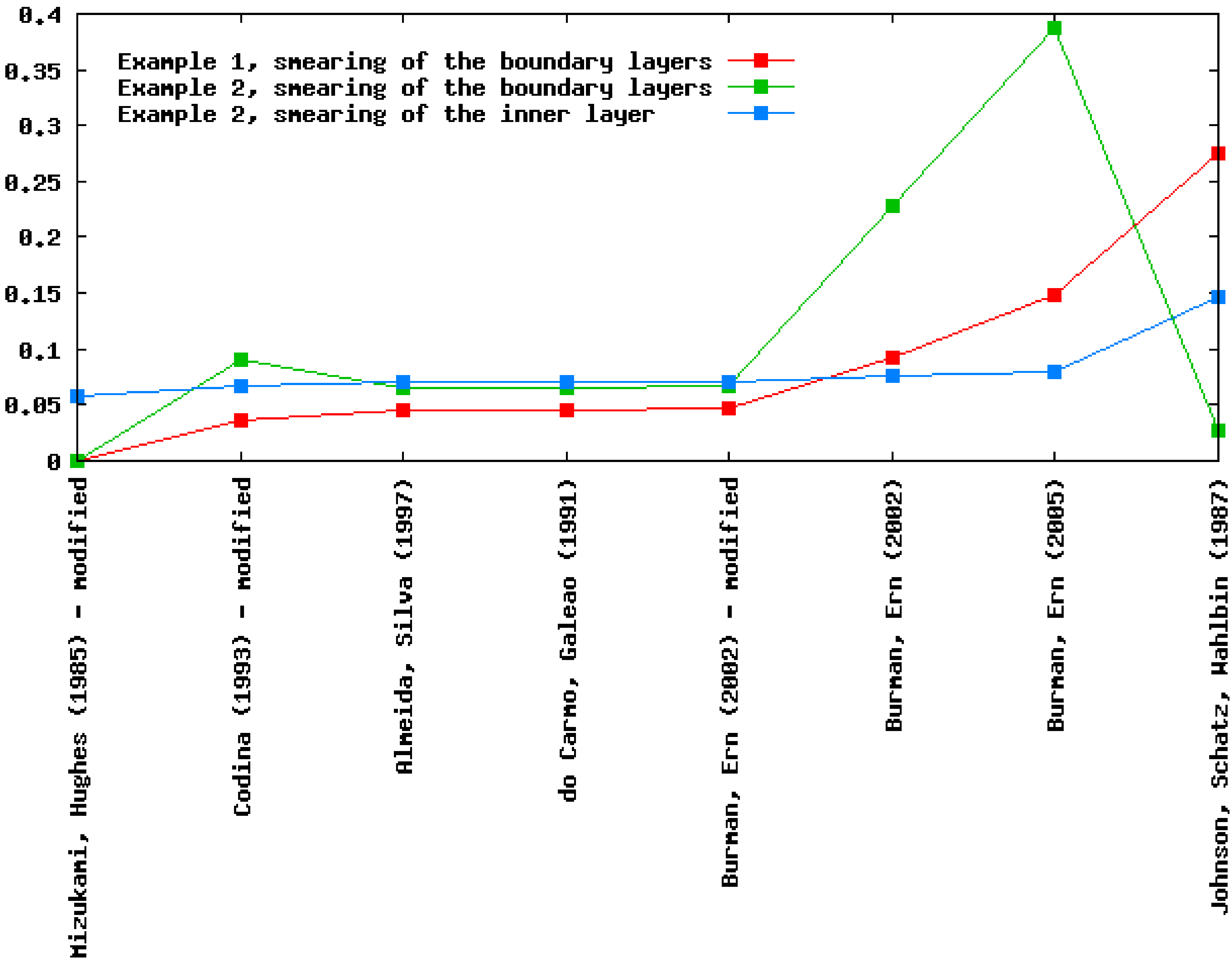




Example 1, oscillations in the boundary layers
Example 2, oscillations in the boundary layers
Example 2, oscillations in the inner layers
0,05
0
0,5
1
1,5
2
SUPG
Tezduyar, Park (1986)
Knopp, Lube, Rapin (2002)
Tezduyar, Park (1986)
Knopp, Lube, Rapin (2002)
Hughes, Mallet, Mizukami (1986)
Johnson (1990)
Codina (1993)
Mizukami, Hughes (1985) - modified
Codina (1993) - modified
Almeida, Silva (1997)
do Carmo, Galeao (1991)
Burman, Ern (2002) - modified
Burman, Ern (2002)
Burman, Ern (2005)
Johnson, Schatz, Wahlbin (1987)
Burman, Hansbo (2004)
Burman, Ern (2005)
Galeao, do Carmo (1988)
do Carmo, Alvarez (2003)

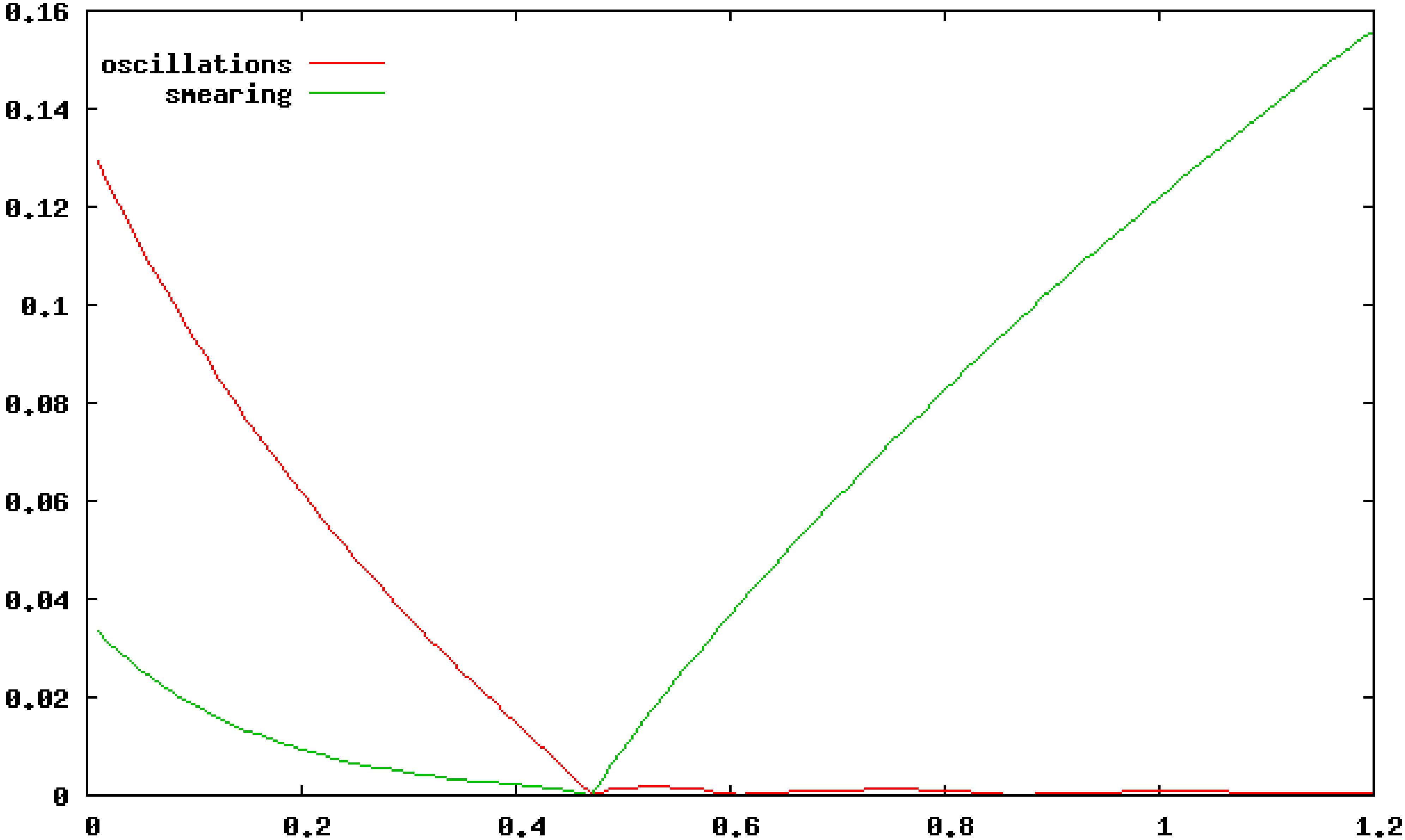


Example 1, oscillations in the boundary layers
Example 2, oscillations in the boundary layers
Example 2, oscillations in the inner layers
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
Mizukami, Hughes (1985) - modified
Codina (1993) - modified
Almeida, Silva (1997)
do Carmo, Galeao (1991)
Burman, Ern (2002) - modified
Burman, Ern (2002)
Burman, Ern (2005)
Johnson, Schatz, Wahlbin (1987)

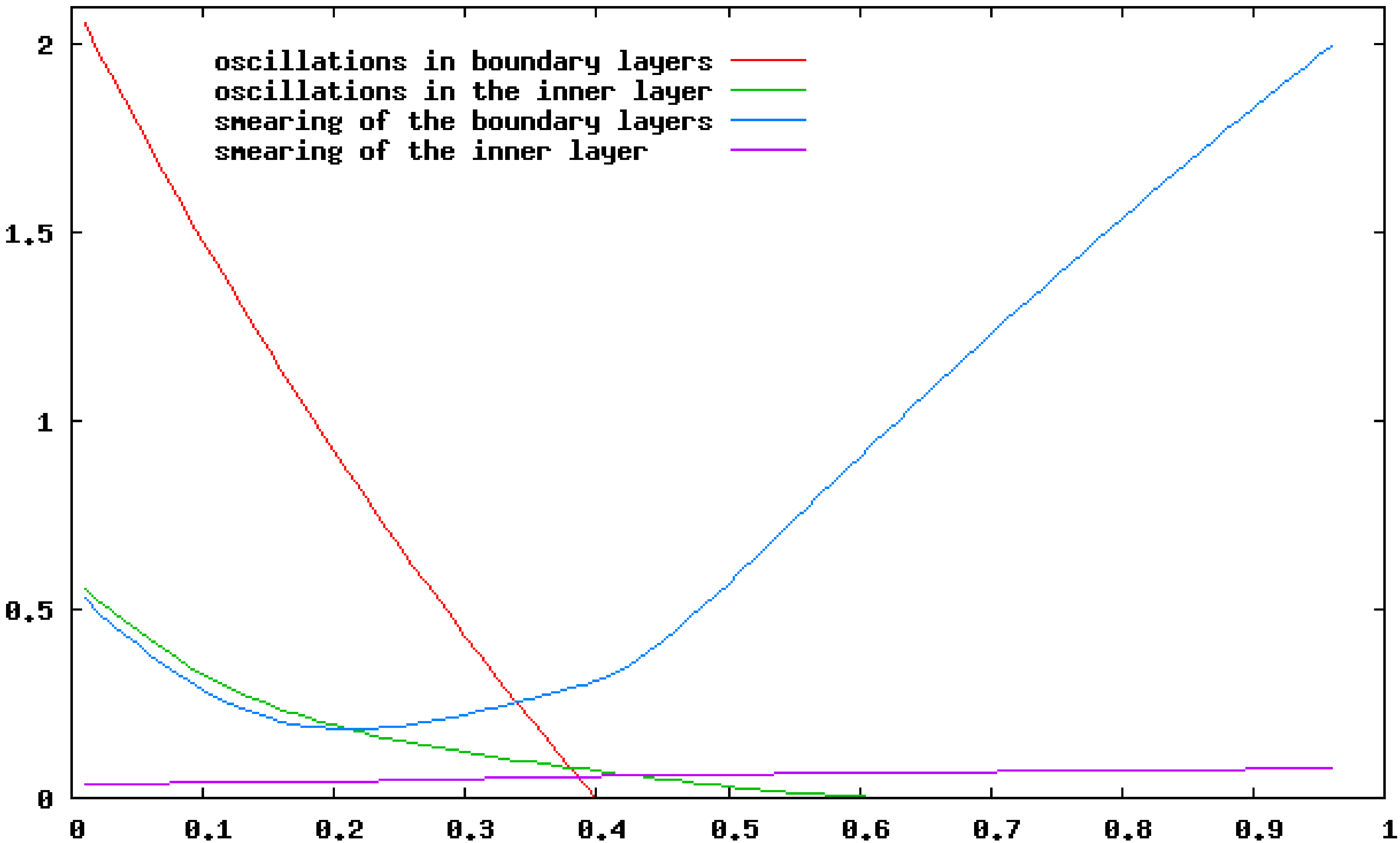


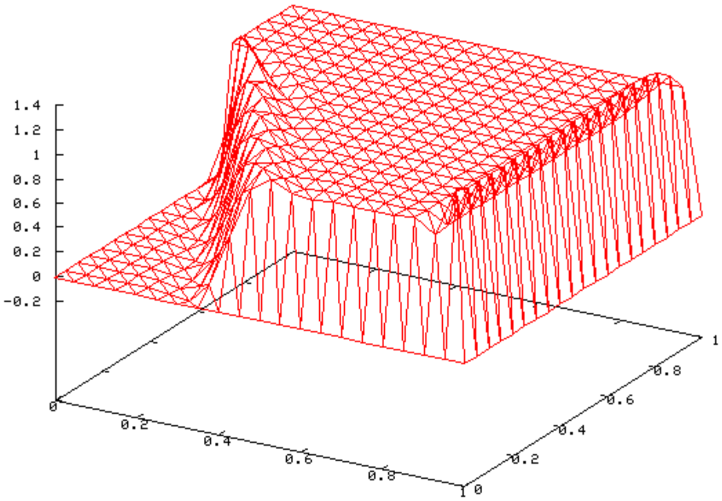
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
Example 1, smearing of the boundary layers
Example 2, smearing of the boundary layers
Example 2, smearing of the inner layer
Mizukami, Hughes (1985) - modified
Codina (1993) - modified
Almeida, Silva (1997)
do Carmo, Galeao (1991)
Burman, Ern (2002) - modified
Burman, Ern (2002)
Burman, Ern (2005)
Johnson, Schatz, Wahlbin (1987)

Modified Codina's method: dependence on C for Ex. 1

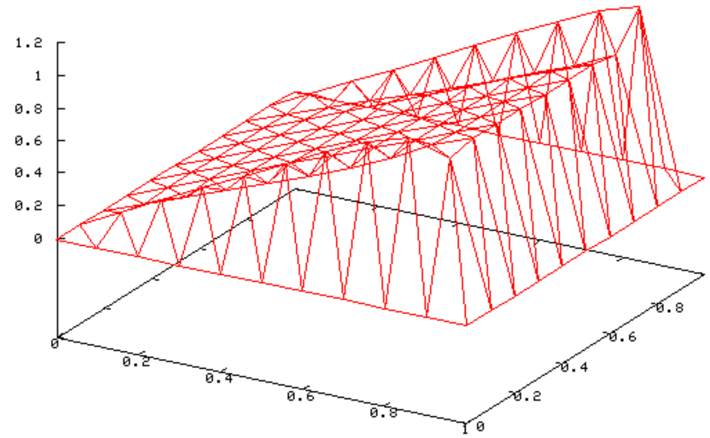


Modified Codina's method: dependence on C for Ex. 2

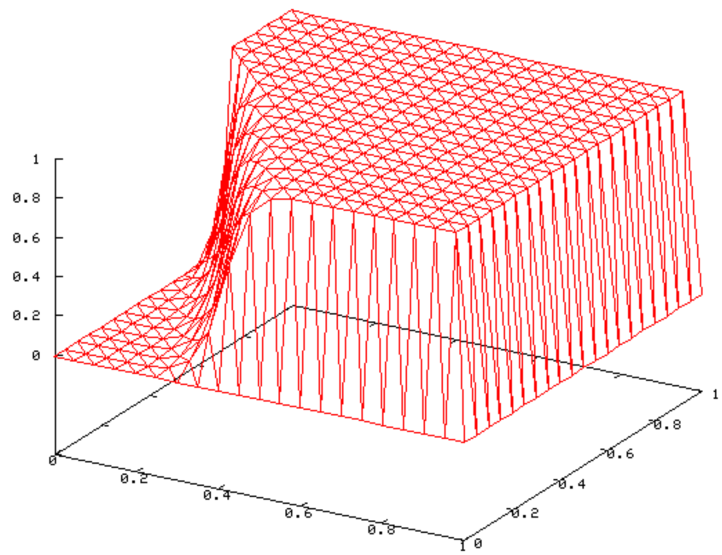




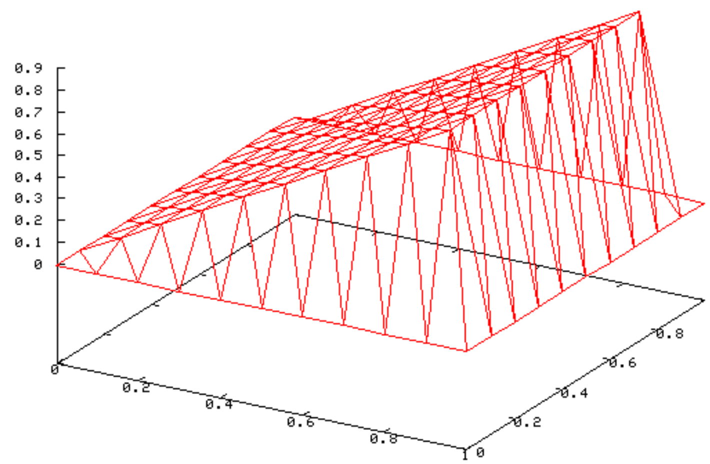
SUPG



SUPG

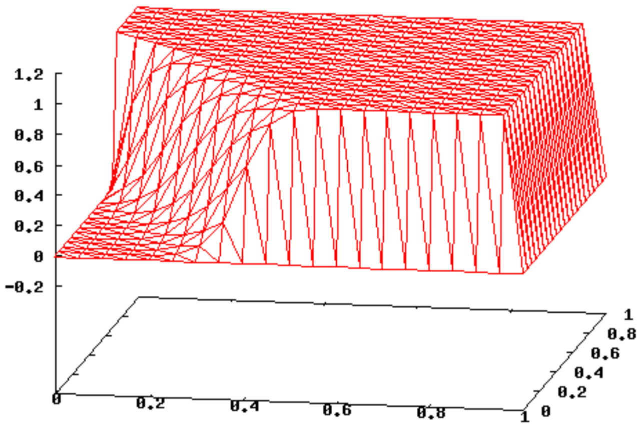


Mizukami, Hughes

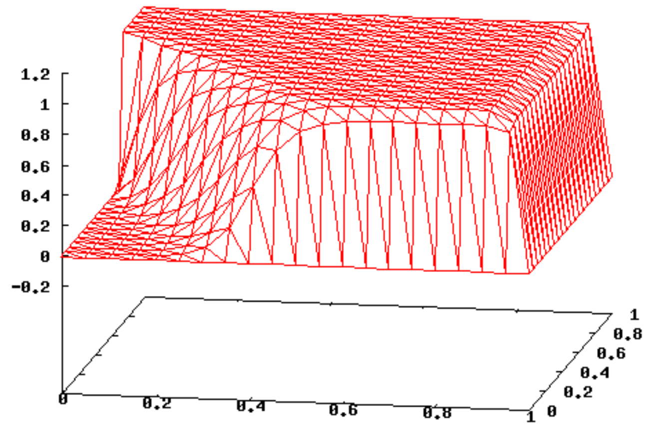


Mizukami, Hughes

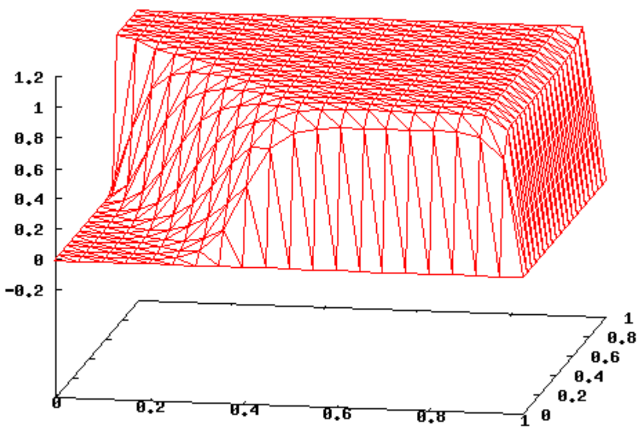
Convection skew to the mesh



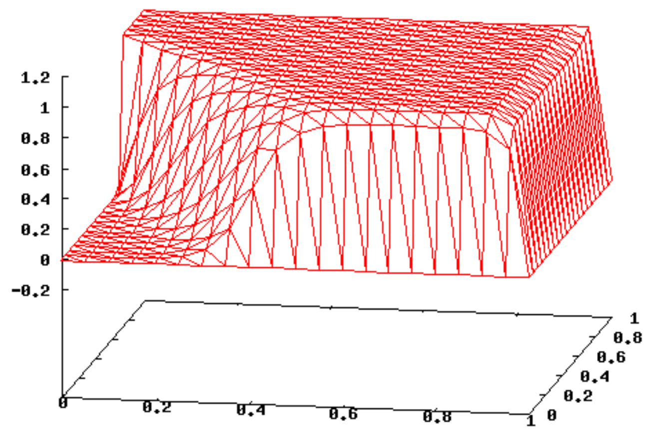
Mizukami, Hughes



do Carmo, Galeão (1991)

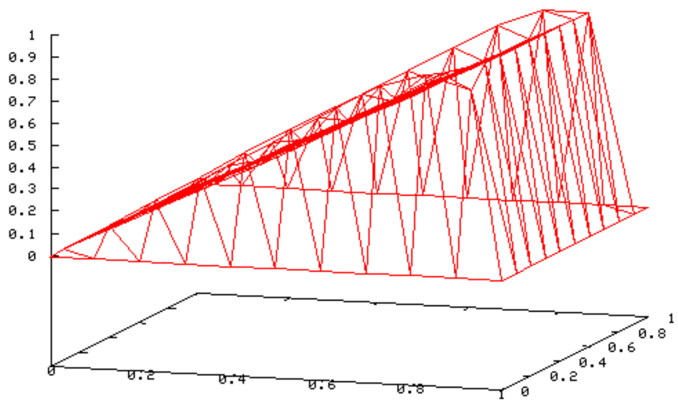


modified Codina

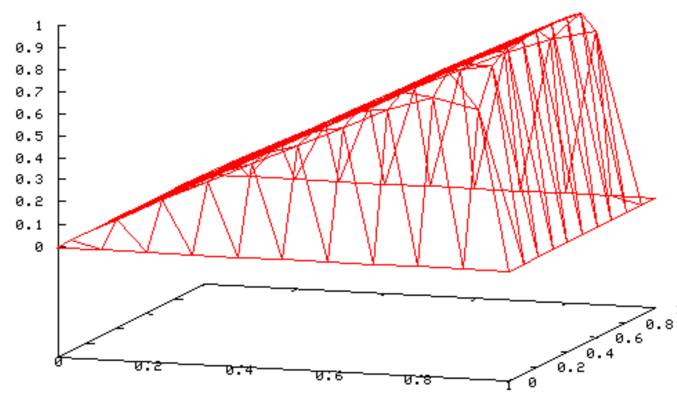


modified Burman, Ern

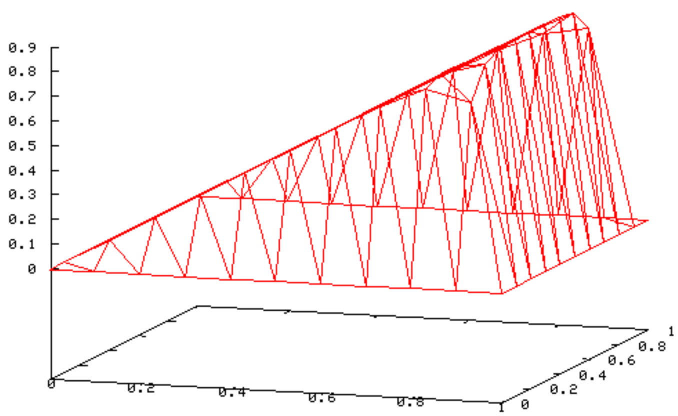
Convection with a source term



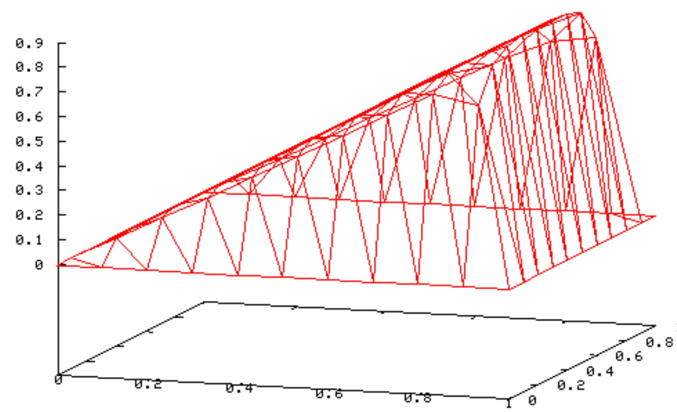
Codina



modified Burman, Ern

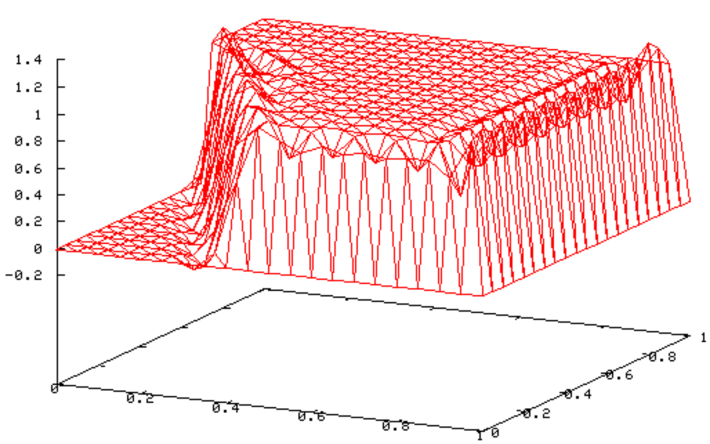


modified Codina C=0.465

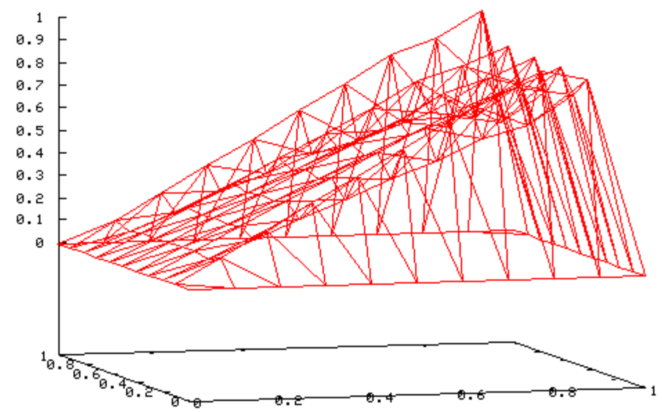


modified Codina C=0.6

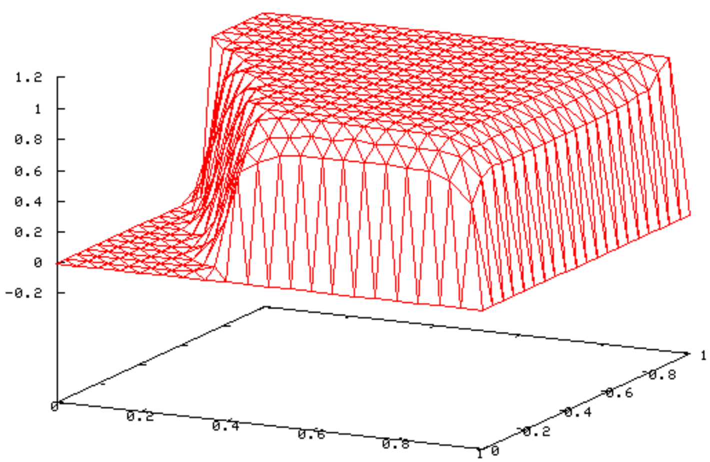
P_2 element



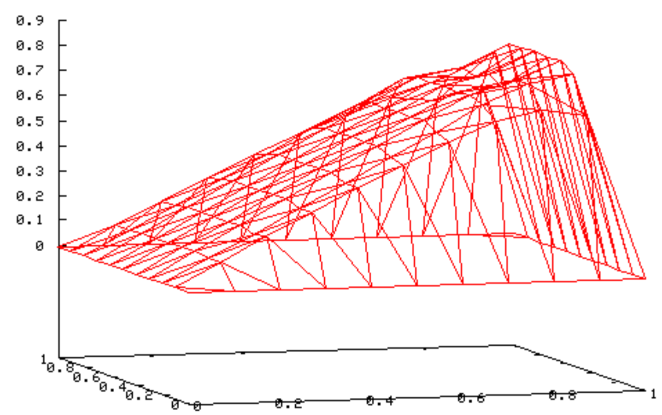
SUPG



SUPG

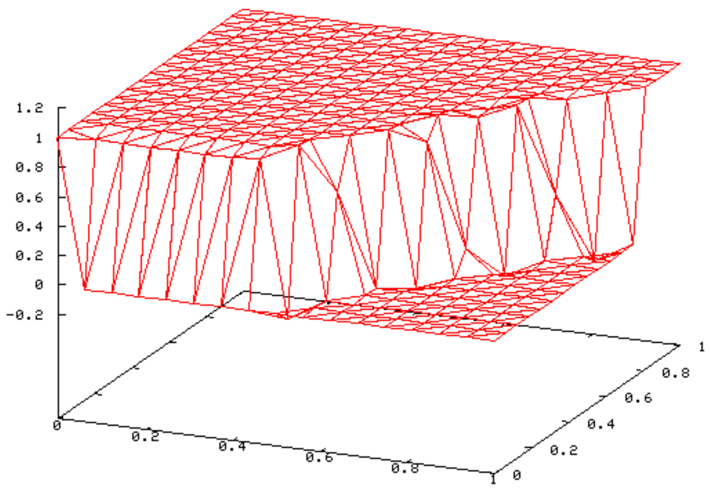


modified Codina C=0.35

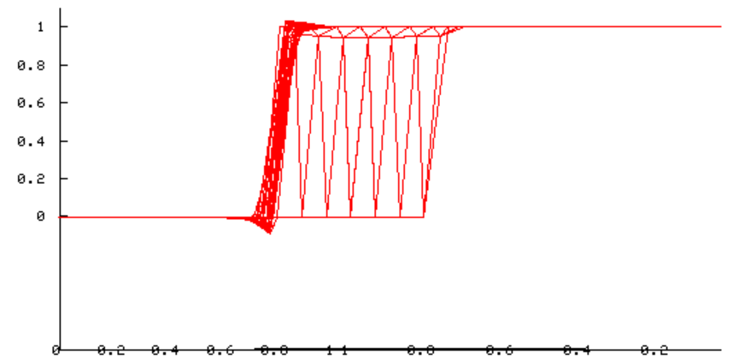


modified Burman, Ern

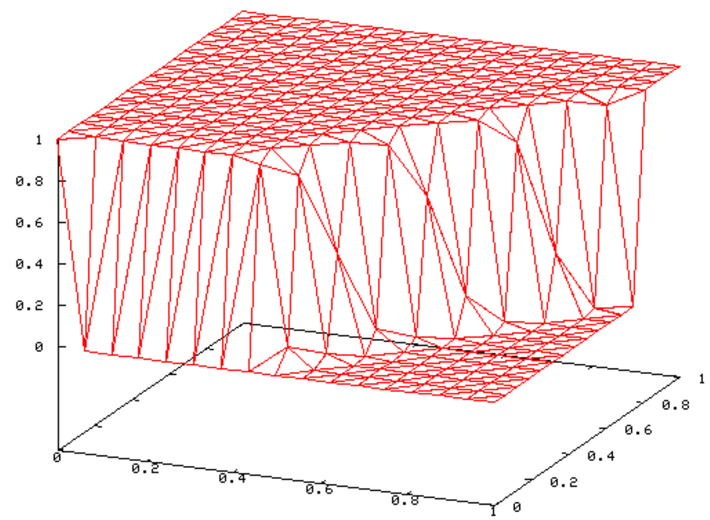
P_4 element



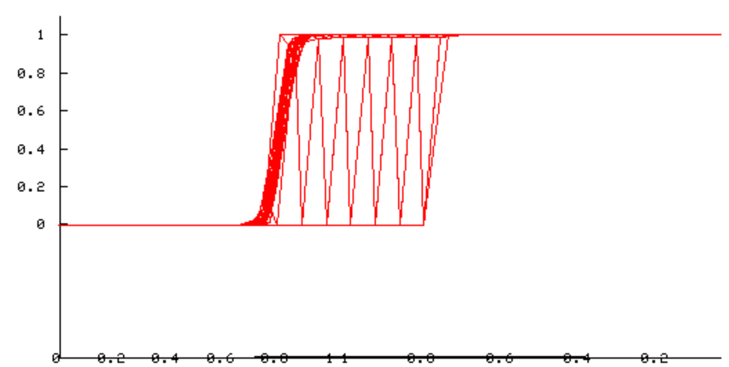
SUPG



SUPG

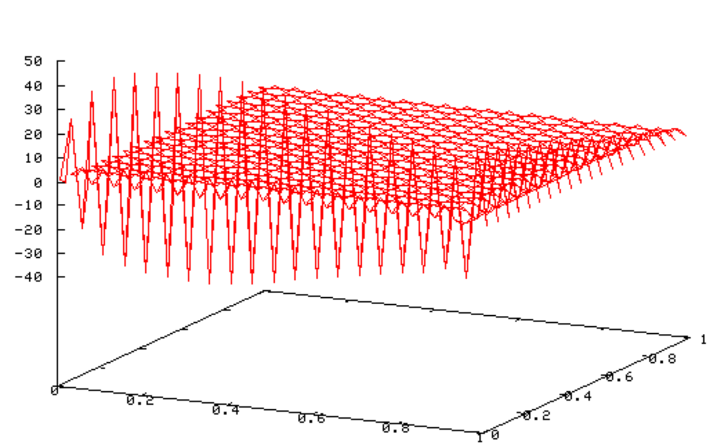


modified Codina C=0.2

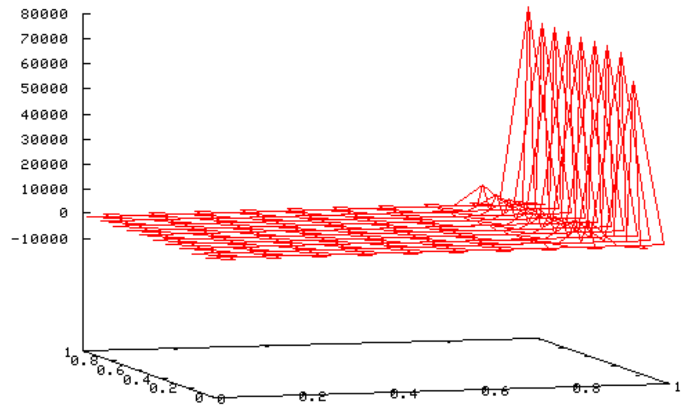


modified Codina C=0.2

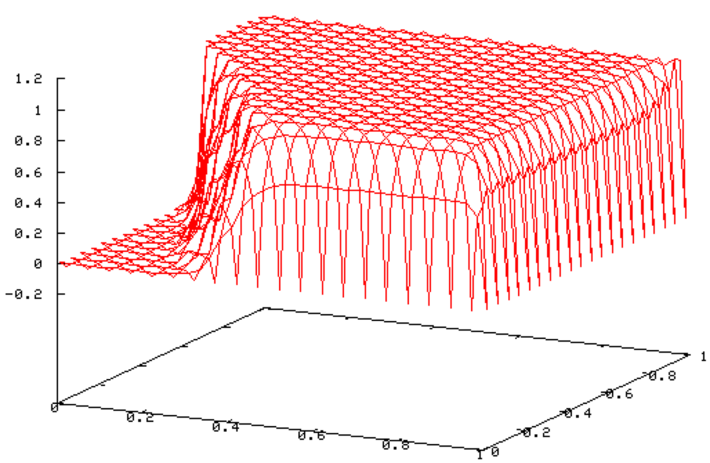
Crouzeix–Raviart element



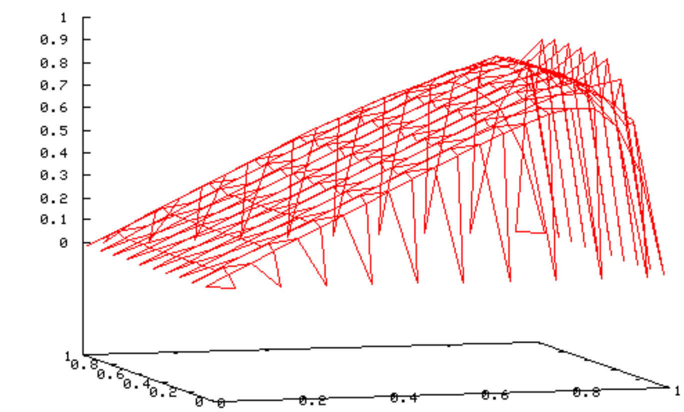
SUPG



SUPG



do Carmo, Galeão (1991)



modified Codina C=0.6

Conclusions

– best methods:

Conclusions

- best methods:
  - Mizukami, Hughes

Conclusions

- best methods:
  - Mizukami, Hughes
  - do Carmo, Galeão (1991)

Conclusions

- best methods:
  - Mizukami, Hughes
  - do Carmo, Galeão (1991)
  - Almeida, Silva

Conclusions

- best methods:
  - Mizukami, Hughes
  - do Carmo, Galeão (1991)
  - Almeida, Silva
  - modified Codina

Conclusions

- best methods:
  - Mizukami, Hughes
  - do Carmo, Galeão (1991)
  - Almeida, Silva
  - modified Codina
  - modified Burman, Ern

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  - higher order elements

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  - do Carmo, Galeão (1991)
  - Almeida, Silva
  - modified Codina
  - modified Burman, Ern
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  - higher order elements
  - nonconforming elements

Conclusions

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  - modified Burman, Ern

BUT: any of the methods may fail

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BUT: any of the methods may fail

there is no reliable method which would provide

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still much research needed!!!