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NUMERICAL MODELLING OF FLOW IN LOWER URINARY TRACT USING HIGH-RESOLUTION METHODS

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Abstract

We propose a new numerical scheme based on the finite volumes to simulate the urethra flow based on hyperbolic balance law. Our approach is based on the Riemann solver designed for the augmented quasilinear homogeneous formulation. The scheme has general semidiscrete wave–propagation form and can be extended to arbitrary high order accuracy. The first goal is to construct the scheme, which is well balanced, i.e. maintains not only some special steady states but all steady states which can occur. The second goal is to use this scheme as the component of the complex model of the urinary tract including chemical reactions and contraction of the bladder.

1. Introduction

The voiding is a very complex process. It consists of the transfer of information about the state of the bladder filling in to the spinal cord. Next part is the sending of the action potentials to the smooth muscle cells of the bladder. Even this process is not simple and includes the spreading of the action potential along the nerve axon and the transmission of the mediator (Ach - acetylcholine) in the synapse. The action potential starts the process of the smooth muscle contraction.

The sliding between actin and myosin causing the change of the form (length) of the muscle cell and its stiffness can be observed as a kind of growth and remodeling. This approach described e.g. in [7] is used in this model. To be able to describe the very complex processes in the SMC in the efficient form it is necessary to use the irreversible thermodynamics. This approach was described in [8].

2. Bladder contraction

The whole model of the bladder contraction is described in [6]. It consists of the following parts:

- Model of the time evolution of the Ca^{2+} concentration. The Ca^{2+} intracellular concentration is the main control parameter for the next processes and finally for the smooth muscle contraction. Its increase depends on the flux $J_{agonist}$ of the mediator (in this case acetylcholine) via the nerve synapse.

$$\begin{aligned}\frac{dc}{dt} &= J_{IP3} - J_{VOCC} + J_{Na/Ca} - J_{SRuptake} + J_{CICR} - J_{extrusion} + J_{leak} \\ &\quad + J_{stretch} \\ \frac{ds}{dt} &= J_{SRuptake} - J_{CICR} - J_{leak} \\ \frac{dv}{dt} &= \gamma(-J_{Na/K} - J_{Cl} - 2J_{VOCC} - J_{Na/Ca} - J_K - J_{stretch}) \\ \frac{dw}{dt} &= \lambda K_{activate} \\ \frac{dI}{dt} &= J_{agonist} - J_{degrad},\end{aligned} \tag{1}$$

  where the unknown functions represents: $c = c(t)$ calcium concentration in cytoplasm, $s = s(t)$ calcium concentration in ER/SR, $v = v(t)$ membrane tension, $w = w(t)$ probability of opening channels activated by Ca^{2+} and $I = I(t)$ IP3 sensitive reservoirs concentration in cytoplasm. For details and complete description of the functions and parameters see [4].

- Model of the time evolution of the phosphorylation of the light myosin chain. The muscle cell contraction is caused by the relative movement of the myosin and actin filaments. For this it is necessary that the phosphorylation of the mentioned light myosin chain on the heads of the myosin occurs.

$$\begin{aligned}\frac{dA_M}{dt} &= k_5 A_{M_p} - (k_7 + k_6) A_M, \\ \frac{dA_{M_p}}{dt} &= k_3 M_p + k_6 A_M - (k_4 + k_5) A_{M_p}, \\ \frac{dM_p}{dt} &= k_1(1 - A_M) + (k_4 - k_1) A_{M_p} - (k_1 + k_2 + k_3) M_p,\end{aligned} \tag{2}$$

  where the unknown functions represent the following: $A_M = A_M(t)$ connected cross-bridges, $A_{M_p} = A_{M_p}(t)$ connected phosphorylated cross-bridges and $M_p = M_p(t)$ unconnected phosphorylated cross-bridges. $k_6 = k_6(c)$, the other terms k_i are constant. For details and complete description of the functions and parameters see [3]. Knowing this process also the time evolution of the ATP consumption (J_{cycl}) can be determined. The ATP (adenosintriphosphate) is the main energy source for the muscle contraction.

$$\frac{dY}{dt} = -Q_Q Y + L J_{cycl}, \tag{3}$$

where $Y = Y(t)$ represents the ATP concentration, Q_Q is the damping parameter and L is the constant.

- Model of the own contraction based on the GRT and the irreversible thermodynamics. The growth and remodelling theory [2] together with the laws of irreversible thermodynamics with internal variables was applied in [8] to describe the mechano-chemical coupling of the smooth muscle cell contraction. The product of the chemical reaction affinity (the ATP hydrolysis) with its rate plays an important role in the discussed model. Further it can be assumed that the rate of the ATP hydrolysis depends on the ATP consumption. The corresponding equations in the non-dimensional form are following:

$$\begin{aligned} \dot{x} &= k_1 \left[\tau - z(x-1)\right], \\ \dot{y} &= \frac{y}{k_2}\left[x\tau - \frac{1}{2}z(x-1)^2 + C'\right], \\ \dot{z} &= \operatorname{sgn}(m) \cdot \left[r - \frac{1}{2}z(x-1)^2\right], \end{aligned} \tag{4}$$

where $x = \frac{l}{l_r}$, $y = \frac{l_r}{l_0}$, l_0 is the initial length of the muscle fibre, l_r its length after stimulation when the fibre is unloaded (s. c. resting length), l the actual length (when the contraction is isometric this is the input value), τ the stress and k is the fibre stiffness, m and r are constants. The non-dimensional values are labeled with the single quote mark. The others symbols are the parameters. The dependence of the single parts of the bladder model is illustrated at the figure 1.

3. Bladder and voiding model

To model the contraction of the bladder during the voiding process we will use the very simple model according [5]. The bladder is modelled as a hollow sphere with the output corresponding to the input into urethra. For the pressure in the bladder the following formula is introduced in [5]

$$p = \frac{V_{sh}}{3V} \cdot \tau, \qquad \tau = \frac{F}{S}, \tag{5}$$

where V_{sh} is the volume of the wall, V the inner volume, S the inner surface, F the force in the muscle cell and τ stress in the muscle fibre, which can be derived as

$$\tau = \frac{\frac{-q}{3\kappa(x\cdot y)^2} + \left[k_1 z y(x-1) + \frac{zyx}{2k_2}(x-1)^2 - \frac{xy}{k_2}C'\right]}{k_1 y + \frac{x^2 y}{k_2}}. \tag{6}$$

This will be putted into the equations for the isotonic contraction.

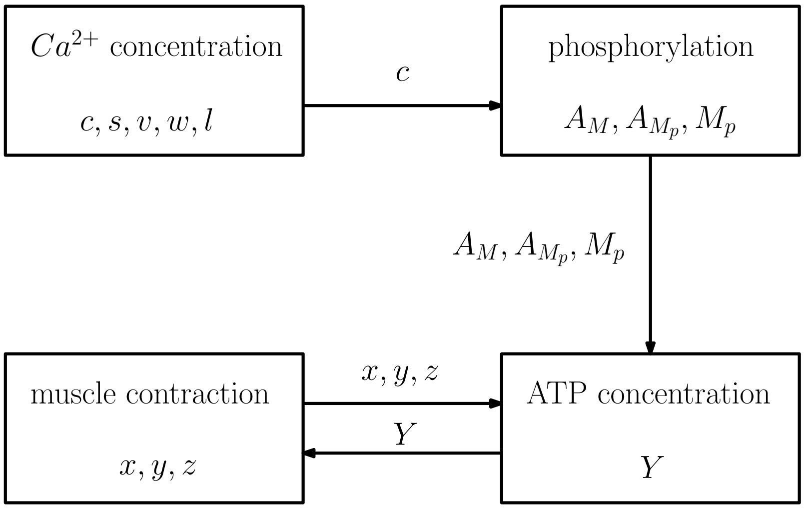


Figure 1: Unknown functions and the dependence of the bladder model parts.

4. Urethra flow

We now briefly introduce a problem describing fluid flow through the elastic tube. In the case of the male urethra, the system based on model in [9] has the following form

$$\begin{aligned} a_t + q_x &= 0, \\ q_t + \left(\frac{q^2}{a} + \frac{a^2}{2\rho\beta}\right)_x &= \frac{a}{\rho}\left(\frac{a_0}{\beta}\right)_x + \frac{a^2}{2\rho\beta^2}\beta_x - \frac{q^2}{4a^2}\sqrt{\frac{\pi}{a}}\lambda(Re), \end{aligned} \tag{7}$$

where $a = a(x,t)$ is the unknown cross-section area, $q = q(x,t)$ is the unknown flow rate (we also denote $v = v(x,t)$ as the fluid velocity, $v = \frac{q}{a}$), ρ is the fluid density, $a_0 = a_0(x)$ is the cross-section of the tube under no pressure, $\beta = \beta(x,t)$ is the coefficient describing tube compliance and $\lambda(Re)$ is the Mooney-Darcy friction factor ($\lambda(Re) = 64/Re$ for laminar flow). Re is the Reynolds number. This model contains constitutive relation between the pressure and the cross section of the tube

$$p = \frac{a - a_0}{\beta} + p_e, \tag{8}$$

where p_e is surrounding pressure.

Presented system (7) can be written in the compact matrix form

$$\mathbf{u}_t + [\mathbf{f}(\mathbf{u}, x)]_x = \boldsymbol{\psi}(\mathbf{u}, x), \tag{9}$$

with $\mathbf{u}(x,t)$ being the vector of conserved quantities, $\mathbf{f}(\mathbf{u}, x)$ the flux function and $\boldsymbol{\psi}(\mathbf{u}, x)$ the source term. This relation represents the balance laws. For the following consideration, we reformulate this problem to the nonconservative form.

4.1. Decompositions based on augmented system

The numerical scheme for solving problems (9) can be written in fluctuation form

$$\frac{\partial \mathbf{U_j}}{\partial t} = -\frac{1}{\Delta x}[\mathbf{A}^-(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j+1/2}) + \mathbf{A}(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j-1/2}) + \mathbf{A}^+(\mathbf{U}^-_{j-1/2}, \mathbf{U}^+_{j-1/2})], \quad (10)$$

where $\mathbf{A}^\pm(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j+1/2})$ are so called fluctuations. They can be defined by the sum of waves moving to the right or to the left. In what follows, we use the notation $\mathbf{U}^+_{j+1/2}$ and $\mathbf{U}^-_{j+1/2}$ for the approximations of limit values at the points $x_{j+1/2}$. The most common choices are based on the minmod function or ENO and WENO techniques [10].

The our approach is based on the extension of the system (7) by other equations. The advantage of this step is in the conversion of the nonhomogeneous system to the homogeneous quasilinear one. The augmented system can be written in the nonconservative form

$$\begin{bmatrix} a \\ q \\ \phi \\ \frac{a_0}{\beta} \\ \beta \end{bmatrix}_t + \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{q^2}{a^2} + \frac{a}{\rho\beta} & 2\frac{q}{a} & 0 & -\frac{a}{\rho} & -\frac{a^2}{\rho\beta^2} \\ 0 & -\frac{q^2}{a^2} + \frac{a}{\rho\beta} & 2\frac{q}{a} & 2\frac{q}{\rho} & -\frac{aq}{\rho\beta^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ q \\ \phi \\ \frac{a_0}{\beta} \\ \beta \end{bmatrix}_x = \mathbf{0}, \quad (11)$$

briefly $\mathbf{w}_t + \mathbf{B}(\mathbf{w})\mathbf{w}_x = \mathbf{0}$, where $\phi = av^2 + \frac{a^2}{2\rho\beta}$.

We have five linearly independent eigenvectors. The approximation is chosen to be able to prove the consistency and provide the stability of the algorithm. In some special cases this scheme is conservative and we can guarantee the positive semidefiniteness, but only under the additional assumptions (see [1]).

The fluctuations are then defined by

$$\begin{aligned} \mathbf{A}^-(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j+1/2}) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \sum_{p=1, s^{p,n}_{j+1/2}<0}^{m} \gamma^p_{j+1/2}\mathbf{r}^p_{j+1/2}, \\ \mathbf{A}^+(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j+1/2}) &= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \sum_{p=1, s^{p,n}_{j+1/2}>0}^{m} \gamma^p_{j+1/2}\mathbf{r}^p_{j+1/2}, \\ \mathbf{A}(\mathbf{U}^+_{j-1/2}, \mathbf{U}^-_{j+1/2}) &= \mathbf{f}(\mathbf{U}^-_{j+1/2}) - \mathbf{f}(\mathbf{U}^+_{j-1/2}) - \mathbf{\Psi}(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j-1/2}), \end{aligned} \quad (12)$$

where $\mathbf{\Psi}(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j-1/2})$ is a suitable approximation of the source term and $\mathbf{r}^p_{j+1/2}$ are suitable approximations of the eigenvectors of Jacobi matrix $\mathbf{f}'(\mathbf{u})$.

4.2. Steady states

It is very important to choose such approximation which conserves steady states, if these states occur exactly. Steady states mean $\mathbf{u}_t = \mathbf{0}$, therefore $[\mathbf{f}(\mathbf{u})]_x = \boldsymbol{\psi}(\mathbf{u}, x)$.

The steady state for the augmented system means $\mathbf{B}(\mathbf{w})\mathbf{w}_x = \mathbf{0}$, therefore $\mathbf{w}_x$ is a linear combination of the eigenvectors corresponding to the zero eigenvalues. The discrete form of the vector $\Delta\mathbf{w}$ corresponds to the certain approximation of these eigenvectors. It can be shown that

$$\Delta \begin{bmatrix} A \\ Q \\ \Phi \\ \frac{a_0}{\beta} \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{\bar{A}}{\rho}\frac{1}{\overline{\lambda^1\lambda^2}} \\ 0 \\ \frac{\bar{A}}{\rho}\frac{\widetilde{\lambda^1\lambda^2}}{\overline{\lambda^1\lambda^2}} \\ 1 \\ 0 \end{bmatrix} \Delta\left(\frac{a_0}{\beta}\right) + \begin{bmatrix} \frac{\bar{A}^2}{\rho\beta_{j+1}\beta_j}\frac{1}{\overline{\lambda^1\lambda^2}} \\ 0 \\ \frac{\bar{A}^2}{\rho\beta_{j+1}\beta_j}\frac{\widetilde{\lambda^1\lambda^2}}{\overline{\lambda^1\lambda^2}} - \frac{\tilde{A}^2}{2\rho\beta_{j+1}\beta_j} \\ 0 \\ 1 \end{bmatrix} \Delta\beta, \tag{13}$$

where for j-th cell $\Delta(.) = (.)_{j+1} - (.)_j, \bar{A} = \frac{A_j + A_{j+1}}{2}, \bar{\beta} = \frac{\beta_j+\beta_{j+1}}{2}$, $\tilde{A}^2 = \frac{A_j^2 + A_{j+1}^2}{2}$, $\tilde{V}^2 = |V_j V_{j+1}|$, $\bar{V}^2 = \left(\frac{V_j+V_{j+1}}{2}\right)^2$, $\widetilde{\lambda^1\lambda^2} = -\tilde{V}^2 + \frac{\bar{A}\bar{\beta}}{\rho\beta_{j+1}\beta_j}$, and $\overline{\lambda^1\lambda^2} = -\bar{V}^2 + \frac{\bar{A}\bar{\beta}}{\rho\beta_{j+1}\beta_j}$.

We use vectors on the RHS of (13) as consistent approximation of the fourth and fifth eigenvectors of the matrix $\mathbf{B}(\mathbf{w})$. The fluctuations (12) are defined with these vectors and the approximation of the source term is defined by the third line in (13)

$$\mathbf{\Psi}(\mathbf{U}^-_{j+1/2}, \mathbf{U}^+_{j-1/2}) = \frac{\bar{A}}{\rho}\frac{\widetilde{\lambda^1\lambda^2}}{\overline{\lambda^1\lambda^2}}\Delta\left(\frac{a_0}{\beta}\right) + \frac{\bar{A}^2}{\rho\beta_{j+1}\beta_j}\frac{\widetilde{\lambda^1\lambda^2}}{\overline{\lambda^1\lambda^2}} - \frac{\tilde{A}^2}{2\rho\beta_{j+1}\beta_j}\Delta\beta, \tag{14}$$

where the values $(.)_j$ and $(.)_{j+1}$ should be replaced by their appropriate reconstructed values $(.)^+_{j-1/2}$ and $(.)^-_{j+1/2}$.

5. Complex model of the bladder and the urethra

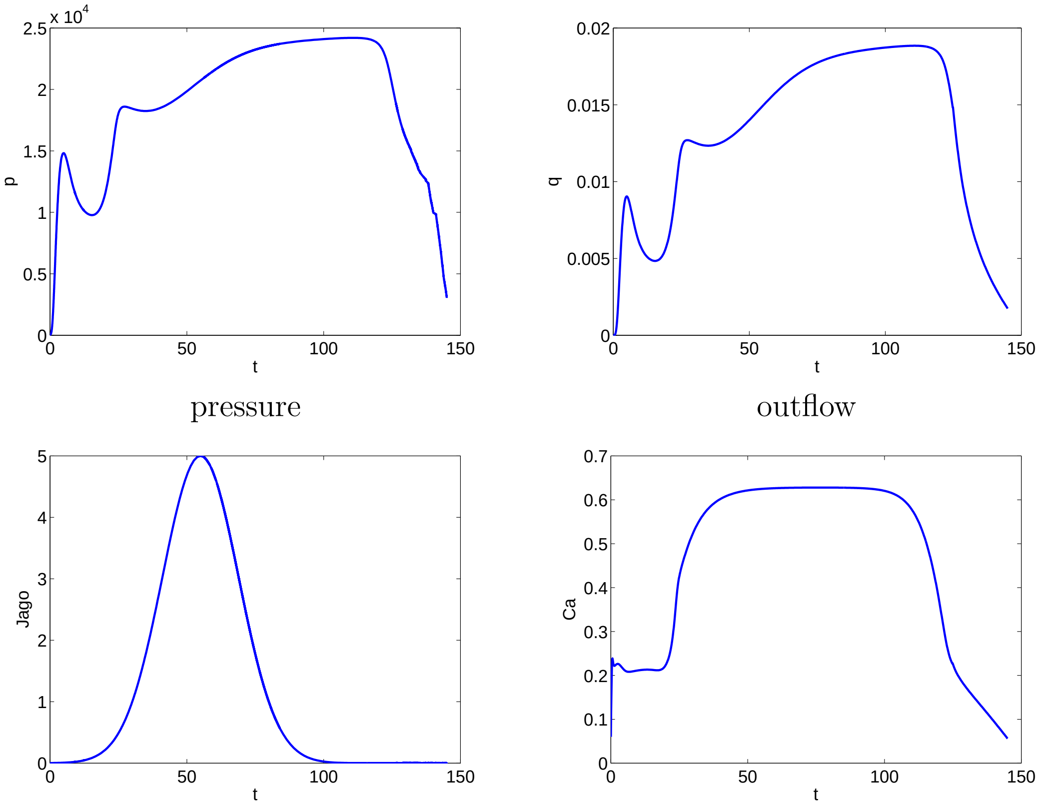
The whole voiding model consists of the detrusor smooth muscle cell model and the model of the urethra flow. It is described by the system of following ordinary differential equations:

- 12 equations describing the bladder model and the detrusor contraction during voiding - the systems (1), (2) and (4).
- $2J$ equations of urethra flow, where J is the number of finite volumes which divide the urethra region

The connection between the detrusor model and urethra flow is implemented by the relation (6) and the constitutive relation (8). The outflow of the bladder is the same as the flow rate in the first finite volume of the urethra region. So the pressure of the bladder is dependent on the flow rate in the tube (6). The cross-section in the first finite volume of the urethra region is then given by the constitutive relation (8). From the view of urethra flow, the inflow boundary condition consists of the given cross-section and extrapolation of the flow rate from the urethra region.

6. Numerical experiment including the complex model of lower urinary tract

Now we present numerical experiment based on the system of differential equations described detrusor smooth muscle cell model (12 equations) and urethral flow



pressure outflow

mediator flux Ca^{2+} concentration

Figure 2: Time evolution of the quantities at the bladder neck.

(30 equations). The parameters used in this experiment are the same as in [6]. The figures 2 illustrate time evolution of the quantities at the bladder neck.

For the simplicity the precious modelling of the synapse is neglected and the mediator flux $J_{agonist}$ is chosen - see Fig. 2. The IC units are used although in the medical paper are used for intravesical pressure cm H_2O (1 cm H_2O = 0.1 kPa) and for the outflow ml/s. The concentration is measured in μM where M = mol/l.

7. Conclusion

We presented the complex model of the lower part of the urinary tract. A simple bladder model and the detrusor contraction model were developed during voiding together with the detailed model of urethra flow. The urethra flow was described by the high-resolution positive semidefiniteness method, which preserves general steady states. For the practical application the identification of the parameters is necessary.

Acknowledgements

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