SCALABLE ALGORITHMS FOR CONTACT PROBLEMS WITH GEOMETRICAL AND MATERIAL NONLINEARITIES

J. Dobiáš, S. Pták

Institute of Thermomechanics, Academy of Sciences of the Czech Republic

Dolejškova 5, 182 00 Praha 8, Czech Republic

Website: http://it.cas.cz/

Z. Dostál, V. Vondrák

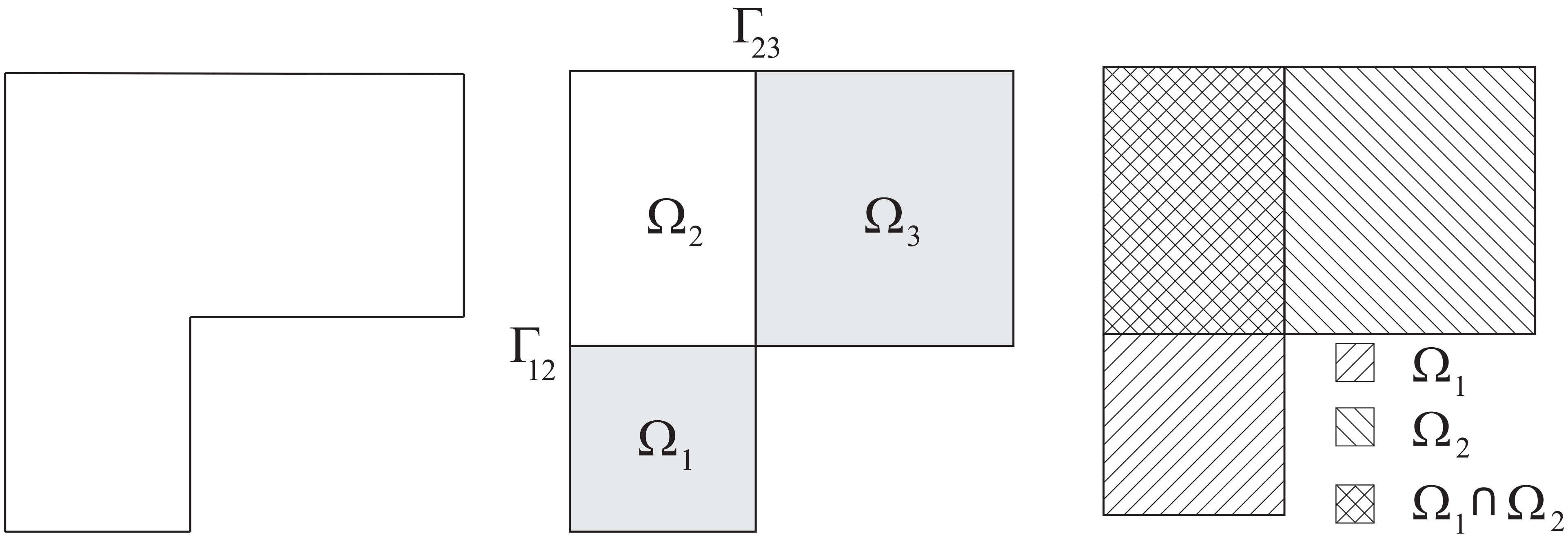
Department of Applied Mathematics, Technical University in Ostrava

17. listopadu 15, 708 00 Ostrava-Poruba, Czech Republic

Supported by the Czech Science Foundation through Grant # 101/05/0423

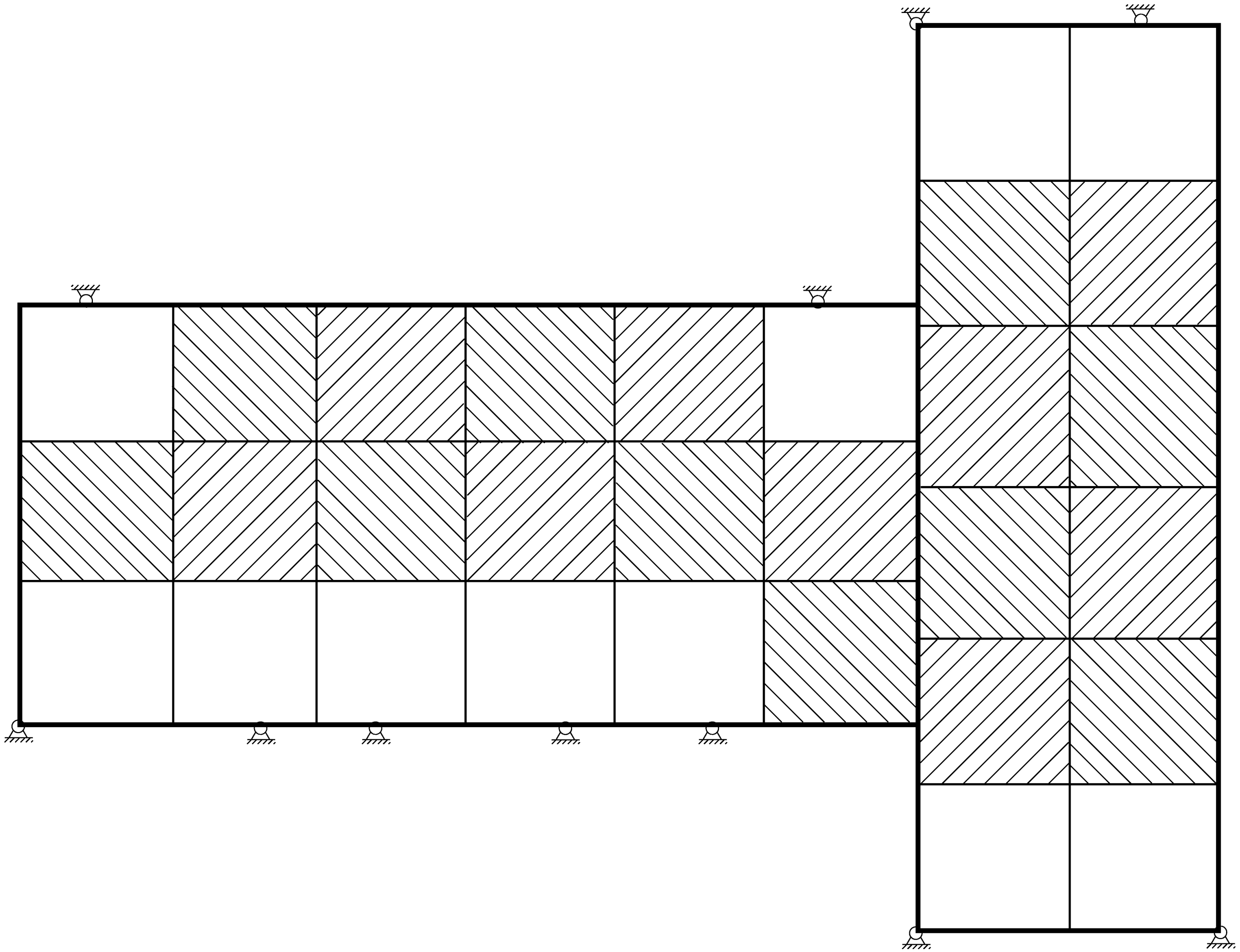
TU Ostrava, CZ — *IT Prague, CZ*

Principle of domain decomposition methods

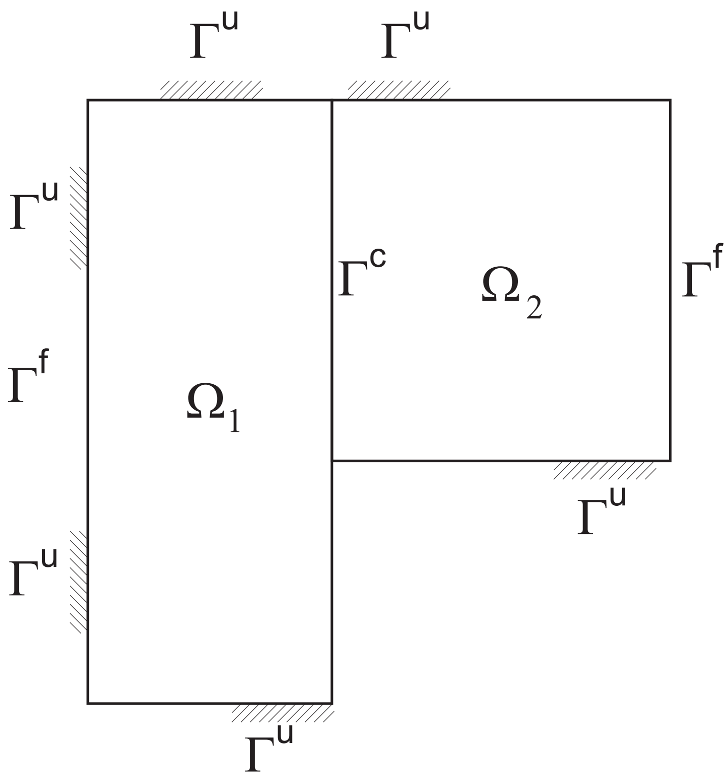


Original problem | Non-overlapping partition | Overlapping partition

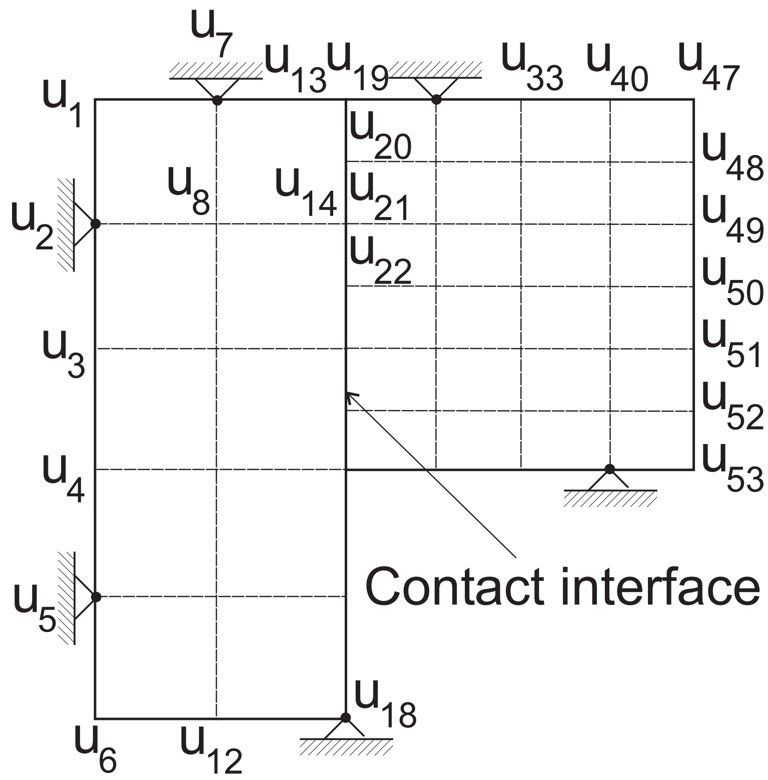
Floating sub-domain



Non-overlapping partition, primal variables

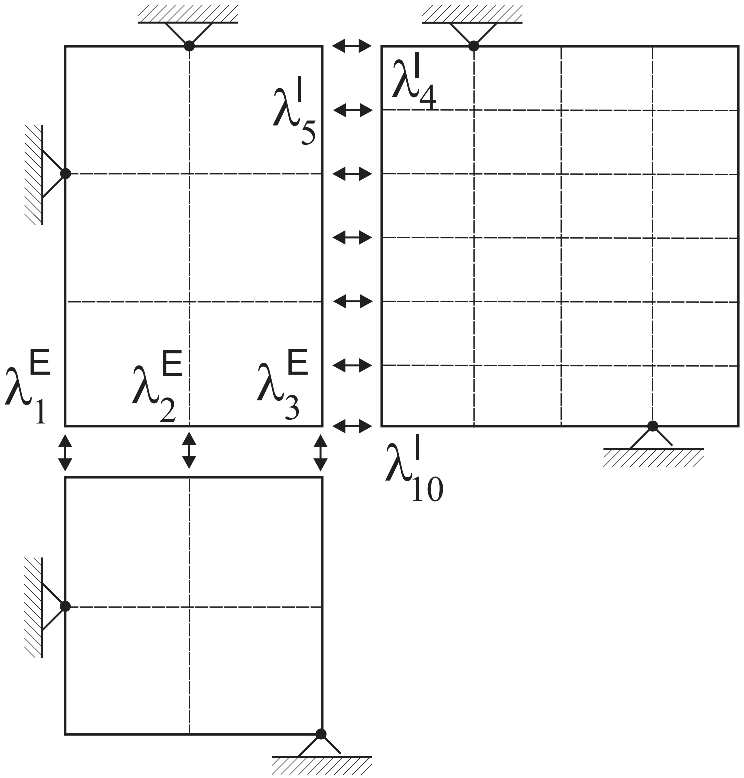


Original problem

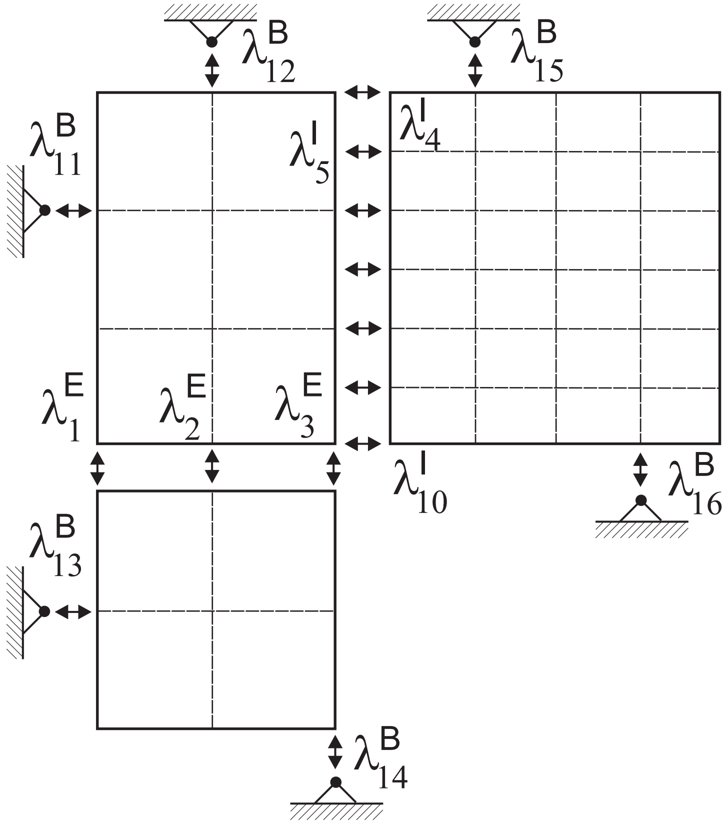


Primal variables (displacements)

Dual variables: FETI and Total FETI



Principle of FETI method



Principle of Total FETI method

FETI domain decomposition method (I)

FETI: Finite Element Tearing and Interconnecting

Solution: minimizer of the energy functional (the 2nd order elliptic bound. v. problem)

$$q(u) = \frac{1}{2}a(u,u) - \ell(u) \quad \text{subject to} \quad u \in \mathcal{K}$$

The numerical approximation by FEM and auxiliary domain decomposition

$$\min \left(\frac{1}{2}\mathsf{u}^\top A\,\mathsf{u} - f^\top \mathsf{u}\right) \quad \text{subject to} \quad B_I\mathsf{u} \leq 0 \quad \text{and} \quad B_E\mathsf{u} = 0$$

Introducing Lagrange multipliers, the discretized form of associated functional

$$L(\mathsf{u},\lambda) = \frac{1}{2}\mathsf{u}^\top A\,\mathsf{u} - f^\top \mathsf{u} + \lambda^\top B\mathsf{u}$$

$$\lambda = \begin{bmatrix} \lambda_I \\ \lambda_E \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_I \\ B_E \end{bmatrix}$$

FETI domain decomposition method (II)

Primal variable elimination

$$\min \left(\frac{1}{2}\lambda^\top B A^\dagger B^\top \lambda - \lambda^\top B A^\dagger f\right) ,$$

$$\text{subject to} \quad \lambda_I \geq 0 \quad \text{and} \quad R^\top (f - B^\top \lambda) = 0$$

$A^\dagger$ ... generalized inverse of A satisfying one of the Penrose axioms

$$A A^\dagger A = A$$

R ... full rank matrix with columns spanning the kernel of A

(rigid body modes of floating sub-domains)

... quadratic programming problem with bound and equality constraints

FETI domain decomposition method (III)

Introduction of new matrices

$$F = B\,A^{\dagger}\,B^{\top}, \quad \tilde{G} = R^{\top}B^{\top}, \quad \tilde{e} = R^{\top}f, \quad \tilde{d} = BA^{\dagger}f$$

$\tilde{\lambda}$ solves $$\tilde{G}\tilde{\lambda} = \tilde{e}$$

Looking for solution in the form $\lambda = \mu + \tilde{\lambda}$

$$\frac{1}{2}\lambda^{\top}F\,\lambda - \lambda^{\top}\,\tilde{d} = \frac{1}{2}\mu^{\top}\,F\,\mu - \mu^{\top}(\tilde{d} - F\,\tilde{\lambda}) + \frac{1}{2}\tilde{\lambda}^{\top}\,F\,\tilde{\lambda} - \tilde{\lambda}^{\top}\,\tilde{d}$$

Returning to the old notation

$$\min\ \left(\frac{1}{2}\lambda^{\top}PFP\lambda - \lambda^{\top}Pd\right), \qquad \text{subject to} \qquad G\lambda = 0 \quad \text{and} \quad \lambda_I \geq -\tilde{\lambda}_I$$

where $P = I - Q$ and $Q = G^{\top}G$

FETI domain decomposition method (IV)

Making use of the (Dual) Penalty Method

$$\min \left(\frac{1}{2} \lambda^\top PFP\lambda - \lambda^\top Pd + \frac{1}{2} \lambda^\top G^\top G \ \lambda\rho \right)$$

ρ ... penalty parameter

or

$$\min \left(\frac{1}{2} \lambda^\top (PFP + \rho Q)\, \lambda - \lambda^\top Pd \right) \qquad \text{subject to} \qquad \lambda_I \geq -\tilde{\lambda}_I$$

... solution in terms of the MPRGP algorithm

FETI with contact and other non-linearities

Equilibrium $res = f_{ext} - f_{int} = 0$

External forces $f_{ext} = f - B^\top \lambda$, Internal forces $f_{int} = \sum_{nelem} \int_{V_e} B_s{}^\top(\epsilon)\, \sigma(\epsilon)\, dV$

Solution algorithm

Initial step: Assemble stiffness matrix $K = diag\{K_1, ..., K_p\}$ and B_E ;

Set $i = 0$; $\mathrm{u}^0 = 0$, $\lambda^0 = 0$, $f_{int}^0 = 0$;

Step 1: Evaluate contact conditions B_I^i ;

Step 2: Solve contact problem by MPRGP for $\Delta\lambda \rightarrow \Delta\mathrm{u}$;

Step 3: $\lambda^i = \lambda^{i-1} + \Delta\lambda$, $\mathrm{u}^i = \mathrm{u}^{i-1} + \Delta\mathrm{u}$;

$$f_{int}^i = \sum_{nelem} \int_{V_e} B_s{}^\top(\epsilon^i)\, \sigma(\epsilon^i)\, dV$$

Assemble residual load vector $res^i = f - B^\top \lambda^i - f_{int}^i$;

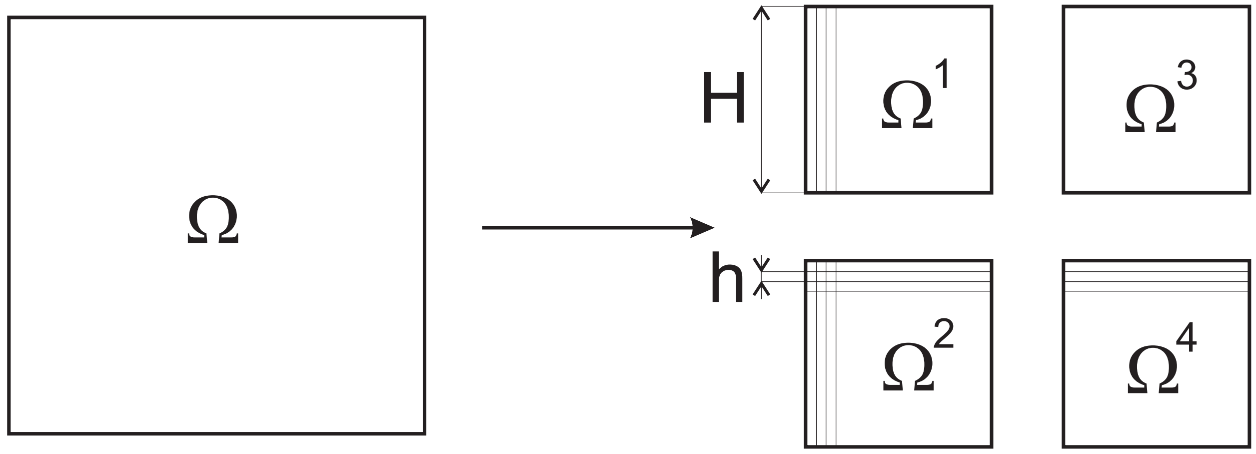
Check on convergence criteria $\frac{\|\Delta\mathrm{u}\|}{\|\mathrm{u}^i\|} < \eta_1$, $\frac{\|res^i\|}{\|f_{ext}^i\|} < \eta_2$;

If fulfilled then STOP,

otherwise set $i \leftarrow i + 1$ and go to **Step 1**

Numerical experiments: Scalability (I)

Numerical scalability: condition number does not grow or grows 'weakly' with $\frac{H}{h}$



Poisson's problem: $\triangle u = 1$ in Ω, $\Omega = (0,1) \times (0,1)$

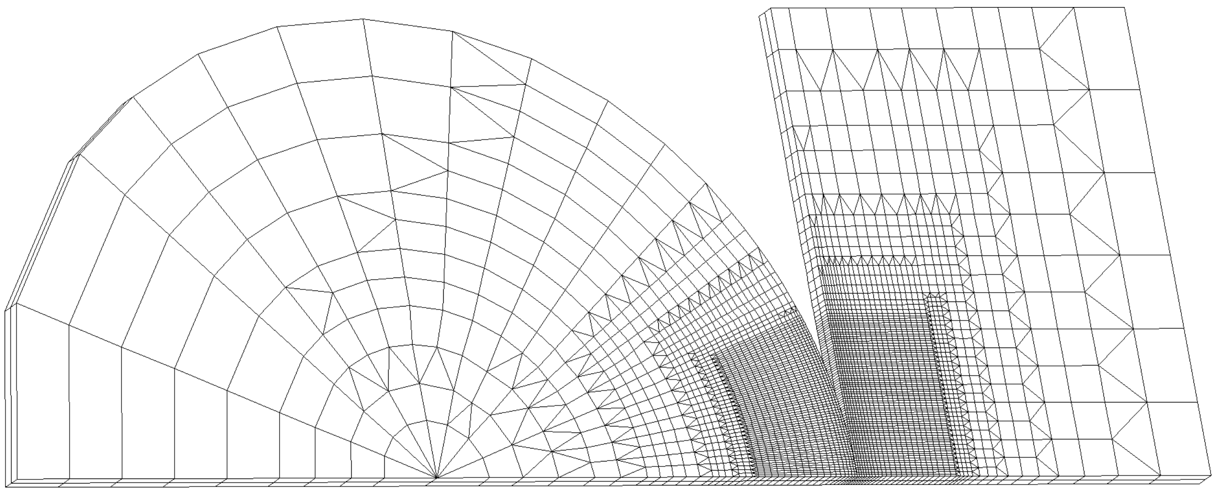
Boundary conditions: $u(0,y) = 0$; $\frac{\delta u(1,y)}{\delta x} = 0$ for $y \in [0,1)$

$\frac{\delta u(x,0)}{\delta y} = 0$; $\frac{\delta u(x,1)}{\delta y} = 0$ for $x \in [0,1)$

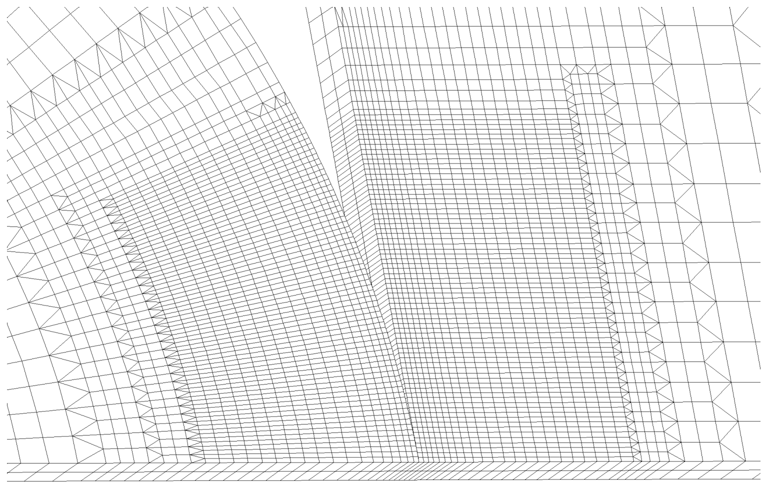
Numerical experiments: Scalability (II)

H	h	prim.	dual FETI	dual TFETI	CG steps FETI	CG steps TFETI
1/2	1/4	36	11	17	7	4
1/4	1/8	144	63	75	12	5
1/8	1/16	576	287	311	13	7
1/16	1/32	2304	1215	1263	15	11
1/2	1/8	100	19	29	9	9
1/4	1/16	400	111	131	16	12
1/8	1/32	1600	511	551	18	16
1/16	1/64	6400	2175	2255	20	21
1/2	1/16	324	35	53	14	9
1/4	1/32	1296	207	243	22	14
1/8	1/64	5184	959	1031	24	20
1/16	1/128	20736	4095	4239	23	23

Two cylinders: mesh, geometry and material



Computational model

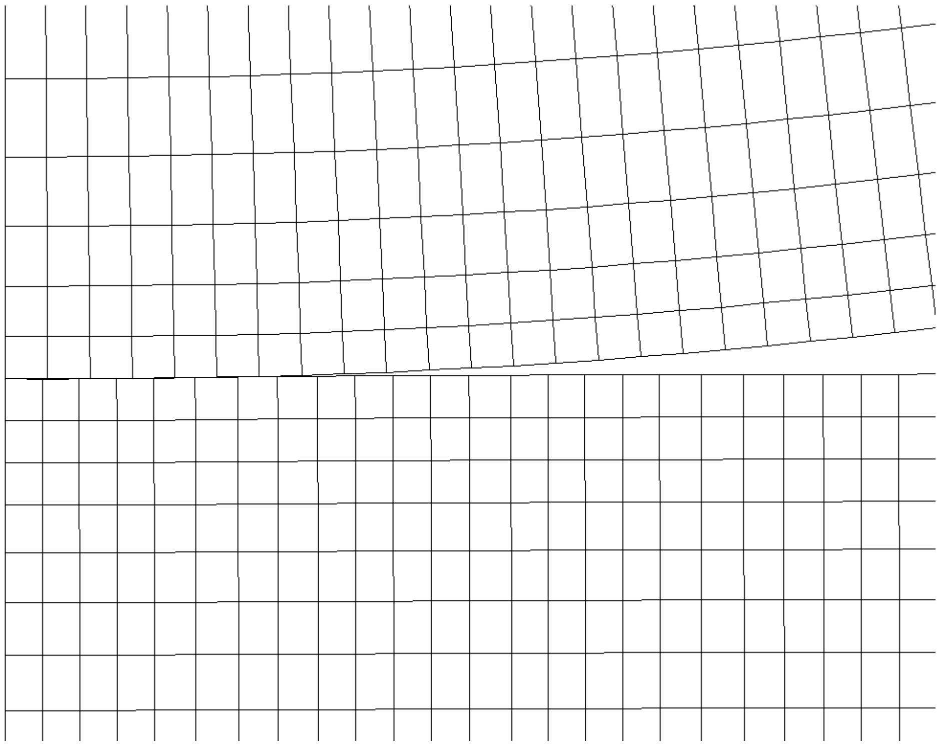


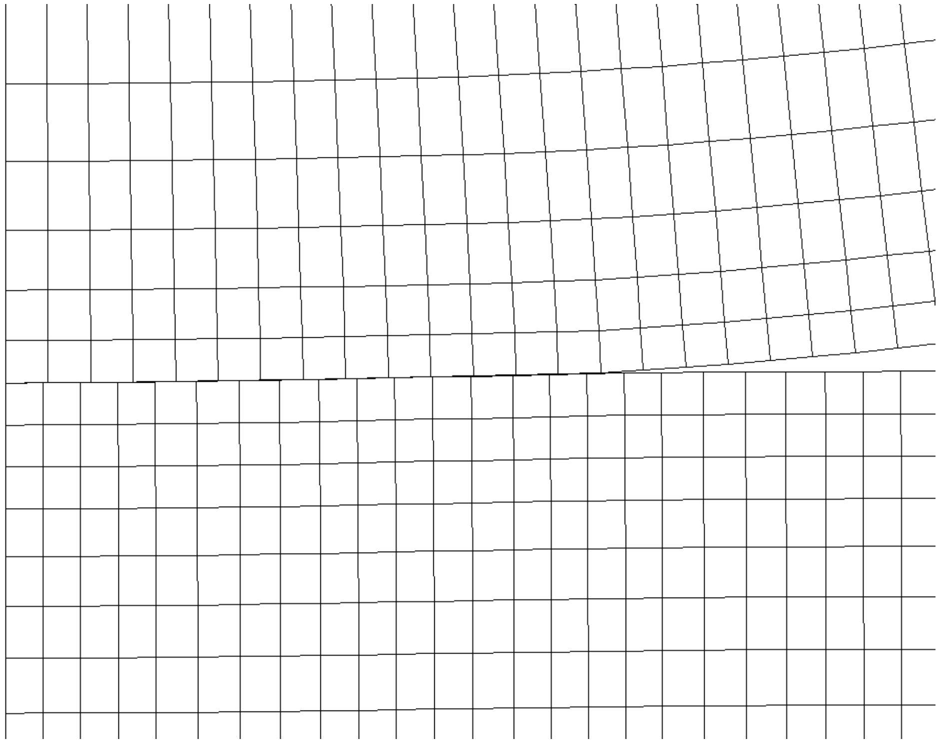
Detail

- Geometry: Radii of cylinders $R_1 = 1$ m, $R_2 = \infty$
- Material properties: Young‘s modulus $E = 2.0 \times 10^{11}$ Pa, Poisson‘s ratio $\nu = 0.3$

Two cylinders: elastic & elastic–plastic problems

Load: $l = 4.0 \times 10^7$ N/m

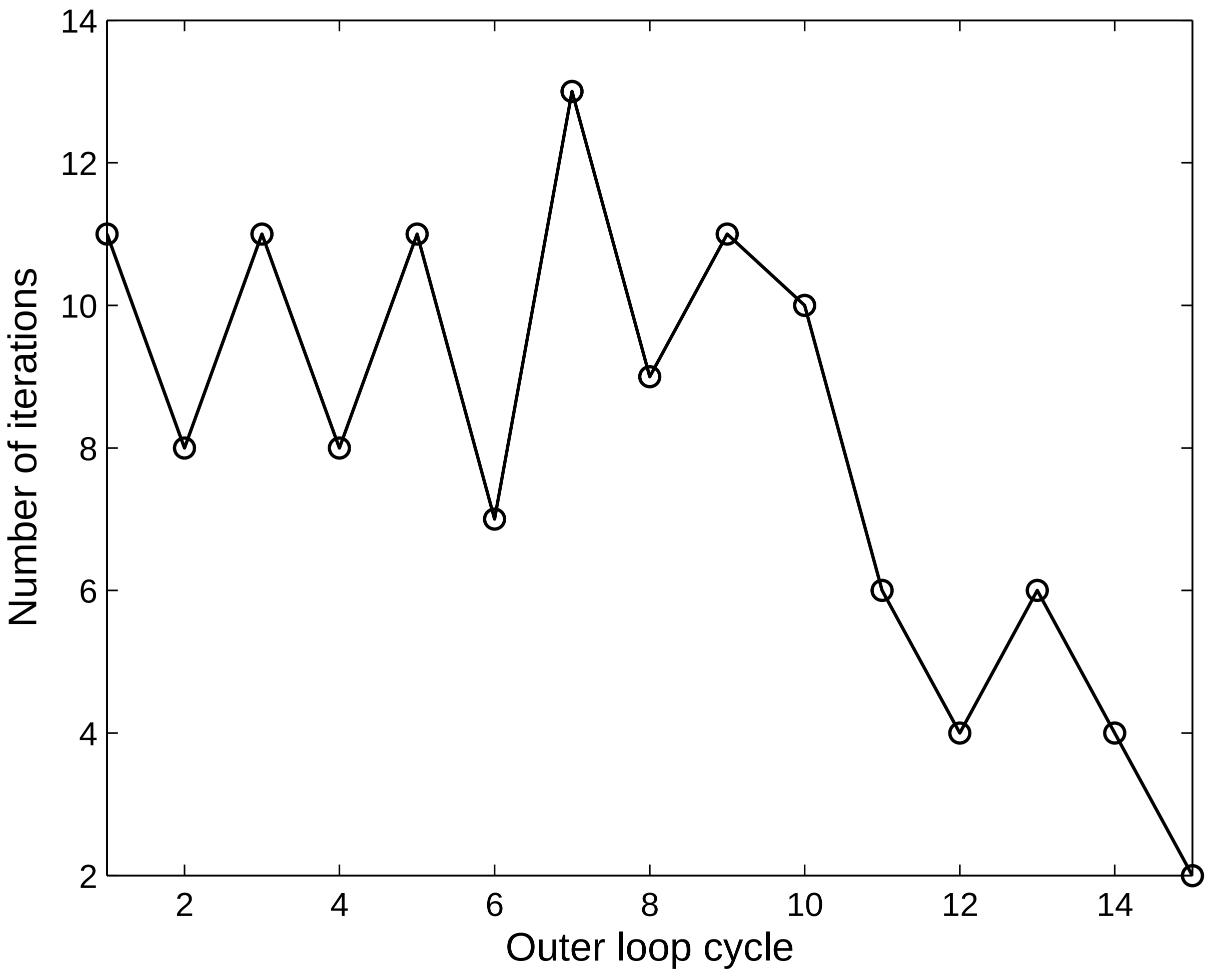




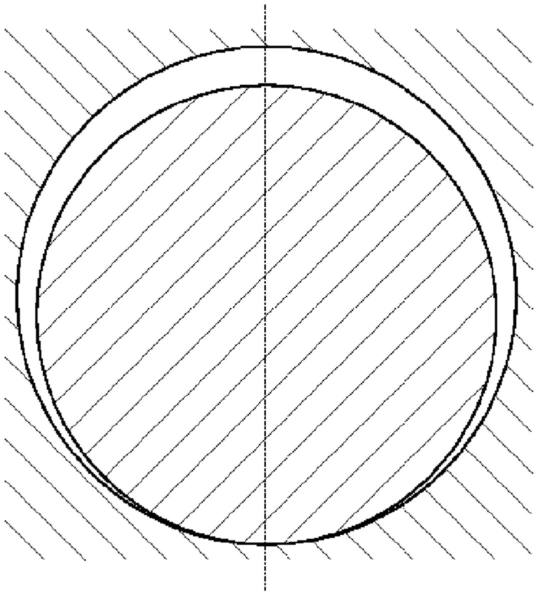
Elastic problem

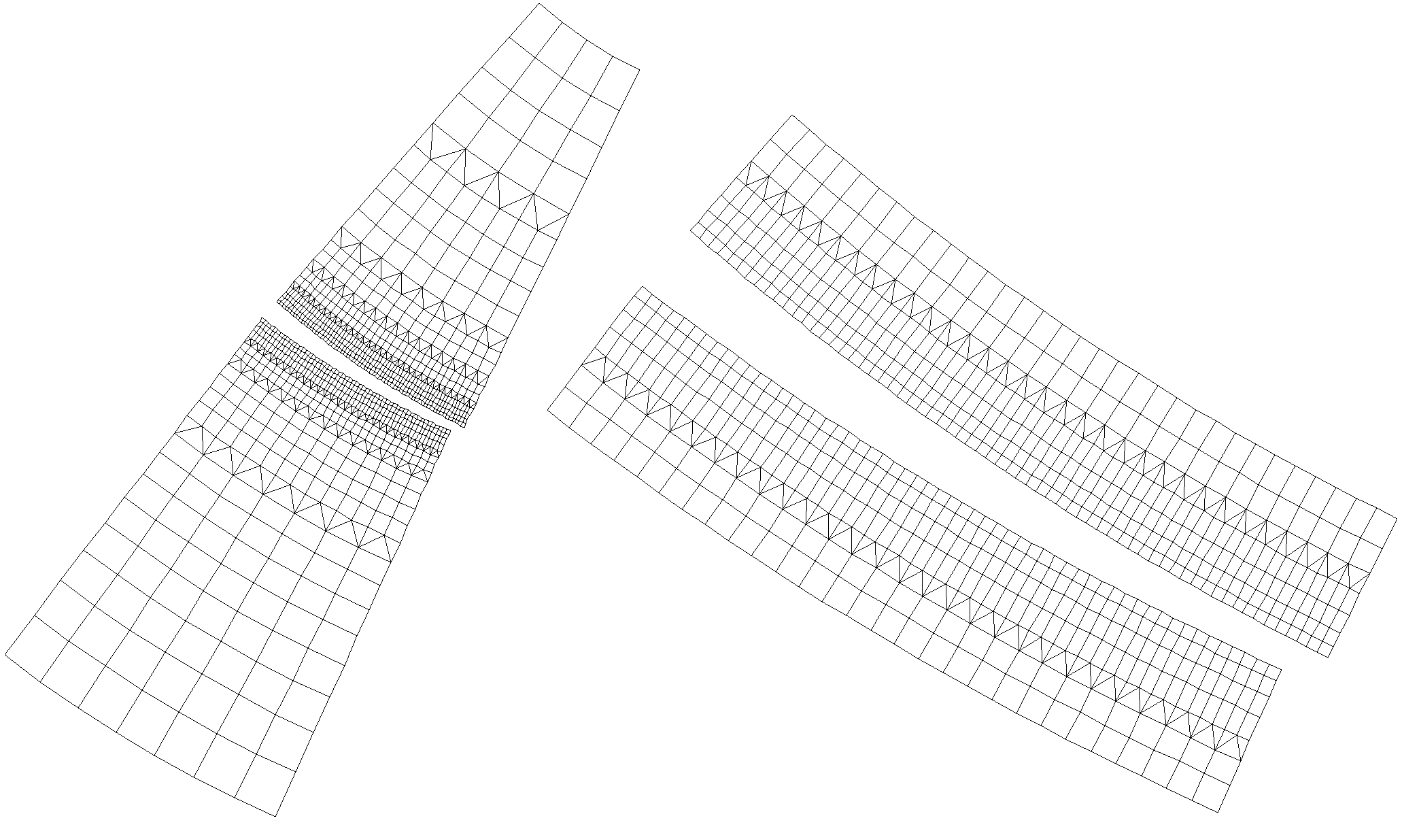
Plastic & geometrically non-linear problem
Yield stress: $\sigma_Y = 8 \times 10^8$ Pa

Two cylinders: convergence rate



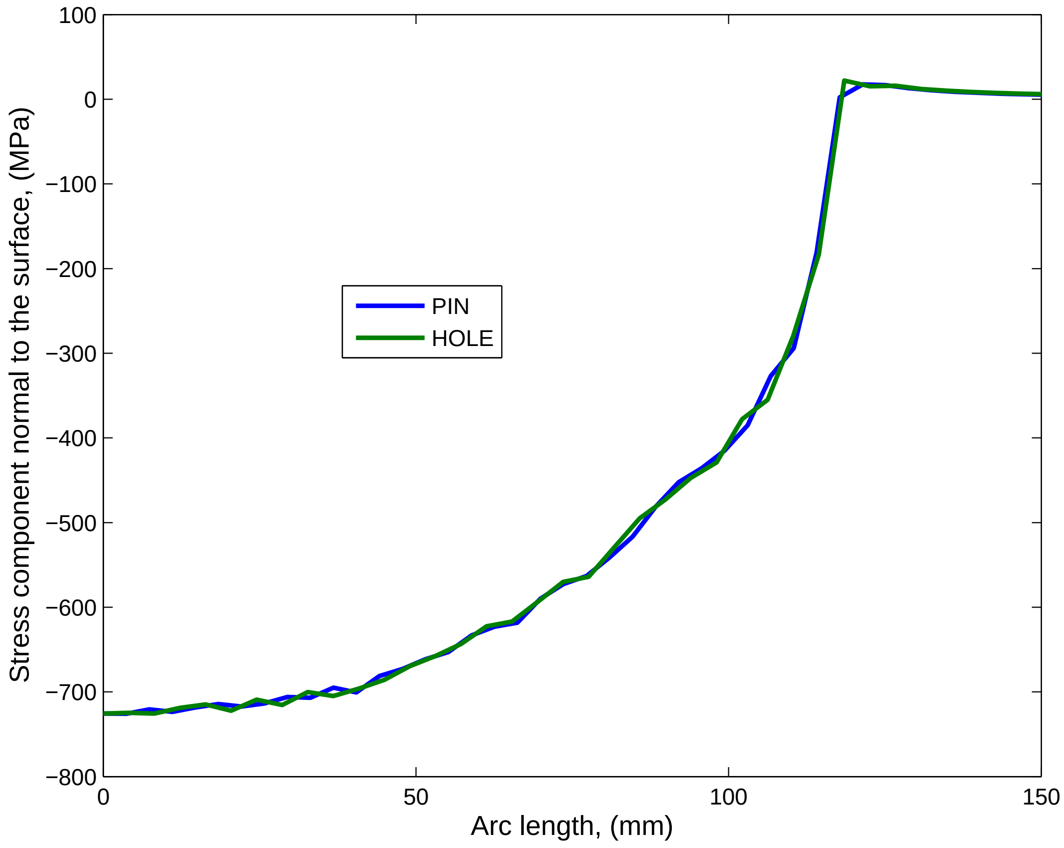
Pin-hole: mesh





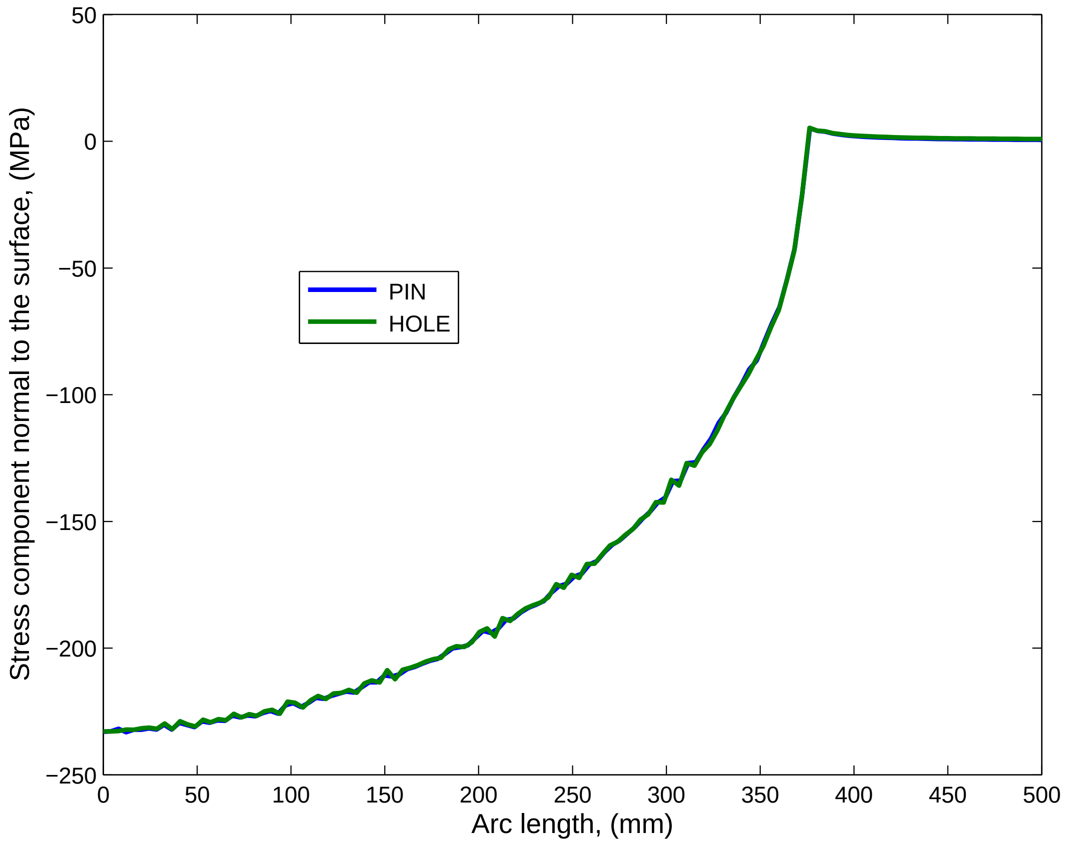
Pin-hole problem: normal stress distribution (I)

$\frac{R_{pin}}{R_{hole}} = 0.9$; Geometrically nonlinear problem; External load $Q = 133$ MN/m



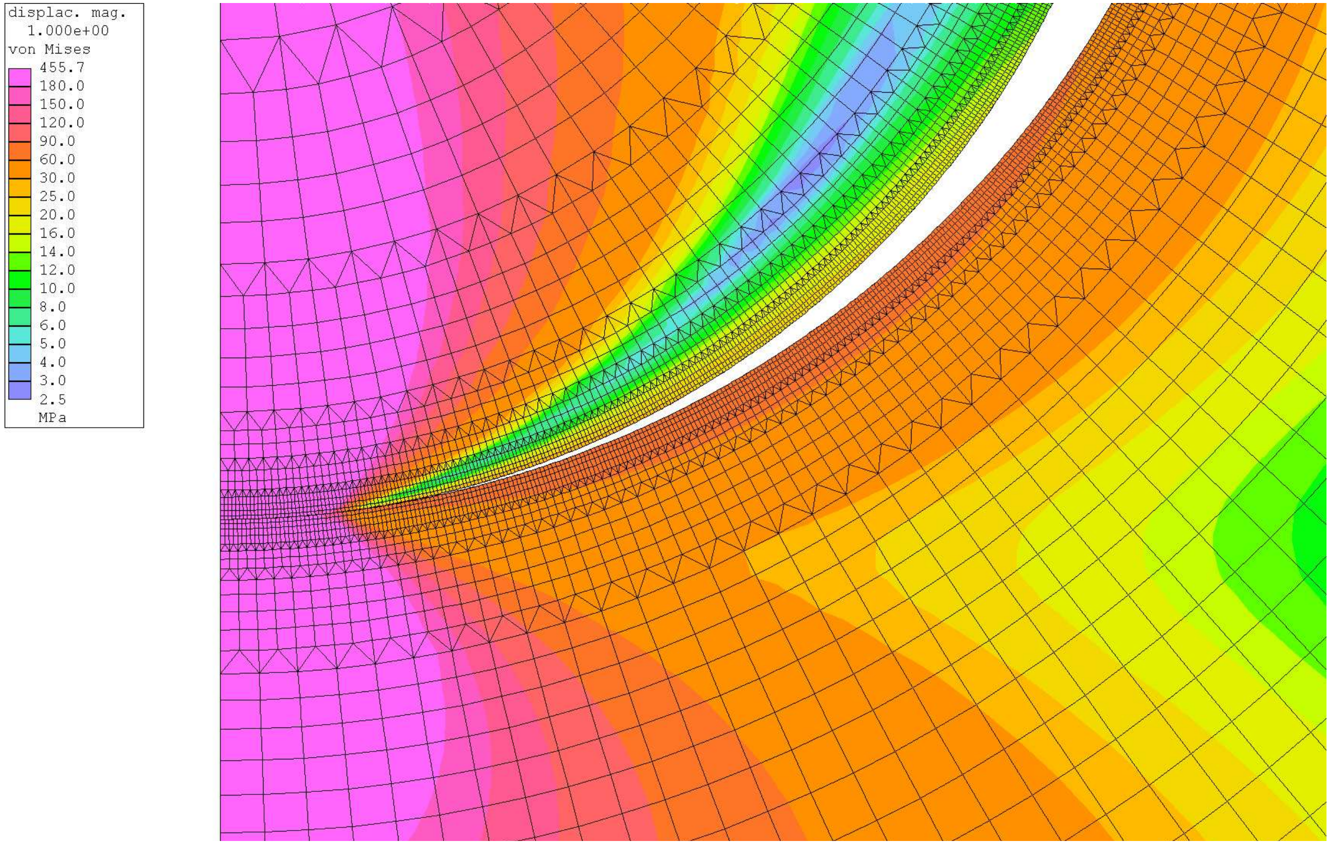
Pin-hole problem: normal stress distribution (II)

$\frac{R_{pin}}{R_{hole}} = 0.99$; Geometrically nonlinear problem; External load $Q = 133$ MN/m



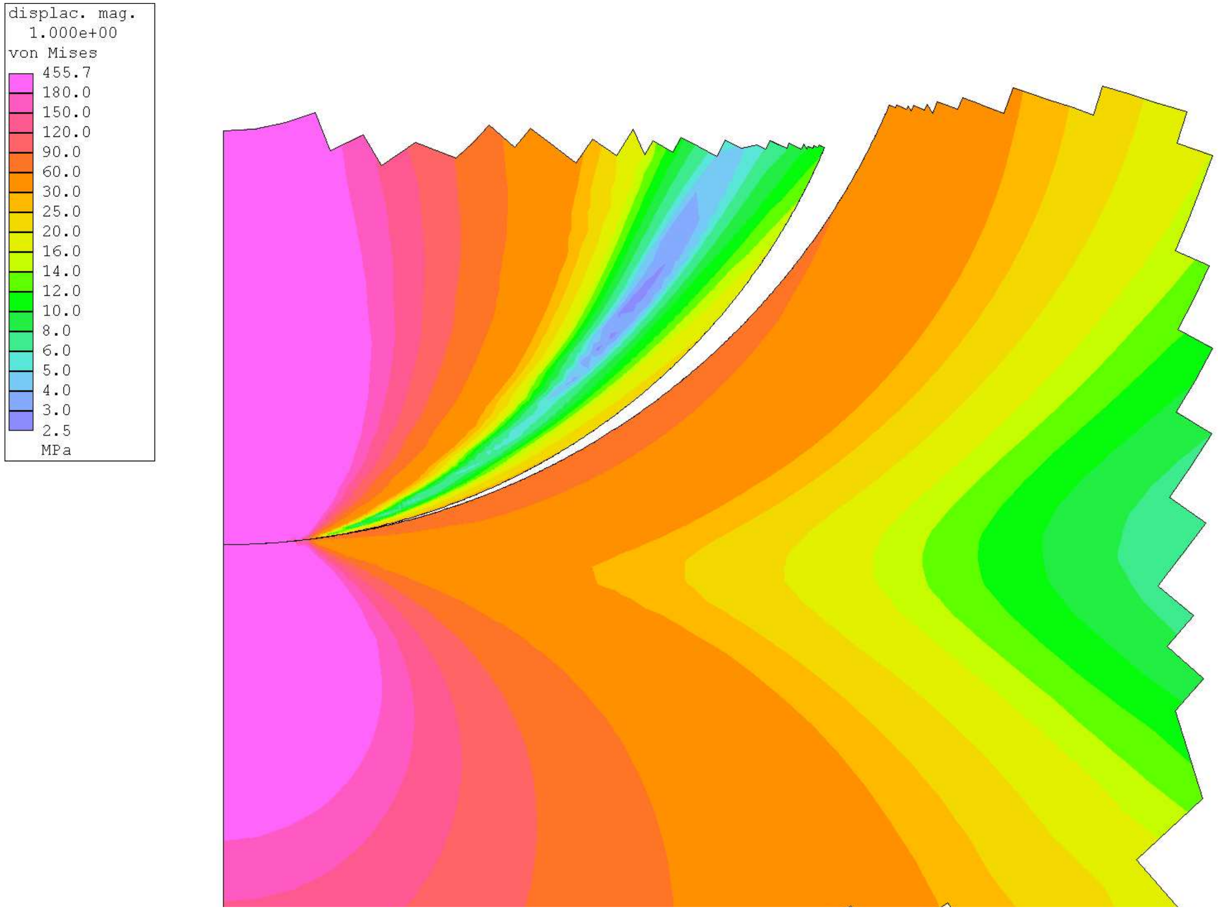
Pin-hole problem: von Mises stress (I)

$\frac{R_{pin}}{R_{hole}} = 0.9$; Geometrically nonlinear problem; External load $Q = 133$ MN/m



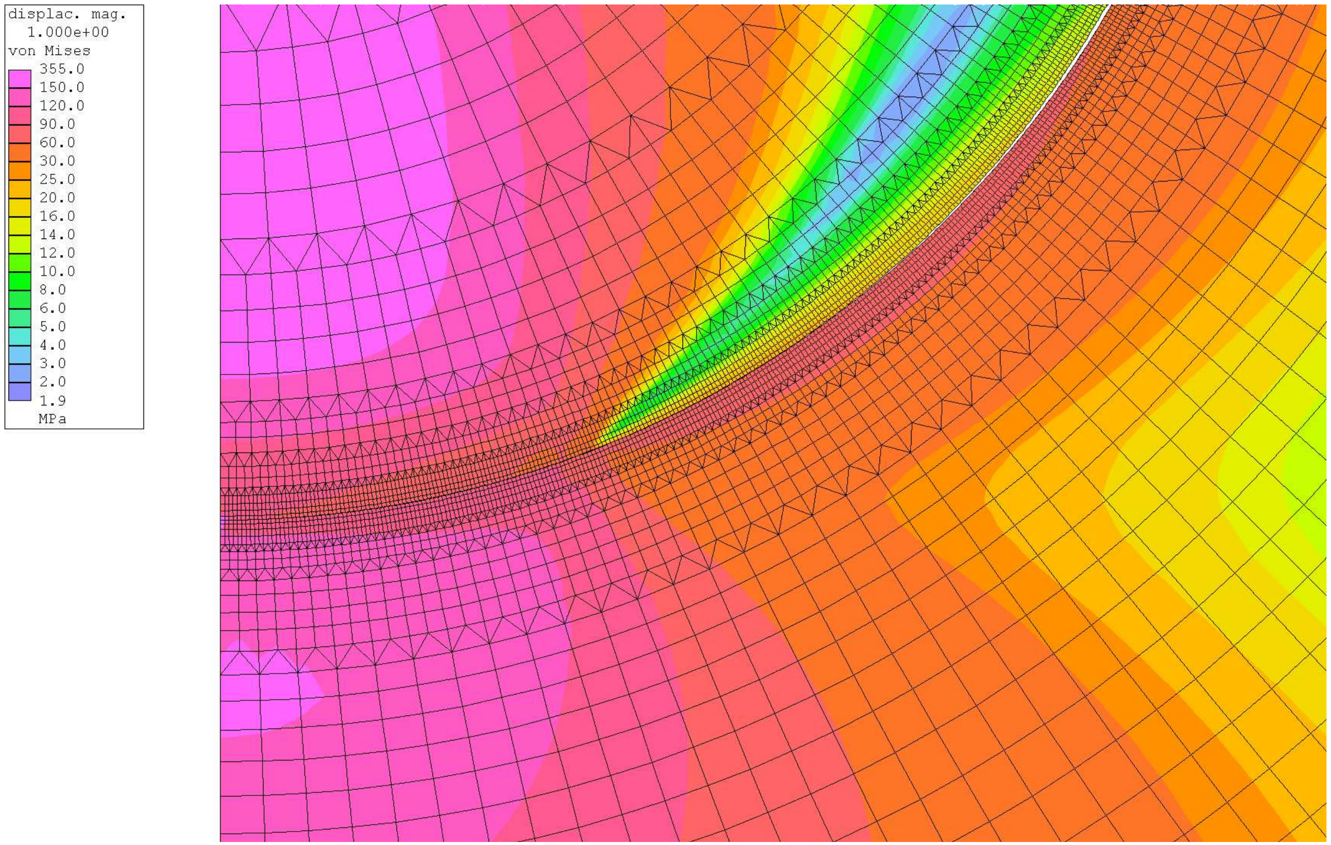
Pin-hole problem: von Mises stress (II)

$\frac{R_{pin}}{R_{hole}} = 0.9$; Geometrically nonlinear problem; External load $Q = 133$ MN/m



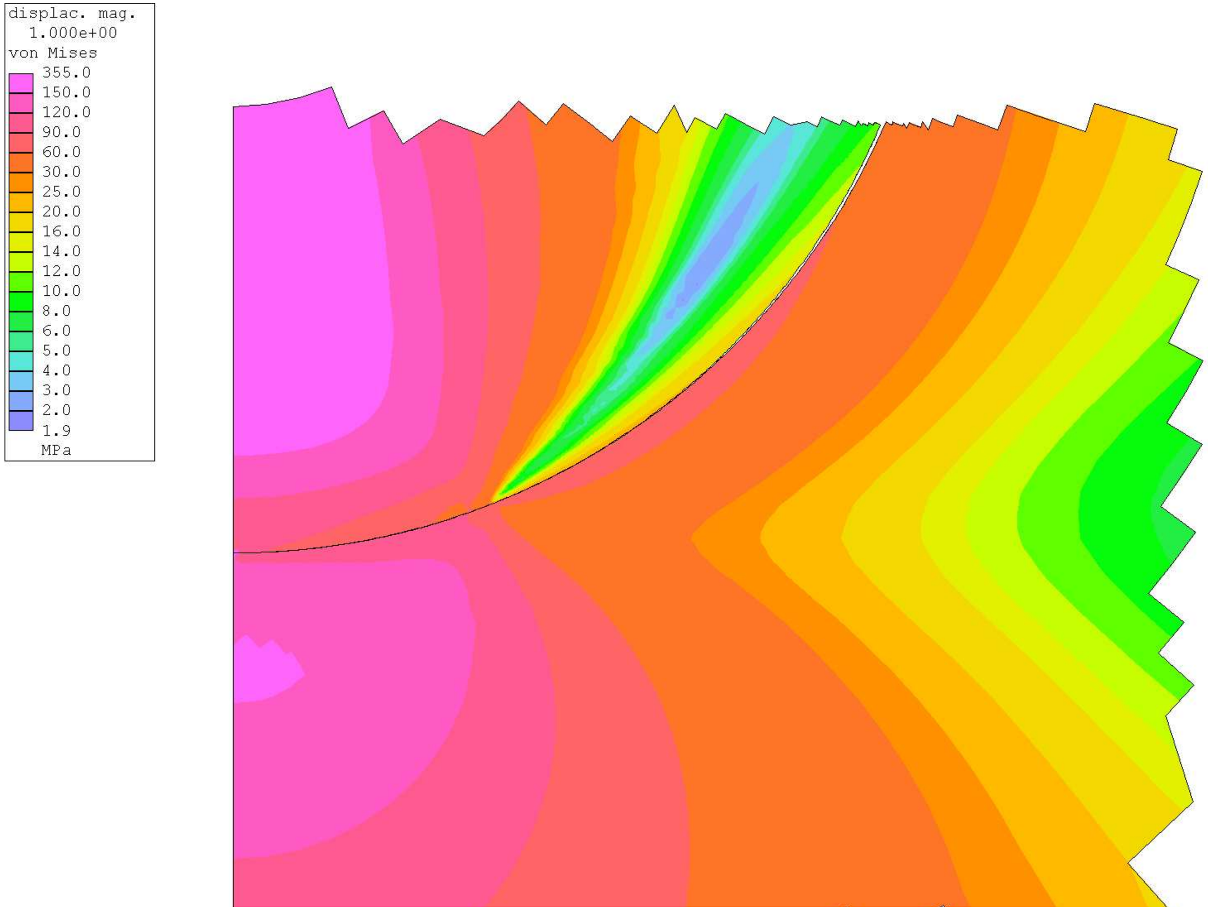
Pin-hole problem: von Mises stress (III)

$\frac{R_{pin}}{R_{hole}} = 0.99$; Geometrically nonlinear problem; External load $Q = 133$ MN/m

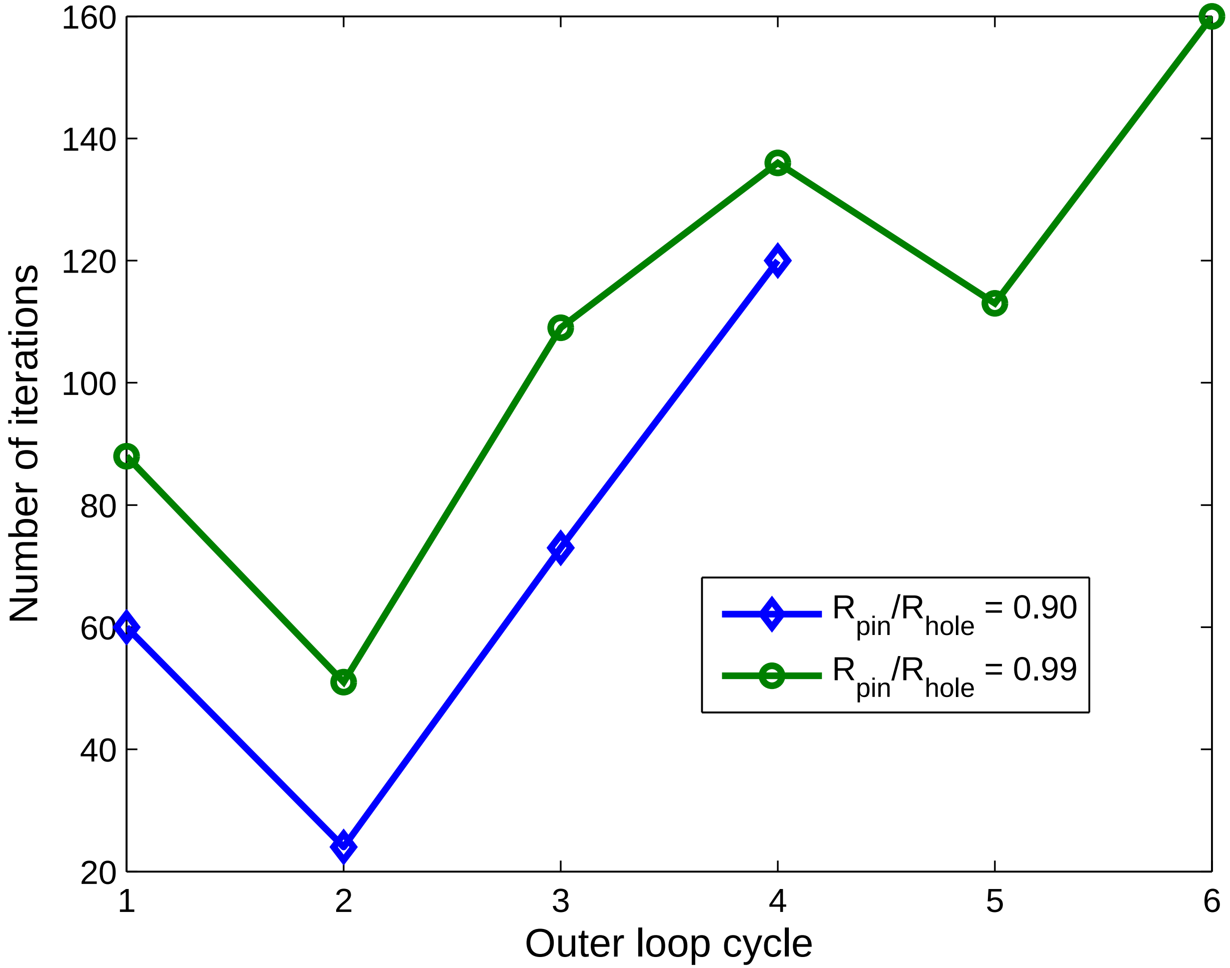


Pin-hole problem: von Mises stress (IV)

$\frac{R_{pin}}{R_{hole}} = 0.99$; Geometrically nonlinear problem; External load $Q = 133$ MN/m



Pin-hole problem: convergence rate



Pin-hole problem: solution norm