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ERROR ESTIMATES FOR NONLINEAR CONVECTIVE PROBLEMS IN THE FINITE ELEMENT METHOD

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Abstract

We describe the basic ideas needed to obtain apriori error estimates for a nonlinear convection diffusion equation discretized by higher order conforming finite elements. For simplicity of presentation, we derive the key estimates under simplified assumptions, e.g. Dirichlet-only boundary conditions. The resulting error estimate is obtained using continuous mathematical induction for the space semi-discrete scheme.

1. Continuous problem

Let $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, be a bounded open polyhedral domain. We treat the following nonlinear convective problem. Find $u : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\text{a)} \quad \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) = g \quad \text{in } \Omega \times (0,T), \tag{1}$$

$$\text{b)} \quad u\big|_{\partial\Omega \times (0,T)} = 0, \tag{2}$$

$$\text{d)} \quad u(x,0) = u^0(x), \quad x \in \Omega. \tag{3}$$

Here $g : \Omega \times (0,T) \to \mathbb{R}$ and $u^0 : \Omega \to \mathbb{R}$ are given functions. We assume that the *convective fluxes* $\mathbf{f} = (f_1, \cdots, f_d) \in (C_b^2(\mathbb{R}))^d = (C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R}))^d$, hence $\mathbf{f}$ and $\mathbf{f}' = (f_1', \cdots, f_d')$ are *globally Lipschitz continuous.*

By $(\cdot\,,\cdot)$ we denote the standard $L^2(\Omega)-$scalar product and by $\|\cdot\|$ the $L^2(\Omega)$-norm. By $\|\cdot\|_\infty$, we denote the $L^\infty(\Omega)$-norm. For simplicity of notation, we shall drop the argument Ω in Sobolev norms, e.g. $\|\cdot\|_{H^{p+1}}$ denotes the $H^{p+1}(\Omega)$-norm. We shall also denote the Bochner norms over the whole interval $[0,T]$ in concise form, e.g. $\|u\|_{L^\infty(H^{p+1})}$ denotes the $L^\infty(0,T;H^{p+1}(\Omega))$-norm.

2. Discretization

Let $\mathcal{T}_h$ be a triangulation of $\overline{\Omega}$, i.e. a partition into a finite number of closed simplexes with mutually disjoint interiors. We assume standard conforming properties: two neighboring elements from $\mathcal{T}_h$ share an entire face, edge or vertex. We set $h = \max_{K\in\mathcal{T}_h} \operatorname{diam}(K)$.

We consider a system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$, $h_0 > 0$, of triangulations of the domain Ω which are shape regular and satisfy the inverse assumption, cf. [2]. Let $p \geq 1$ be an integer. The approximate solution will be sought in the space of globally continuous piecewise polynomial functions $S_h = \{v \in C(\overline{\Omega});\, v|_{\Gamma_D} = 0,\, v|_K \in P^p(K) \forall K \in \mathcal{T}_h\}$, where $P^p(K)$ denotes the space of polynomials on K of degree $\leq p$.

We discretize the continuous problem in a standard way. Multiply (1) by a test function $\varphi_h \in S_h$, integrate over Ω and apply Green's theorem.

Definition 1. *We say that $u_h \in C^1([0,T]; S_h)$ is the space-semidiscretized finite element solution of problem (1)–(3), if $u_h(0) = u_h^0 \approx u^0$ and*

$$\frac{d}{dt}\big(u_h(t), \varphi_h\big) + b\big(u_h(t), \varphi_h\big) = l\big(\varphi_h\big)(t), \quad \forall \varphi_h \in S_h,\ t \in (0,T). \tag{4}$$

Here, we have introduced an approximation $u_h^0 \in S_h$ of the initial condition u^0 and the *convective* and *right-hand side forms* defined for $v, \varphi \in H^1(\Omega)$:

$$b(v,\varphi) = -\int_\Omega \mathbf{f}(v)\cdot \nabla\varphi \,\mathrm{d}x, \qquad l(\varphi)(t) = \int_\Omega g(t)\varphi \,\mathrm{d}x.$$

We note that a sufficiently regular exact solution u of problem (1) satisfies

$$\frac{d}{dt}\big(u(t), \varphi_h\big) + b\big(u(t), \varphi_h\big) = l\big(\varphi_h\big)(t), \quad \forall \varphi_h \in S_h,\ \forall t \in (0,T), \tag{5}$$

which implies the *Galerkin orthogonality property* of the error.

3. Key estimates of the convective terms

As usual in apriori error analysis, we assume that the weak solution u is sufficiently regular, namely

$$u, u_t \in L^2\big(0,T; H^{p+1}(\Omega)\big), \quad u \in L^\infty(0,T; W^{1,\infty}(\Omega)), \tag{6}$$

where $u_t := \frac{\partial u}{\partial t}$. For $v \in L^2(\Omega)$ we denote by $\Pi_h v$ the $L^2(\Omega)$-projection of v on S_h:

$$\Pi_h v \in S_h, \quad (\Pi_h v - v,\, \varphi_h) = 0, \qquad \forall\, \varphi_h \in S_h.$$

Let $\eta_h(t) = u(t) - \Pi_h u(t) \in H^{p+1}(\Omega)$ and $\xi_h(t) = \Pi_h u(t) - u_h(t) \in S_h$ for $t \in (0,T)$. Then we can write the error e_h as $e_h(t) := u(t) - u_h(t) = \eta_h(t) + \xi_h(t)$. By C we denote a generic constant independent of h, which may have different values in different parts of the text. Also, for simplicity of notation, we shall usually omit the argument (t) and subscript h in $\xi_h(t)$ and $\eta_h(t)$. In our analysis, we shall need the following standard inverse inequalities and approximation properties of η, (cf. [2]):

Lemma 1. *There exists a constant $C_I > 0$ independent of h s.t. for all $v_h \in S_h$*

$$|v_h|_{H^1} \leq C_I h^{-1} \|v_h\|,$$
$$\|v_h\|_\infty \leq C_I h^{-d/2} \|v_h\|.$$

Lemma 2. *There exists a constant $C > 0$ independent of h s.t. for all $h \in (0, h_0)$*

$$\begin{aligned} \|\eta_h(t)\| &\leq Ch^{p+1}|u(t)|_{H^{p+1}}, \\ \left\|\tfrac{\partial \eta_h(t)}{\partial t}\right\| &\leq Ch^{p+1}\left|\tfrac{\partial u(t)}{\partial t}\right|_{H^{p+1}}, \\ \|\eta_h(t)\|_\infty &\leq Ch|u(t)|_{W^{1,\infty}}. \end{aligned}$$

Lemma 3. *There exists a constant $C \geq 0$ independent of h, t, such that*

$$b\big(u_h(t), \xi(t)\big) - b\big(u(t), \xi(t)\big) \leq C\Big(1 + \frac{\|e_h(t)\|_\infty}{h}\Big)\big(h^{2p+2}|u(t)|^2_{H^{p+1}} + \|\xi(t)\|^2\big). \quad (7)$$

Proof. The proof follows the arguments of [5], where similar estimates are derived for periodic boundary conditions or compactly supported solutions in 1D. The proof for mixed Dirichlet-Neumann boundary conditions is contained in [4]. We write

$$b(u_h, \xi) - b(u, \xi) = \int_\Omega \big(\mathbf{f}(u) - \mathbf{f}(u_h)\big)\cdot \nabla\xi \,\mathrm{d}x. \quad (8)$$

By the Taylor expansion of $\mathbf{f}$ with respect to u, we have

$$\mathbf{f}(u) - \mathbf{f}(u_h) = \mathbf{f}'(u)\xi + \mathbf{f}'(u)\eta - \frac{1}{2}\mathbf{f}''_{u,u_h} e_h^2, \quad (9)$$

where $\mathbf{f}''_{u,u_h}$ is the Lagrange form of the remainder of the Taylor expansion, i.e. $\mathbf{f}''_{u,u_h}(x,t)$ has components $f''_s\big(\vartheta_s(x,t)u(x,t) + (1-\vartheta_s(x,t))u_h(x,t)\big)$ for some $\vartheta_s(x,t) \in [0,1]$ and $s = 1, \cdots, d$. Substituting (9) into (8), we obtain

$$b(u_h, \xi) - b(u, \xi) = \underbrace{\int_\Omega \mathbf{f}'(u)\xi\cdot \nabla\xi \,\mathrm{d}x}_{Y_1} + \underbrace{\int_\Omega \mathbf{f}'(u)\eta\cdot \nabla\xi \,\mathrm{d}x}_{Y_2} \underbrace{-\frac{1}{2}\int_\Omega \mathbf{f}''_{u,u_h} e_h^2\cdot \nabla\xi \,\mathrm{d}x}_{Y_3}. \quad (10)$$

We shall estimate these terms individually.
(A) Term $\mathbf{Y_1}$: Due to Green's theorem and the boundedness of $\mathbf{f}''$ and the regularity of u, we have

$$\int_\Omega \mathbf{f}'(u)\xi\cdot \nabla\xi \,\mathrm{d}x = -\frac{1}{2}\int_\Omega \mathrm{div}\big(\mathbf{f}'(u)\big)\xi^2 \,\mathrm{d}x \leq C\|\xi\|^2.$$

(B) Term $\mathbf{Y_2}$: We define $\Pi_h^1 : (L^2(\Omega))^d \to (S_h^1)^d = \{\mathbf{v} \in (C(\overline{\Omega}))^d; \mathbf{v}|_{\Gamma_D} = 0, \mathbf{v}|_K \in (P^1(K))^d, \forall K \in \mathcal{T}_h\}$, the $(L^2(\Omega))^d$-projection onto the space of continuous piecewise linear vector functions. From standard approximation results (similar to those of Lemma 2, cf. [2]), we obtain

$$\|\mathbf{f}'(u) - \Pi_h^1(\mathbf{f}'(u))\|_\infty \leq Ch|\mathbf{f}'(u)|_{W^{1,\infty}} \leq Ch\|\mathbf{f}''\|_{L^\infty(\mathbb{R})}|u|_{L^\infty(W^{1,\infty})} = \tilde{C}h.$$

Furthermore, due to the definition of η, we have $\int_\Omega \Pi_h^1(\mathbf{f}'(u))\cdot \nabla\xi\,\eta\,\mathrm{d}x = 0$, since $\Pi_h^1(\mathbf{f}'(u))\cdot \nabla\xi \in S_h$. Therefore, by Lemmas 1, 2 and Young's inequality

$$|Y_2| = \Big|\int_\Omega \big(\mathbf{f}'(u) - \Pi_h^1(\mathbf{f}'(u))\big)\cdot \nabla\xi\,\eta\,\mathrm{d}x\Big| \le \|\mathbf{f}'(u) - \Pi_h^1(\mathbf{f}'(u))\|_\infty C_I h^{-1}\|\xi\|\|\eta\|$$
$$\le \tilde{C} h C_I h^{-1}\|\xi\|\|\eta\| \le \|\xi\|^2 + Ch^{2p+2}|u(t)|^2_{H^{p+1}}.$$

(C) Term $\mathbf{Y_3}$: We apply Lemmas 1, 2 and Young's inequality:

$$|Y_3| \le C\|e_h\|_\infty\|e_h\|C_I h^{-1}\|\xi\| \le Ch^{-1}\|e_h\|_\infty\big(Ch^{2p+2}|u(t)|^2_{H^{p+1}} + \|\xi\|^2\big).$$

□

4. Error analysis of the semidiscrete scheme

We proceed similarly as for a parabolic equation. By Galerkin orthogonality, we subtract (5) and (4) and set $\varphi_h := \xi_h(t) \in S_h$. Since $\big(\frac{\partial \xi_h}{\partial t}, \xi_h\big) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_h\|^2$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_h(t)\|^2 = b\big(u_h(t),\xi_h(t)\big) - b\big(u(t),\xi_h(t)\big) - \Big(\frac{\partial \eta_h(t)}{\partial t}, \xi_h(t)\Big).$$

For the last right-hand side term, we use the Cauchy and Young's inequalities and Lemma 2 and Lemma 3 for the convective terms. We integrate from 0 to $t \in [0,T]$,

$$\|\xi_h(t)\|^2 \le C\int_0^t \Big(1 + \tfrac{\|e_h(\vartheta)\|_\infty}{h}\Big)\Big(h^{2p+1}|u(\vartheta)|^2_{H^{p+1}} + h^{2p+2}|u_t(\vartheta)|^2_{H^{p+1}} + \|\xi_h(\vartheta)\|^2\Big)\mathrm{d}\vartheta, \quad (11)$$

where $C \ge 0$ is independent of h, t. For simplicity, we have assumed that $\xi_h(0) = 0$, i.e. $u_h^0 = \Pi_h u^0$. Otherwise we must assume e.g. $\|\xi_h(0)\|^2 \le Ch^{2p+1}|u^0|^2_{H^{p+1}}$ and include this term in the estimate.

We notice that if we knew *apriori* that $\|e_h\|_\infty = O(h)$ then the unpleasant term $h^{-1}\|e_h\|_\infty$ in (11) would be $O(1)$. Thus we could simply apply the standard Gronwall lemma to obtain the desired error estimates. We state this formally:

Lemma 4. *Let $t \in [0,T]$ and $p \ge d/2$. If $\|e_h(\vartheta)\| \le h^{1+d/2}$ for all $\vartheta \in [0,t]$, then there exists a constant C_T independent of h, t such that*

$$\max_{\vartheta\in[0,t]} \|e_h(\vartheta)\|^2 \le C_T^2 h^{2p+1}. \quad (12)$$

Proof. The assumptions imply, by the inverse inequality and estimates of η, that

$$\|e_h(\vartheta)\|_\infty \le \|\eta_h(\vartheta)\|_\infty + \|\xi_h(\vartheta)\|_\infty \le Ch|u(t)|_{W^{1,\infty}} + C_I h^{-d/2}\|\xi_h(\vartheta)\| \quad (13)$$
$$\le Ch + C_I h^{-d/2}\|e_h(\vartheta)\| + C_I h^{-d/2}\|\eta_h(\vartheta)\| \le Ch + Ch^{p+1-d/2}|u(\vartheta)|_{H^{p+1}(\Omega)} \le Ch,$$

where the constant C is independent of h, ϑ, t. Using this estimate in (11) gives us

$$\|\xi_h(t)\|^2 \le \tilde{C}h^{2p+1} + C\int_0^t \|\xi_h(\vartheta)\|^2\,\mathrm{d}\vartheta, \quad (14)$$

where the constants $\widetilde{C}, C$ are independent of h, t. Gronwall's inequality applied to (14) states that there exists a constant $\widetilde{C}_T$, independent of h, t, such that

$$\max_{\vartheta\in[0,t]} \|\xi_h(\vartheta)\|^2 + \frac{1}{2}\int_0^t |\xi_h(\vartheta)|^2_{\Gamma_N}\, \mathrm{d}\vartheta \le \widetilde{C}_T h^{2p+1},$$

which allong with similar estimates for η gives us (12). $\square$

Now it remains to get rid of the *apriori* assumption $\|e_h\|_\infty = O(h)$. In [5] this is done for an explicit scheme using mathematical induction. Starting from $\|e_h^0\| = O(h^{p+1/2})$, the following induction step is proved:

$$\|e_h^n\| = O(h^{p+1/2}) \quad\Longrightarrow\quad \|e_h^{n+1}\|_\infty = O(h) \quad\Longrightarrow\quad \|e_h^{n+1}\| = O(h^{p+1/2}). \tag{15}$$

For the method of lines we have continuous time and hence cannot use mathematical induction straightforwardly. However, we can divide $[0, T]$ into a finite number of sufficiently small intervals $[t_n, t_{n+1}]$ on which "e_h *does not change too much*" and use induction with respect to n. This is essentially a *continuous mathematical induction* argument, a concept introduced in [1], which has many generalizations, cf. [3].

Lemma 5 (Continuous mathematical induction)**.** *Let $\varphi(t)$ be a propositional function depending on $t \in [0, T]$ such that*

(*i*) $\varphi(0)$ *is true,*

(*ii*) $\exists\delta_0 > 0:\ \varphi(t)$ *implies* $\varphi(t+\delta)$, $\forall t \in [0, T]\ \forall\delta \in [0, \delta_0] : t + \delta \in [0, T]$.

Then $\varphi(t)$ holds for all $t \in [0, T]$.

Remark 1 Due to the regularity assumptions, the functions $u(\cdot)$, $u_h(\cdot)$ are continuous mappings from $[0, T]$ to $L^2(\Omega)$. Since $[0, T]$ is a compact set, $e_h(\cdot)$ is a *uniformly continuous* function from $[0, T]$ to $L^2(\Omega)$. By definition,

$$\forall\epsilon > 0\ \exists\delta > 0:\ s, \bar{s} \in [0, T], |s - \bar{s}| \le \delta \implies \|e_h(s) - e_h(\bar{s})\| \le \epsilon.$$

Theorem 6 (Semidiscrete error estimate)**.** *Let $p > (1 + d)/2$. Let $h_1 > 0$ be such that $C_T h_1^{p+1/2} = \frac{1}{2} h_1^{1+d/2}$, where C_T is the constant from Lemma 4. Then for all $h \in (0, h_1]$ we have the estimate*

$$\max_{\vartheta\in[0,T]} \|e_h(\vartheta)\|^2 \le C_T^2 h^{2p+1}. \tag{16}$$

Proof. Since $p > (1 + d)/2$, h_1 is uniquely determined and $C_T h^{p+1/2} \le \frac{1}{2} h^{1+d/2}$ for all $h \in (0, h_1]$. We define the propositional function φ by

$$\varphi(t) \equiv \Big\{ \max_{\vartheta\in[0,t]} \|e_h(\vartheta)\|^2 \le C_T^2 h^{2p+1} \Big\}.$$

We shall use Lemma 5 to show that φ holds on $[0, T]$, hence $\varphi(T)$ holds, which is equivalent to (16).

(i) $\varphi(0)$ holds, since this is the error of the initial condition.
(ii) *Induction step*: We fix an arbitrary $h \in (0, h_1]$. By Remark 1, there exists $\delta_0 > 0$, such that if $t \in [0, T), \delta \in [0, \delta_0]$, then $\|e_h(t+\delta) - e_h(t)\| \le \frac{1}{2}h^{1+d/2}$. Now let $t \in [0, T)$ and assume $\varphi(t)$ holds. Then $\varphi(t)$ implies $\|e_h(t)\| \le C_T h^{p+1/2} \le \frac{1}{2}h^{1+d/2}$. Let $\delta \in [0, \delta_0]$, then by uniform continuity

$$\|e_h(t+\delta)\| \le \|e_h(t)\| + \|e_h(t+\delta) - e_h(t)\| \le \tfrac{1}{2}h^{1+d/2} + \tfrac{1}{2}h^{1+d/2} = h^{1+d/2}.$$

This and $\varphi(t)$ implies that $\|e_h(s)\| \le h^{1+d/2}$ for $s \in [0, t] \cup [t, t+\delta] = [0, t+\delta]$. By Lemma 4, φ holds on $[0, t+\delta]$. As a special case, we obtain the "induction step" $\varphi(t) \Longrightarrow \varphi(t+\delta)$ for all $\delta \in [0, \delta_0]$. □

5. Conclusion

We have presented the basic ideas behind the apriori analysis of nonlinear convective problems. To keep things as simple as possible, we have presented the analysis only for a space-semidiscrete scheme, with Dirichlet boundary conditions only. The extension to mixed boundary conditions, the extension to implicit schemes via continuation, derivation of improved estimates under the assumption $\mathbf{f} \in (C_b^3(\mathbb{R}))^d$ and the generalization to *locally Lipschitz* $\mathbf{f} \in (C^2(\mathbb{R}))^d$ can be found in [4].

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