A Second Order Unconditionally Positive Space-Time Residual Distribution Method for Solving Compressible Flows on Moving Meshes

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- Long term research effort in the design of numerical methods
  - Arbitrary unstructured meshes (Generation of structured mesh is extremely difficult for complex problems)
  - Complex local features of the flow resolved in an accurate and monotone manner (Accuracy and monotonicity are contradicting)
  - The accuracy should not depend dramatically on the mesh quality (Less requirements on mesh quality)
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- Multidimensional Upwind Residual Distribution Schemes (RDS)
  - State of the art method for steady problems
    - 3D turbulent Navier-Stokes equations
    - Magneto-Hydro-Dynamic equations

- Roe ’81; van der Weide ’98, Abgrall ’01, Abgrall & Mezine ’04
- Unsteady: Maerz ’96, Ferrante ’97
- Space-time schemes since 2001:
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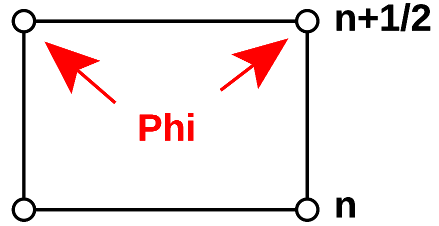
- Two layer space-time scheme of Abgrall & Mezine
  - Lower layer $t \in [t^n, t^{n+1/2}]$: N scheme + Crank-Nicholson time integration + limiting procedure (modification of distibution coefficients)
    - Second order and positive method
      - time-step restriction



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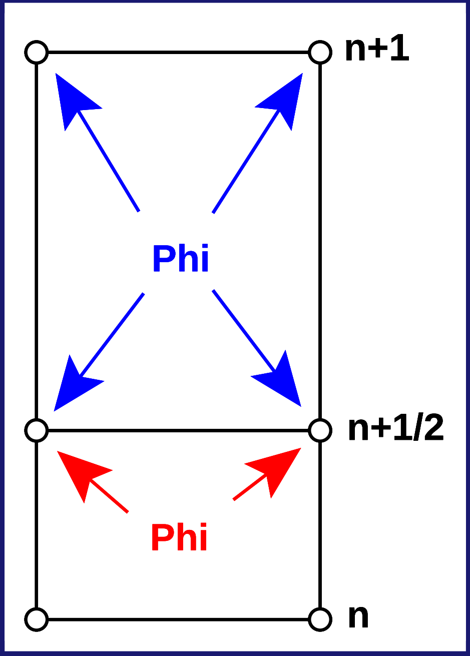

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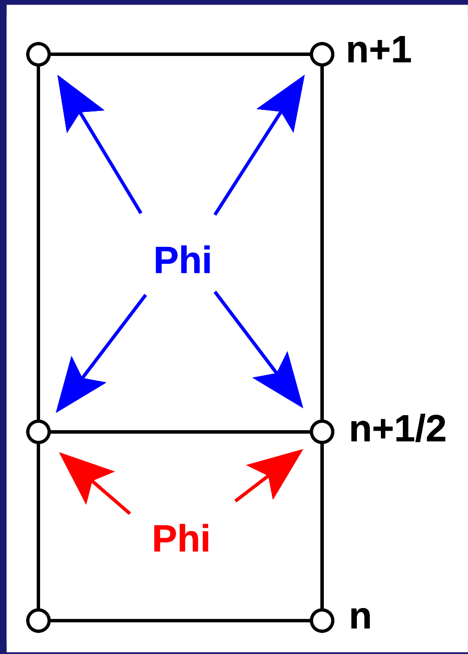
- Can be written as a scheme, distributing space-time residual φ^{ST} via distribution coefficients β_i



- **Problem:**

$$\frac{1}{J_{\mathcal{A}_t}} \frac{\partial J_{\mathcal{A}_t} \mathbf{u}}{\partial t}\bigg|_{\vec{Y}} + \vec{\nabla}_x \cdot [\vec{\mathbf{f}}(\mathbf{u}) - \mathbf{u}\vec{w}] = 0, \qquad \mathbf{u} : R^d \times R^+ \rightarrow R^q, \qquad \vec{\mathbf{f}} : R^q \rightarrow R^{d\times q}$$

+ Initial and boundary conditions

- Discretized on unstructured grid consisting of simplex elements
- Unknowns stored in vertices
- Quasi-linear form (+conservative linearization for the scheme)

$$\frac{1}{J_{\mathcal{A}_t}} \frac{\partial J_{\mathcal{A}_t} \mathbf{u}}{\partial t}\bigg|_{\vec{Y}} + \left(\frac{\partial \vec{\mathbf{f}}}{\partial \mathbf{u}} - \mathbf{I}\vec{w} \right) \cdot \vec{\nabla}_x \mathbf{u} - \mathbf{u} \nabla_x \cdot \vec{w} = 0$$

- with identity

$$\nabla_x \cdot \vec{w}(\vec{x}, t) = \frac{1}{J_{\mathcal{A}_t}(\vec{Y}, t)} \frac{\partial J_{\mathcal{A}_t}}{\partial t}(\vec{Y}, t)$$

- **Contribution from one element:**

$$\frac{\mu(E^{n+1/2})\mathbf{u}_i^{n+1/2} - \mu(E^n)\mathbf{u}_i^n}{d+1} + \frac{\Delta t_1}{2}\left(\varphi_i^{N,n+1/2} + \varphi_i^{N,n}\right)$$

$$-\frac{\sum_{j\in E}\mathbf{u}_j^{n+1/2} + \mathbf{u}_j^n}{2(d+1)}\,\frac{\mu(E^{n+1/2}) - \mu(E^n)}{d+1} = \varphi_i^{E^{ST},n+1/2,\text{lower}}.$$

- **Contribution from the (spatial) N scheme**

$$\varphi_i^N = \bar{\mathbf{k}}_i^+(\bar{\mathbf{u}}_i - \mathbf{u}_{\text{in}}), \quad \mathbf{u}_{\text{in}} = -\mathbf{N}\sum_{j\in E}\bar{\mathbf{k}}_j^-\bar{\mathbf{u}}_j, \quad \mathbf{N} = \left(\sum_{j\in E}\bar{\mathbf{k}}_j^+\right)^{-1}.$$

- **Upwind matrices**

$$\mathbf{k}_i = \left(\frac{\partial\vec{\mathbf{f}}}{\partial\mathbf{u}} - \mathbf{I}\vec{w}\right)\cdot\frac{\vec{n}_i}{d}.$$

with $\vec{n}_i$ normal to the face opposite to node i.

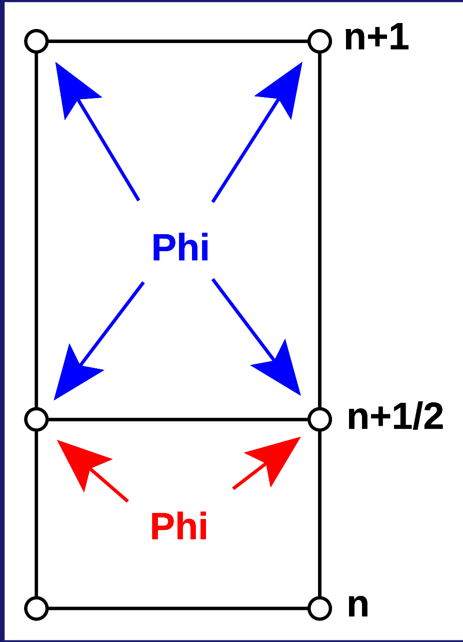
- Contribution to the $n+1$ level:

$$\frac{\mu(E^{n+1})\mathbf{u}_i^{n+1} - \mu(E^{n+1/2})\mathbf{u}_i^{n+1/2}}{d+1} + \frac{\Delta t_2}{2}\varphi_i^{N,n+1}$$

$$-\frac{\sum_{j\in E}\mathbf{u}_j^{n+1} + \mathbf{u}_j^{n+1/2}}{2(d+1)}\frac{\mu(E^{n+1}) - \mu(E^{n+1/2})}{d+1} = \varphi_i^{E^{ST},n+1,\text{upper}}.$$

- Contribution to the $n+1/2$ level:

$$\frac{\Delta t_2}{2}\varphi_i^{N,n+1/2} = \varphi_i^{E^{ST},n+1/2,\text{upper}}.$$



- Properties:
  - + Positive under arbitrary time step Δt_2
  - + Respects Geometrical Conservation Law (GCL)
  - − Not second order accurate

- **Second order accuracy:**
  - ✓ **Sufficient order of accuracy for residual computation**

$$\varphi^{ST} = \sum \varphi_i$$

  - × **Uniformly bounded distribution coefficients**

$$\beta_i \varphi^{ST} = \varphi_i$$

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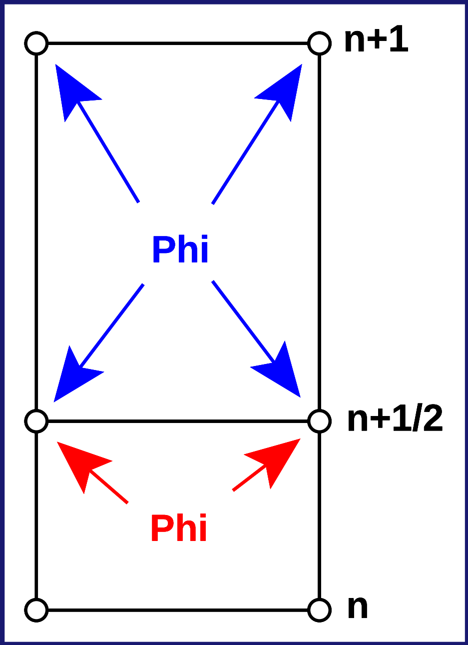
- Modification procedure:
  - Bound distribution coefficients without changing their sign (preserve positivity)

$$\beta_i^{\mathrm{modif}} = \frac{\beta_i^+}{\sum_{j \in E} \beta_j^+}$$

  - ✚ Positive, second order accurate scheme

- The problem is solved in pseudo-time (or dual time)
  - Level $n+1$:

$$\frac{\mathbf{u}_i^{n+1,m+1} - \mathbf{u}_i^{n+1,m}}{\Delta\tau} = \alpha_i \sum_{T \in i} \varphi_i^{ST,T,modif,upper,n+1}$$



  - Level $n+1/2$:

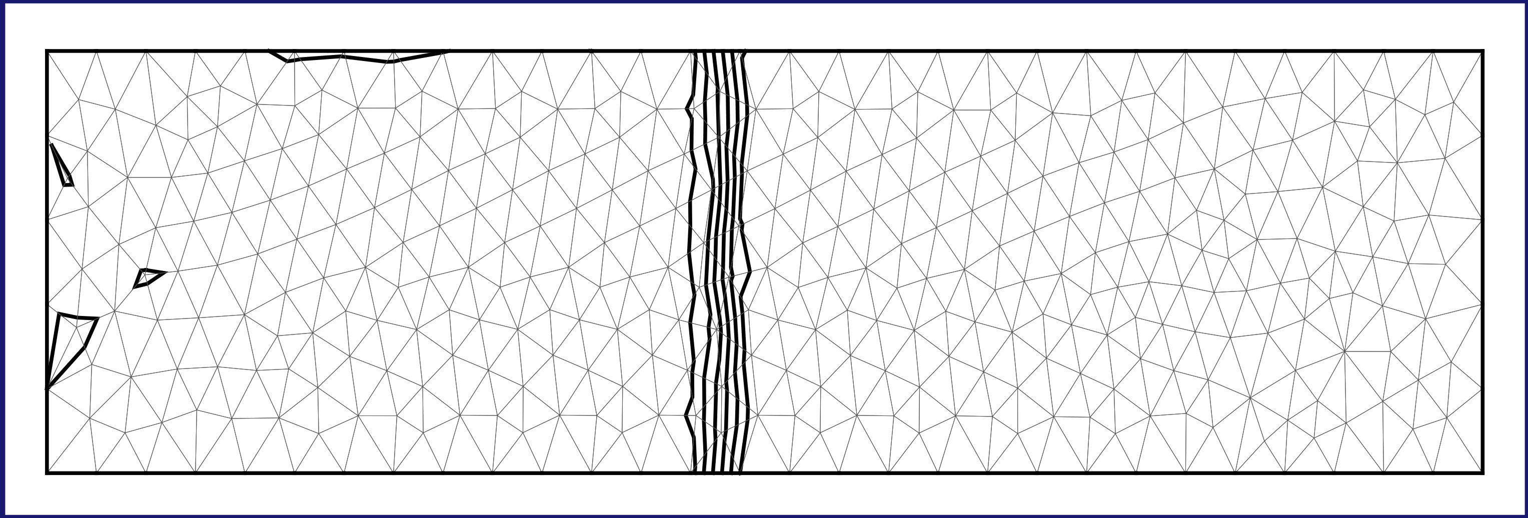
$$\frac{\mathbf{u}_i^{n+1/2,m+1} - \mathbf{u}_i^{n+1/2,m}}{\Delta\tau} =$$

$$\alpha_i \sum_{T \in i} \left(\varphi_i^{ST,T,modif,upper,n+1/2} + \varphi_i^{ST,T,modif,lower,n+1/2} \right),$$

    - Relaxation coefficient α_i from stability analysis

Piston problem – shock

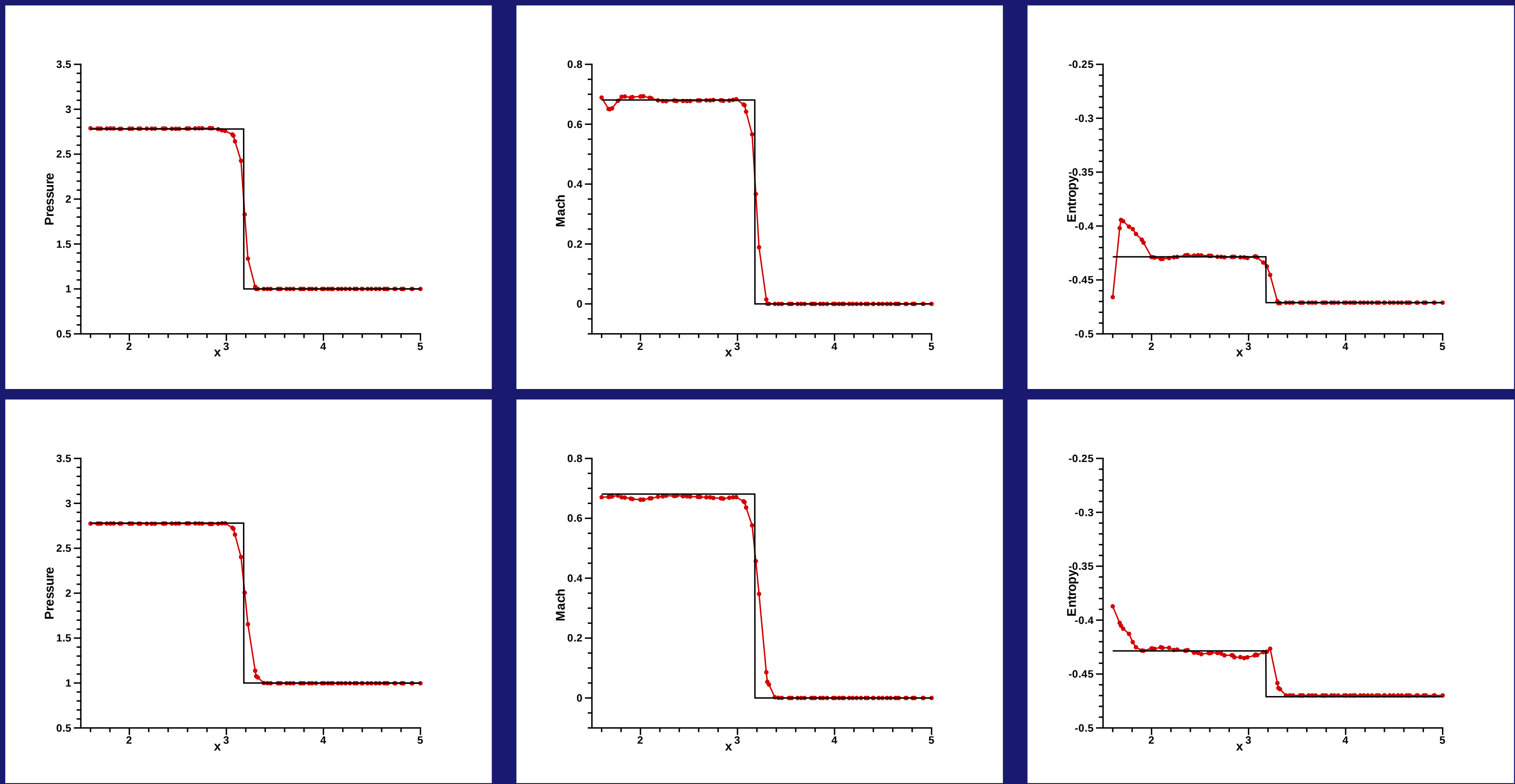
- Domain size 5×1
- A gas at the rest is enclosed between four walls, $(\rho, u, v, p)_0 = (1.4, 0, 0, 1)$
- Left wall instantaneously starts to move with uniform speed $b = 0.8$
- Shock speed is $s = 0.79461$, after shock conditions $(\rho, u, v, p) = (2.8191, 0.8, 0, 2.78)$
- Mesh – 372 nodes, 674 triangles, 60 nodes along longes side, 6 along shorter side
- Solution at $t = 2$ is shown. Mach number isolines:



Numerical results

Piston problem – shock

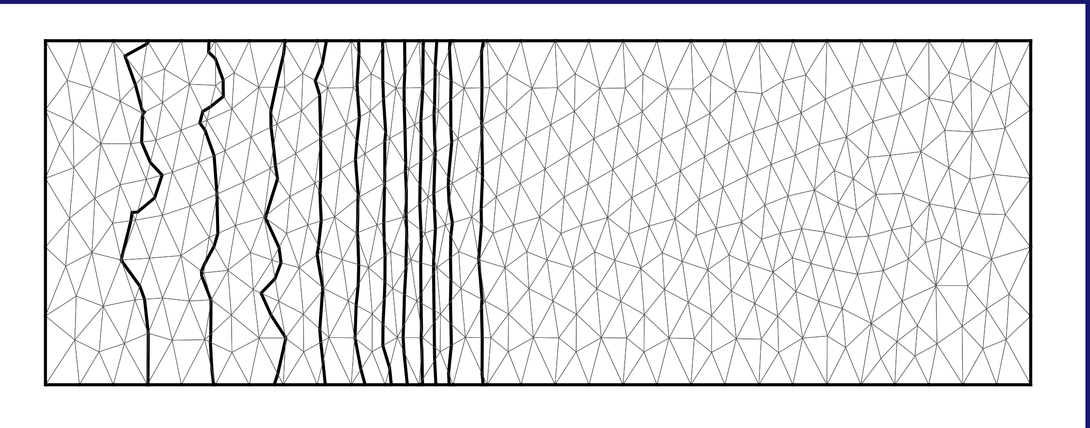
- **Top: presented scheme. Bottom: FV+Barth+Farhat's scheme A**
- **Cut in the middle:** p, Ma, s



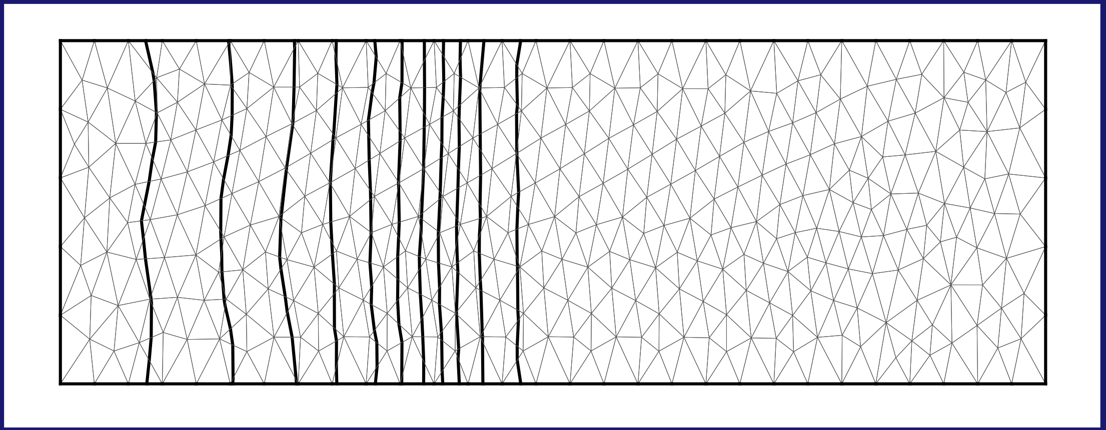
- **Both schemes: monotonne shock capturing, conservativity**
- **problems on the boundary**

Piston problem – smooth

- Similar problem as previous
- Smooth movement of the wall ($\ddot{x} = 0.2$)
- Analytical solution using characteristics theory
- Solution at time $t = 4$
- Mach number isolines:



2 layer N-modified scheme

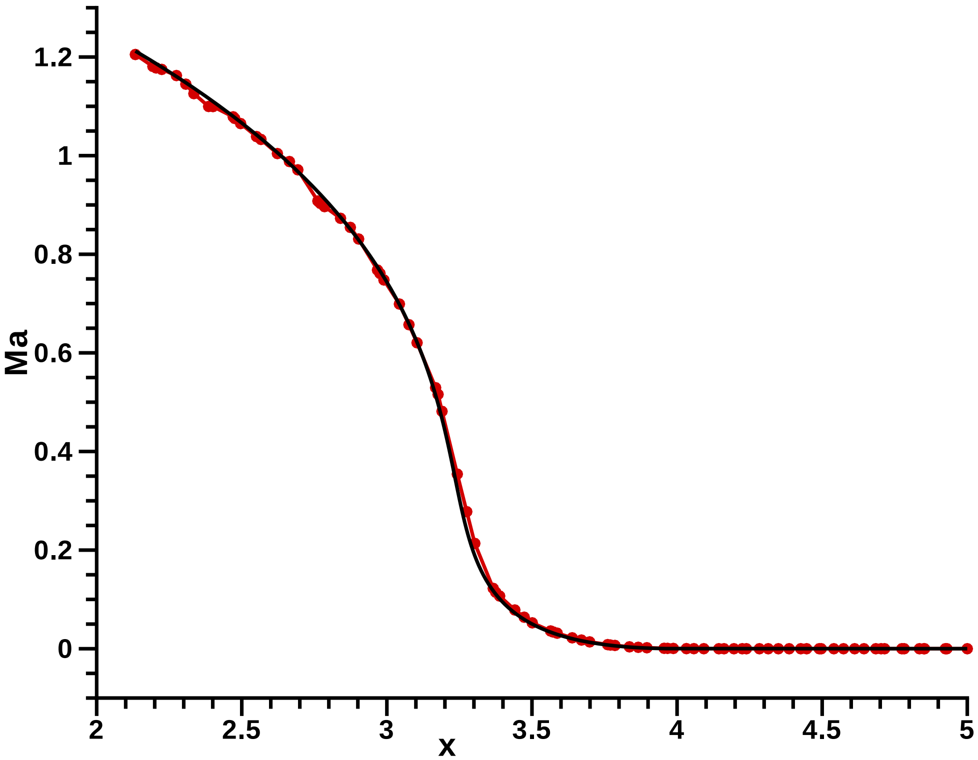


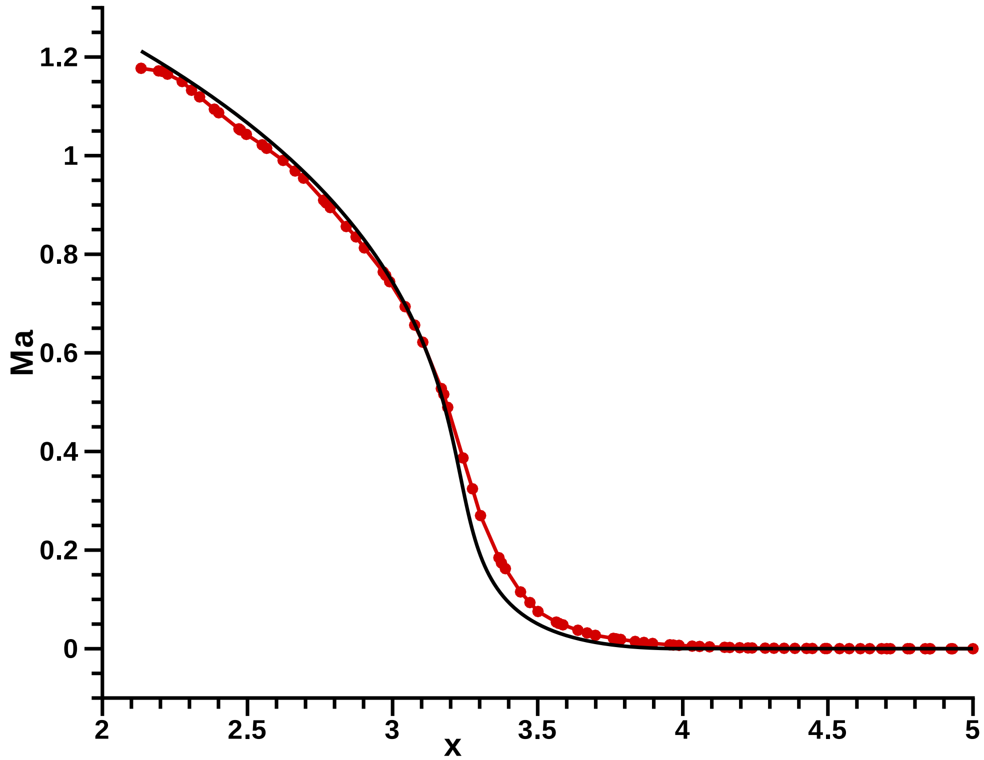
FV+Barth+Farhat's scheme A

Numerical results

Piston problem – smooth

- **Cut in the middle – Mach number:**





2 layer N-modified scheme **FV+Barth+Farhat's scheme A**

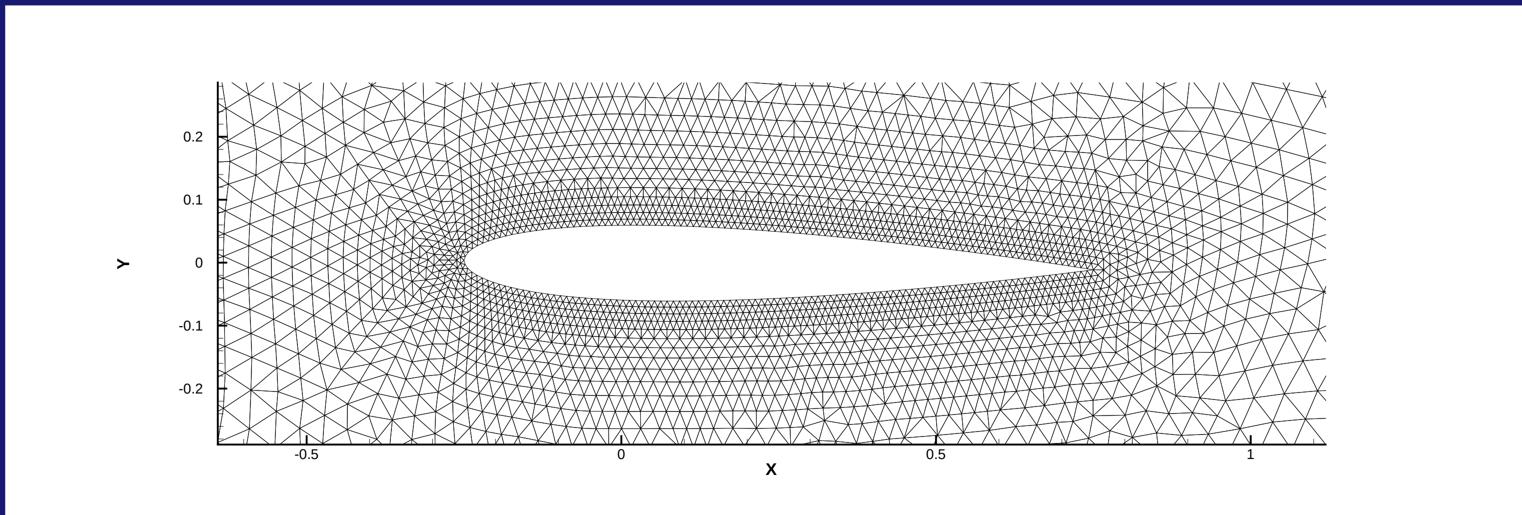
Numerical results

NACA 0012

- **Test case AGARD CT 5 (Landon 1982)**
- $Ma_\infty = 0.755$

$$\alpha = 2.51 \sin(2kt) + 0.016, \qquad k = \frac{\omega c}{2u_\infty} = 0.0814$$

- **Mesh 5711 nodes and 11153 elems, 206 nodes around the airfoil**
  - **FV uses $\approx 2\times$ more unknowns**
  - **C-N time integration, both schemes $CFL = 5$**



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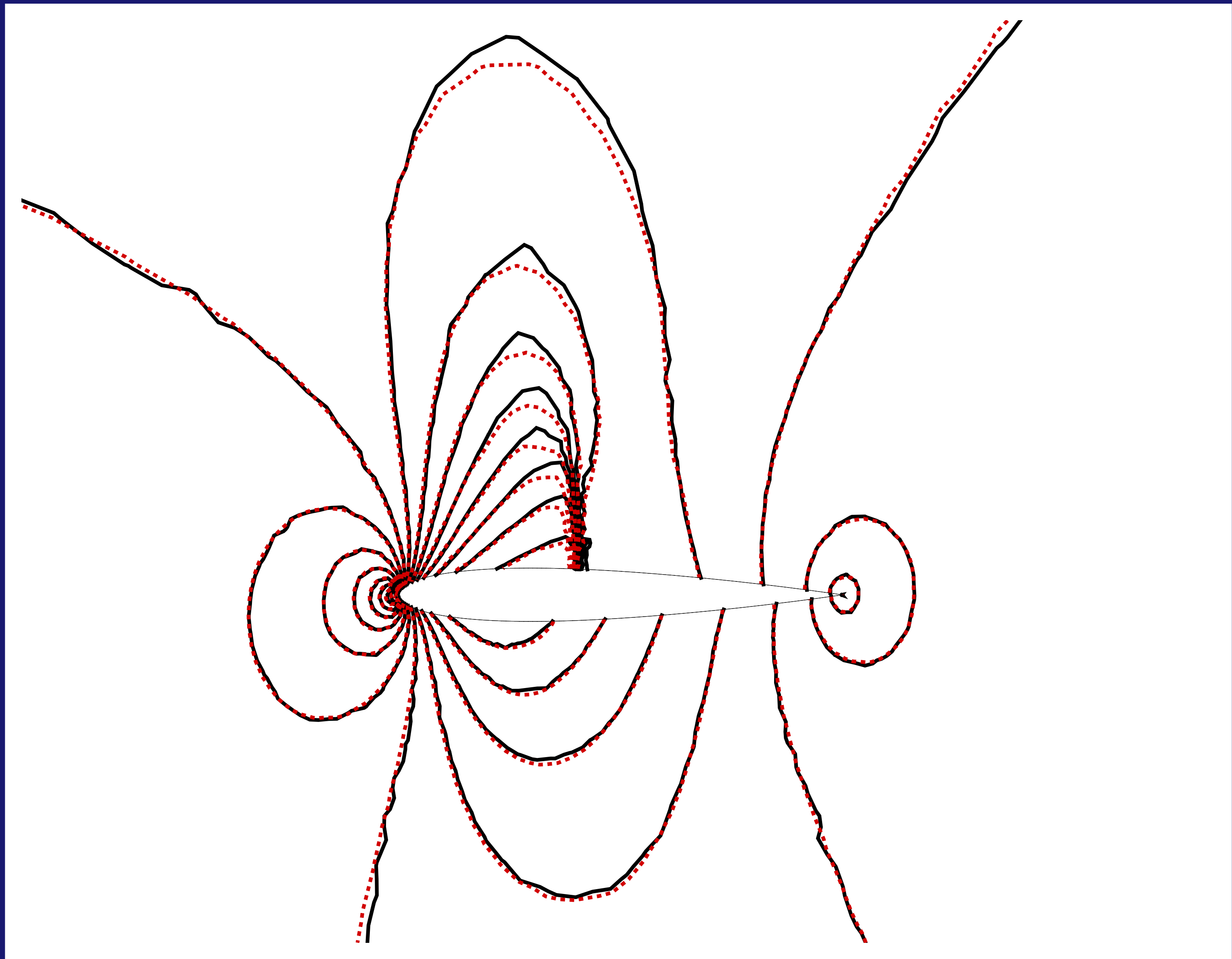

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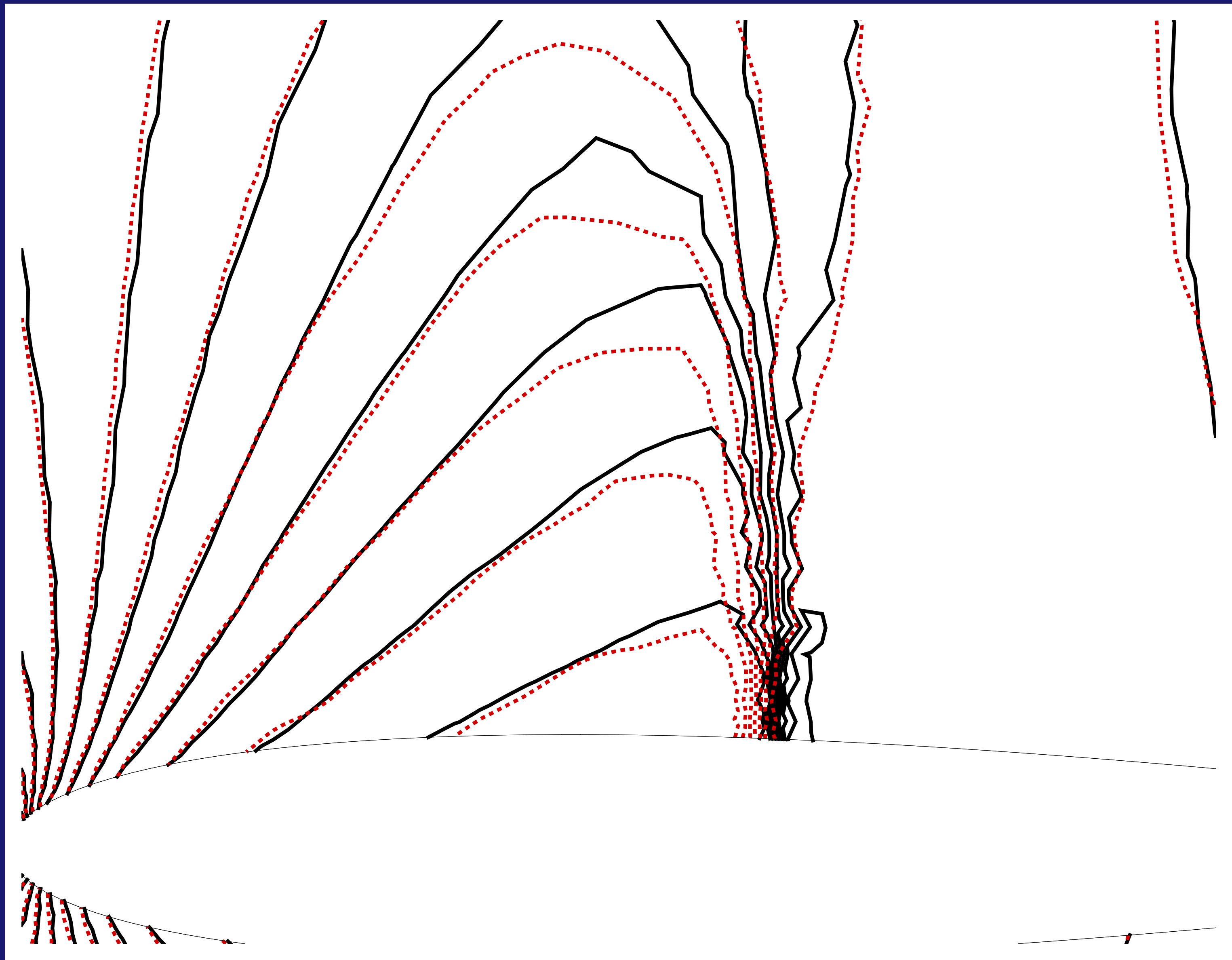

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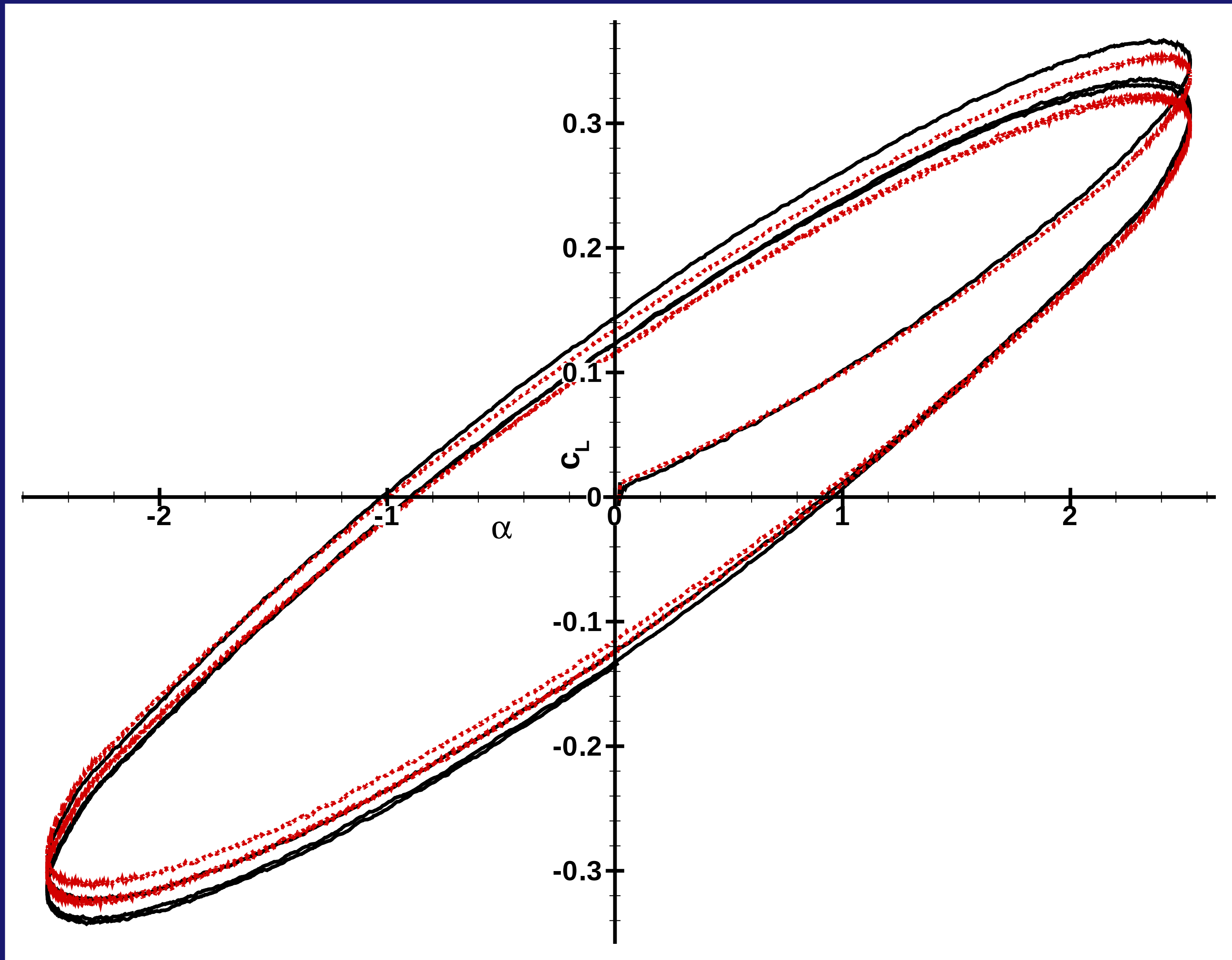

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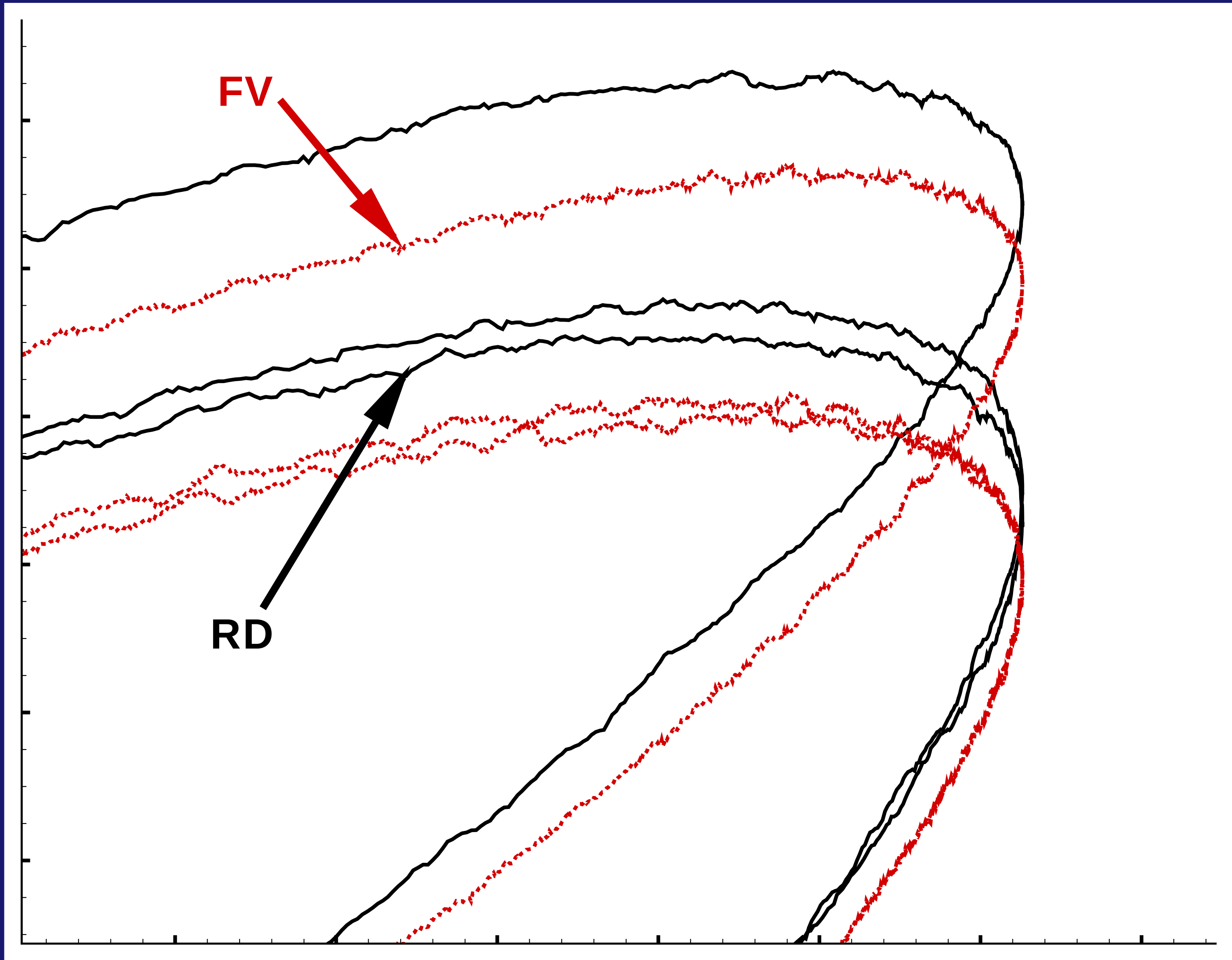

Conclusions

- An extension of 2 layer space-time RD N-modified scheme of Mezine & Abgrall for moving meshes
- Properties
  - Second order accurate
  - Positive
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Thank you for attention