On a Multigrid Preconditioned Augmented Lagrangians Applied to the Stokes and Optimization Problems

Z. Dostál and D. Lukáš[1]
Department of Applied Mathematics, VŠB–Technical University of Ostrava

Let V, Z be Hilbert spaces. Denote their respective inner products by $(.,.)_V$, $(.,.)_Z$, their dual spaces by V', Z' and the duality pairings by $\langle .,. \rangle_V$, $\langle .,. \rangle_Z$. Let further $A : V \to V'$ be a self–adjoint positive definite linear bounded operator, i.e. $\langle A.,. \rangle_V$ is an equivalent inner product on V, let $B : V \to Z'$ be another linear bounded operator and let $b \in V'$. Denote by $B^* : Z \to V'$ the adjoint operator of B. We consider the following equality constrained quadratic programming problem:

$$\min_{u \in V} h(u) \quad \text{s.t.} Bu = 0,$$

where $h(u) := (1/2)\langle Au, u \rangle_V - \langle b, u \rangle_V$.

Our development is based on the well–known augmented Lagrangian algorithm. It generates approximations of the Lagrange multipliers in the outer loop while following unconstrained auxiliary subproblems are solved in the inner loop of the k-th iteration:

$$\min_{u \in V} \left\{ h(u) + \langle Bu, p^{(k)} \rangle_Z + (\rho^{(k)}/2) \langle B^* M^{-1} Bu, u \rangle_V \right\}, \tag{1}$$

where $M : Z \to Z'$ represents an easily invertible inner product on Z. In particular, we will treat a variant of the augmented Lagrangian method proposed by Dostál, see [1], which uses an adaptive updating formula for the augmented Lagrange parameter ρ and which turned out to be very robust with respect to the additional algorithmic parameters. We will couple this outer iterative method with the conjugate gradients method for the solution of the inner subproblems (1), for which we will use one multigrid preconditioner throughout all the outer iterations. From [1] it follows that the overall algorithm is proven to have optimal, i.e. linear with respect to the dimension of V, computational complexity, which will be presented for an application to the Stokes problem. At the end, we will discuss an application of the method to a topology optimization problem in electromagntism, for which the multigrid preconditioning is based on [2].

References

[1] Dostál, Z.: Semi-monotonic inexact augmented Lagrangians for quadratic programming with equality constraints. Optim. Meth. and Soft. To appear.

[2] Schöberl, J., Zulehner, W.: On Schwarz-type smoothers for saddle-point problems. Numer. Math. **95**, 377–399 (2003).

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