The analytic singular value decomposition

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A singular value decomposition (SVD) of a real matrix $A \in \mathbb{R}^{m\times n}$, $m \geq n$, is a factorization $A = U\Sigma V^T$, where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices and $\Sigma = \operatorname{diag}(s_1, \ldots, s_n) \in \mathbb{R}^{m\times n}$. Let A depend smoothly on a parameter $t \in \mathbb{R}$, $t \in [a, b]$. The aim is to construct a path of SVD's

$$A(t) = U(t)\Sigma(t)V(t)^T \,,$$

where $U(t)$, $\Sigma(t)$ and $V(t)$ depend smoothly on $t \in [a, b]$. If A is a real analytic matrix function on $[a, b]$ then there exists *Analytic Singular Value Decomposition* (ASVD). Diagonal entries $s_i(t) \in \mathbb{R}$ of $\Sigma(t)$ are called *singular values*. Due to the requirement of smoothness, singular values may be negative and also their ordering may by arbitrary. Under certain assumptions, ASVD may be uniquely determined by the factors at $t = a$.

Our aim is to develop an efficient tool for a computation of the ASVD. In particular, the computational technique has to be suited for large sparse matrices $A(t)$. In this respect, the current numerical methods should be improved.

The idea is to trace just p, $p \leq n$, *selected* singular values $s_i(t)$ (and the relevant left/right singular vectors) only. The problem is formulated as a continuation of a set of *overdetermined nonlinear equations* that depend on $t \in \mathbb{R}$. Predictor-corrector technique is considered.

In the corrector step, we apply Gauss-Newton method and hence approximate the solution by a sequence of linear least-squares problems. The linear least-squares are computed via CGNR or CGNE. The point is that differential of the defining nonlinear equations is available by its *action* only. We need to transform CGNR and CGNE accordingly.

The resulting method exploits sparsity, guarantees a prescribed precision and computes selected singular values. Experiments show that the technique resolves quite well possible clusters of singular values.