On the existence analysis of fluids whose viscosity depends on the pressure and the shear rate

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Problem definition

$$
\begin{aligned}
\operatorname{div} \boldsymbol{v} &= 0 \\
-\operatorname{div} \mathbf{T} &= \boldsymbol{f}\,, \qquad \mathbf{T}^T = \mathbf{T} \\
\mathbf{T} &= -\pi \mathbf{I} + \mathbf{S}(\pi, \mathbf{D}\boldsymbol{v})
\end{aligned}
$$

Problem definition

$$\begin{aligned} \operatorname{div} \boldsymbol{v} &= 0 \\ -\operatorname{div} \mathbf{T} &= \boldsymbol{f}, \qquad \mathbf{T}^T = \mathbf{T} \\ \mathbf{T} &= -\pi \mathbf{I} + \mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}) \end{aligned}$$

$$\boldsymbol{v}\big|_{\Gamma_D} = \mathbf{0} \qquad \text{and} \qquad -\mathbf{T}\boldsymbol{n}\big|_{\Gamma_P} = \boldsymbol{b}$$

Weak formulation

$$\begin{aligned} (q, \operatorname{div} \boldsymbol{v})_\Omega &= 0 & \forall q \in \ldots \ni \pi \\ (\mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}), \mathbf{D}\boldsymbol{w})_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega &= \langle \boldsymbol{f}, \boldsymbol{w} \rangle_\Omega - \langle \boldsymbol{b}, \boldsymbol{w} \rangle_{\Gamma_P} & \forall \boldsymbol{w} \in \ldots \ni \boldsymbol{v} \end{aligned}$$

$$
\begin{aligned}
(q, \operatorname{div} \boldsymbol{v})_\Omega &= 0 & \forall \boldsymbol{q} \in \mathrm{L}^{p'}(\Omega) \quad &\ni \pi \\
(\mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}), \mathbf{D}\boldsymbol{w})_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega &= \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w} \rangle & \forall \boldsymbol{w} \in \mathbf{W}^{1,p}_{\mathrm{b.c}}(\Omega) \quad &\ni \boldsymbol{v}
\end{aligned}
$$

Assumption (A1)

$$
\frac{\partial \mathbf{S}(\pi, \mathbf{D})}{\partial \mathbf{D}} \sim (1 + |\mathbf{D}|^2)^{\frac{p-2}{2}} \qquad p \in (1, 2)
$$

$\Longrightarrow$

$$
\begin{aligned}
|\mathbf{S}(\pi, \mathbf{D})| &\leq \frac{\sigma_1}{p-1}(1 + |\mathbf{D}|)^{p-1} \\
\mathbf{S}(\pi, \mathbf{D}) : \mathbf{D} &\geq \frac{\sigma_0}{2p}(|\mathbf{D}|^p - 1)
\end{aligned}
$$

$$
\begin{aligned}
(q_h, \operatorname{div} \boldsymbol{v}_h)_\Omega &= 0 && \forall q_h \in Q_h \ \ni \pi_h \\
(\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h), \mathbf{D}\boldsymbol{w}_h)_\Omega - (\pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega &= \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w}_h \rangle && \forall \boldsymbol{w}_h \in \boldsymbol{X}_h \ \ni \boldsymbol{v}_h
\end{aligned}
$$

Discrete solutions

$$
\begin{aligned}
\pi_h = \textstyle\sum_{i=1}^{N_h} c_h^i \alpha_h^i \quad &\in \quad Q_h \subset \mathrm{L}^{p'}(\Omega) \\
\boldsymbol{v}_h = \textstyle\sum_{i=1}^{N_h} d_h^i \boldsymbol{a}_h^i \quad &\in \quad \boldsymbol{X}_h \subset \mathbf{W}^{1,p}_{\mathrm{b.c.}}(\Omega)
\end{aligned}
$$

$$
\mathcal{P} : \mathbb{R}^{2N_h} \to \mathbb{R}^{2N_h}
$$

$$
\begin{aligned}
\mathcal{P}_i(\boldsymbol{c}_h, \boldsymbol{d}_h) \quad &:= \quad \left(\alpha_h^i, \operatorname{div} \boldsymbol{v}_h\right)_\Omega \\
\mathcal{P}_{N_h+i}(\boldsymbol{c}_h, \boldsymbol{d}_h) \quad &:= \quad \left(\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h), \mathbf{D}\boldsymbol{a}_h^i\right)_\Omega - \left(\pi_h, \operatorname{div} \boldsymbol{a}_h^i\right)_\Omega - \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{a}_h^i \rangle \\
\mathcal{P}(\boldsymbol{c}_h, \boldsymbol{d}_h) \quad &= \quad \mathbf{0}
\end{aligned}
$$

$$\begin{aligned} (q_h, \operatorname{div} \boldsymbol{v}_h)_\Omega &= 0 && \forall q_h \in Q_h \ \ni \pi_h \\ (\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h), \mathbf{D}\boldsymbol{w}_h)_\Omega - (\pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega &= \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w}_h \rangle && \forall \boldsymbol{w}_h \in \boldsymbol{X}_h \ \ni \boldsymbol{v}_h \end{aligned}$$

Test by solution

$$\begin{aligned} (\pi_h, \operatorname{div} \boldsymbol{v}_h)_\Omega - (\pi_h, \operatorname{div} \boldsymbol{v}_h)_\Omega &= 0 \\ (\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h), \mathbf{D}\boldsymbol{v}_h)_\Omega &\geq C\|\mathbf{D}\boldsymbol{v}_h\|_p^p - C \\ \Longrightarrow \ \mathcal{P}(\boldsymbol{c}_h, \boldsymbol{d}_h) \cdot (\boldsymbol{c}_h, \boldsymbol{d}_h) &\geq C\|\mathbf{D}\boldsymbol{v}_h\|_p^p - C. \end{aligned}$$

no information about pressure

$$\delta^2 \left(q_h, \pi_h^\delta\right)_\Omega + \left(q_h, \operatorname{div} \boldsymbol{v}_h^\delta\right)_\Omega = 0 \qquad \forall q_h \in Q_h \ \ni \pi_h^\delta$$

$$\left(\mathbf{S}(\pi_h^\delta, \mathbf{D}\boldsymbol{v}_h^\delta), \mathbf{D}\boldsymbol{w}_h\right)_\Omega - \left(\pi_h^\delta, \operatorname{div} \boldsymbol{w}_h\right)_\Omega = \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w}_h \rangle \qquad \forall \boldsymbol{w}_h \in \boldsymbol{X}_h \ \ni \boldsymbol{v}_h^\delta$$

$$\mathcal{P}_i(\boldsymbol{c}_h, \boldsymbol{d}_h) \ := \ \left(\alpha_h^i, \operatorname{div} \boldsymbol{v}_h\right)_\Omega + \delta^2 \left(\alpha_h^i, \pi_h\right)_\Omega$$

$\Longrightarrow$

$$\mathcal{P}(\boldsymbol{c}_h, \boldsymbol{d}_h) \cdot (\boldsymbol{c}_h, \boldsymbol{d}_h) \ \geq \ C\|\mathbf{D}\boldsymbol{v}_h^\delta\|_p^p - C + \|\delta\pi_h^\delta\|_2^2$$

$\Longrightarrow$

$$\exists \, (\pi_h^\delta, \boldsymbol{v}_h^\delta) \ \in \ Q_h \times \boldsymbol{X}_h$$

$$\|\boldsymbol{v}_h^\delta\|_{1,p}^p + \|\mathbf{S}(\pi_h^\delta, \mathbf{D}\boldsymbol{v}_h^\delta)\|_{p'}^{p'} + \|\delta\pi_h^\delta\|_2^2 \ \leq \ C \neq C(\delta, h)$$

$$\delta^2 \left(q_h, \pi_h^\delta\right)_\Omega + \left(q_h, \operatorname{div} \boldsymbol{v}_h^\delta\right)_\Omega = 0 \qquad \forall q_h \in Q_h \ \ni \pi_h^\delta$$
$$\left(\mathbf{S}(\pi_h^\delta, \mathbf{D}\boldsymbol{v}_h^\delta), \mathbf{D}\boldsymbol{w}_h\right)_\Omega - \left(\pi_h^\delta, \operatorname{div} \boldsymbol{w}_h\right)_\Omega = \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w}_h \rangle \qquad \forall \boldsymbol{w}_h \in \boldsymbol{X}_h \ \ni \boldsymbol{v}_h^\delta$$

Assumption: inf-sup condition

$$0 < \beta \leq \inf_{q_h \in Q_h} \sup_{\boldsymbol{w}_h \in \boldsymbol{X}_h} \frac{(q_h, \operatorname{div} \boldsymbol{w}_h)_\Omega}{\|q_h\|_{t'} \|\boldsymbol{w}_h\|_{1,t}} \qquad t = p, 2$$

$$\delta^2 \left(q_h, \pi_h^\delta\right)_\Omega + \left(q_h, \operatorname{div} \boldsymbol{v}_h^\delta\right)_\Omega = 0 \qquad \forall q_h \in Q_h \ni \pi_h^\delta$$
$$\left(\mathbf{S}(\pi_h^\delta, \mathbf{D}\boldsymbol{v}_h^\delta), \mathbf{D}\boldsymbol{w}_h\right)_\Omega - \left(\pi_h^\delta, \operatorname{div} \boldsymbol{w}_h\right)_\Omega = \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w}_h \rangle \qquad \forall \boldsymbol{w}_h \in \boldsymbol{X}_h \ni \boldsymbol{v}_h^\delta$$

Assumption: inf-sup condition

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$\Longrightarrow$ uniform estimate for pressure

$$\beta \|\pi_h^\delta\|_{p'} \leq \|\mathbf{S}(\pi_h^\delta, \mathbf{D}\boldsymbol{v}_h^\delta)\|_{p'} + \|-\boldsymbol{b} + \boldsymbol{f}\| \leq C$$

$$\delta^2 \left(q_h, \pi_h^\delta\right)_\Omega + \left(q_h, \operatorname{div} \boldsymbol{v}_h^\delta\right)_\Omega = 0 \qquad \forall q_h \in Q_h \ \ni \pi_h^\delta$$

$$\left(\mathbf{S}(\pi_h^\delta, \mathbf{D}\boldsymbol{v}_h^\delta), \mathbf{D}\boldsymbol{w}_h\right)_\Omega - \left(\pi_h^\delta, \operatorname{div} \boldsymbol{w}_h\right)_\Omega = \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w}_h\rangle \qquad \forall \boldsymbol{w}_h \in \boldsymbol{X}_h \ \ni \boldsymbol{v}_h^\delta$$

Existence of a discrete solution

Finite dimension!

$$\boldsymbol{v}_h^\delta \to \boldsymbol{v}_h \qquad \text{strongly in } \boldsymbol{X}_h$$

$$\pi_h^\delta \to \pi_h \qquad \text{strongly in } Q_h$$

$$\mathbf{S}(\pi_h^\delta, \mathbf{D}\boldsymbol{v}_h^\delta) \to \mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h) \qquad \text{strongly in } \mathrm{L}^{p'}(\Omega)^{d\times d}$$

$$
\begin{aligned}
(q_h, \operatorname{div} \boldsymbol{v}_h)_\Omega &= 0 && \forall q_h \in Q_h \ \ni \pi_h \\
(\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h), \mathbf{D}\boldsymbol{w}_h)_\Omega - (\pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega &= \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w}_h \rangle && \forall \boldsymbol{w}_h \in \boldsymbol{X}_h \ \ni \boldsymbol{v}_h
\end{aligned}
$$

Existence of a discrete solution

Finite dimension!

$$
\begin{aligned}
\boldsymbol{v}_h^\delta &\to \boldsymbol{v}_h && \text{strongly in } \boldsymbol{X}_h \\
\pi_h^\delta &\to \pi_h && \text{strongly in } Q_h \\
\mathbf{S}(\pi_h^\delta, \mathbf{D}\boldsymbol{v}_h^\delta) &\to \mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h) && \text{strongly in } \mathrm{L}^{p'}(\Omega)^{d \times d}
\end{aligned}
$$

$$(q, \operatorname{div} \boldsymbol{v})_\Omega = 0 \qquad \forall q \in Q \ \ni \pi$$
$$(\mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}), \mathbf{D}\boldsymbol{w})_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{X} \ \ni \boldsymbol{v}$$

Uniqueness — formally:

Let π^1, π^2 be two pressure field:

$$\begin{aligned} \pi^1 - \pi^2 &= (-\Delta)^{-1} \operatorname{div} \operatorname{div} \left(\mathbf{S}(\pi^1, \mathbf{D}\boldsymbol{v}) - \mathbf{S}(\pi^2, \mathbf{D}\boldsymbol{v}) \right) \\ &= (-\Delta)^{-1} \operatorname{div} \operatorname{div} \left((\pi^1 - \pi^2) \int_0^1 \frac{\partial}{\partial \pi} \mathbf{S}(\hat{\pi}(s), \mathbf{D}\boldsymbol{v}) \, \mathrm{d}s \, \mathrm{d}\boldsymbol{x} \right) \end{aligned}$$

$\Longrightarrow$

$$\|\pi^1 - \pi^2\|_q \lesssim \|\tfrac{\partial}{\partial \pi} \mathbf{S}(\hat{\pi}(s), \mathbf{D}\boldsymbol{v})\|_\infty \|\pi^1 - \pi^2\|_q$$

$$(q, \operatorname{div} \boldsymbol{v})_\Omega = 0 \qquad \forall q \in Q \ni \pi$$
$$(\mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}), \mathbf{D}\boldsymbol{w})_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{X} \ni \boldsymbol{v}$$

Assumption (A2) $$\left| \frac{\partial \mathbf{S}(\pi, \mathbf{D})}{\partial \pi} \right| \leq \gamma_0 (1 + |\mathbf{D}|^2)^{\frac{p-2}{4}} \leq \gamma_0$$

$$\Longrightarrow \quad d(\boldsymbol{v}, \boldsymbol{w})^2 := \int_\Omega \int_0^1 \left(1 + |\mathbf{D}\boldsymbol{w} + s(\mathbf{D}\boldsymbol{v} - \mathbf{D}\boldsymbol{w})|^2\right)^{\frac{p-2}{2}} |\mathbf{D}\boldsymbol{v} - \mathbf{D}\boldsymbol{w}|^2 \,\mathrm{d}s \,\mathrm{d}\boldsymbol{x}$$

$$d(\boldsymbol{v}, \boldsymbol{w})^2 \leq \tfrac{2}{\sigma_0} (\mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}) - \mathbf{S}(q, \mathbf{D}\boldsymbol{w}), \mathbf{D}\boldsymbol{v} - \mathbf{D}\boldsymbol{w})_\Omega + \tfrac{\gamma_0^2}{\sigma_0^2} \|\pi - q\|_2^2$$

$$\|\mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}) - \mathbf{S}(q, \mathbf{D}\boldsymbol{w})\|_2 \leq \sigma_1 d(\boldsymbol{v}, \boldsymbol{w}) + \gamma_0 \|\pi - q\|_2$$

$$\|\mathbf{D}\boldsymbol{v} - \mathbf{D}\boldsymbol{w}\|_p^2 \leq d(\boldsymbol{v}, \boldsymbol{w})^2 \|1 + |\mathbf{D}\boldsymbol{v}| + |\mathbf{D}\boldsymbol{w}|\|_p^{2-p} \leq C\, d(\boldsymbol{v}, \boldsymbol{w})^2$$

$$(q, \operatorname{div} \boldsymbol{v})_\Omega = 0 \qquad \forall q \in Q \ \ni \pi$$
$$(\mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}), \mathbf{D}\boldsymbol{w})_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{X} \ \ni \boldsymbol{v}$$

Uniqueness

$$\pi^1, \pi^2 \in Q \subseteq \mathrm{L}^{p'}(\Omega)$$
$$\boldsymbol{v}^1, \boldsymbol{v}^2 \in \boldsymbol{X} \subseteq \mathbf{W}^{1,p}_{\mathrm{b.c.}}(\Omega)$$

$$(q, \operatorname{div} \boldsymbol{v})_\Omega = 0 \qquad \forall q \in Q \ \ni \pi$$
$$(\mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}), \mathbf{D}\boldsymbol{w})_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{X} \ \ni \boldsymbol{v}$$

Uniqueness

$$\left(\mathbf{S}(\pi^1, \mathbf{D}\boldsymbol{v}^1) - \mathbf{S}(\pi^2, \mathbf{D}\boldsymbol{v}^2), \mathbf{D}\boldsymbol{w}\right)_\Omega = \left(\pi^1 - \pi^2, \operatorname{div} \boldsymbol{w}\right)_\Omega \qquad \forall \boldsymbol{w} \in \boldsymbol{X}$$

$$
\begin{aligned}
(q, \operatorname{div} \boldsymbol{v})_\Omega &= 0 && \forall q \in Q \ \ni \pi \\
(\mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}), \mathbf{D}\boldsymbol{w})_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega &= \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w} \rangle && \forall \boldsymbol{w} \in \boldsymbol{X} \ \ni \boldsymbol{v}
\end{aligned}
$$

Uniqueness

$$
\left(\mathbf{S}(\pi^1, \mathbf{D}\boldsymbol{v}^1) - \mathbf{S}(\pi^2, \mathbf{D}\boldsymbol{v}^2), \mathbf{D}\boldsymbol{w}\right)_\Omega = \left(\pi^1 - \pi^2, \operatorname{div} \boldsymbol{w}\right)_\Omega \qquad \forall \boldsymbol{w} \in \boldsymbol{X}
$$

Test by solution

$$
\left(q, \operatorname{div} \boldsymbol{v}^1\right)_\Omega = \left(q, \operatorname{div} \boldsymbol{v}^2\right)_\Omega = 0 \qquad \forall q \in Q
$$

$$
\left(\mathbf{S}(\pi^1, \mathbf{D}\boldsymbol{v}^1) - \mathbf{S}(\pi^2, \mathbf{D}\boldsymbol{v}^2), \mathbf{D}\boldsymbol{v}^1 - \mathbf{D}\boldsymbol{v}^2\right)_\Omega = 0
$$

$$
d(\boldsymbol{v}^1, \boldsymbol{v}^2) \le \frac{\gamma_0}{\sigma_0} \|\pi^1 - \pi^2\|_2
$$

$$
\begin{aligned}
(q, \operatorname{div} \boldsymbol{v})_\Omega &= 0 && \forall q \in Q \ \ni \pi \\
(\mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}), \mathbf{D}\boldsymbol{w})_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega &= \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w} \rangle && \forall \boldsymbol{w} \in \boldsymbol{X} \ \ni \boldsymbol{v}
\end{aligned}
$$

Uniqueness

$$
\left(\mathbf{S}(\pi^1, \mathbf{D}\boldsymbol{v}^1) - \mathbf{S}(\pi^2, \mathbf{D}\boldsymbol{v}^2), \mathbf{D}\boldsymbol{w}\right)_\Omega = \left(\pi^1 - \pi^2, \operatorname{div} \boldsymbol{w}\right)_\Omega \qquad \forall \boldsymbol{w} \in \boldsymbol{X}
$$

$$
C\, \|\mathbf{D}\boldsymbol{v}^1 - \mathbf{D}\boldsymbol{v}^2\|_p \le d(\boldsymbol{v}^1, \boldsymbol{v}^2) \le \frac{\gamma_0}{\sigma_0} \|\pi^1 - \pi^2\|_2
$$

Use inf-sup condition

$$
\begin{aligned}
\beta\, \|\pi^1 - \pi^2\|_2 &\le \|\mathbf{S}(\pi^1, \mathbf{D}\boldsymbol{v}^1) - \mathbf{S}(\pi^2, \mathbf{D}\boldsymbol{v}^2)\|_2 \\
&\le \sigma_1 d(\boldsymbol{v}^1, \boldsymbol{v}^2) + \gamma_0 \|\pi^1 - \pi^2\|_2 \\
&\le \gamma_0 \left(1 + \frac{\sigma_1}{\sigma_0}\right) \|\pi^1 - \pi^2\|_2
\end{aligned}
$$

$$(q, \operatorname{div} \boldsymbol{v})_\Omega = 0 \qquad \forall q \in Q \ \ni \pi$$

$$(\mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}), \mathbf{D}\boldsymbol{w})_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \boldsymbol{X} \ \ni \boldsymbol{v}$$

Uniqueness

$$\gamma_0 < \beta \frac{\sigma_0}{\sigma_0 + \sigma_1} \quad \Longrightarrow \quad \|\pi^1 - \pi^2\|_2 = \|\mathbf{D}\boldsymbol{v}^1 - \mathbf{D}\boldsymbol{v}^2\|_p = 0$$

$$
\begin{aligned}
(q_h, \operatorname{div} \boldsymbol{v}_h)_\Omega &= 0 && \forall q_h \in Q_h \ \ni \pi_h \\
(\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h), \mathbf{D}\boldsymbol{w}_h)_\Omega - (\pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega &= \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w}_h \rangle && \forall \boldsymbol{w}_h \in \boldsymbol{X}_h \ \ni \boldsymbol{v}_h
\end{aligned}
$$

Convergence of discrete solutions

We observed

$$
\|\boldsymbol{v}_h\|_{1,p} + \|\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h)\|_{p'} + \|\pi_h\|_{p'} \le C \ne C(h)
$$

$$
\Longrightarrow \qquad
\begin{aligned}
\boldsymbol{v}_h &\rightharpoonup \boldsymbol{v} && \text{weakly in } \mathbf{W}^{1,p}_{\mathrm{b.c.}}(\Omega) \\
\pi_h &\rightharpoonup \pi && \text{weakly in } \mathrm{L}^{p'}(\Omega) \\
\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h) &\rightharpoonup \overline{\mathbf{S}} && \text{weakly in } \mathrm{L}^{p'}(\Omega)^{d\times d}
\end{aligned}
$$

$$
\Longrightarrow \qquad
\begin{aligned}
\operatorname{div} \boldsymbol{v} &= 0 && \text{a.e. in } \Omega \\
\left(\overline{\mathbf{S}}, \mathbf{D}\boldsymbol{w}\right)_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega &= \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w} \rangle && \forall \boldsymbol{w} \in \mathbf{W}^{1,p}_{\mathrm{b.c.}}(\Omega)
\end{aligned}
$$

$$
\begin{aligned}
(q_h, \operatorname{div} \boldsymbol{v}_h)_\Omega &= 0 && \forall q_h \in Q_h \ \ni \pi_h \\
(\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h), \mathbf{D}\boldsymbol{w}_h)_\Omega - (\pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega &= \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w}_h \rangle && \forall \boldsymbol{w}_h \in \boldsymbol{X}_h \ \ni \boldsymbol{v}_h
\end{aligned}
$$

Convergence of discrete solutions

$$
\begin{aligned}
&(\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h) - \mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}), \mathbf{D}\boldsymbol{v}_h - \mathbf{D}\boldsymbol{v})_\Omega \\
&\qquad = (\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h), \mathbf{D}\boldsymbol{v}_h)_\Omega - \left(\overline{\mathbf{S}}, \mathbf{D}\boldsymbol{v}\right)_\Omega + o(1) \\
&\qquad = \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{v} \rangle - \left(\overline{\mathbf{S}}, \mathbf{D}\boldsymbol{v}\right)_\Omega + o(1) \\
&\qquad = o(1)
\end{aligned}
$$

$$
\Longrightarrow \qquad d(\boldsymbol{v}_h, \boldsymbol{v}) \leq \frac{\gamma_0}{\sigma_0} \|\pi_h - \pi\|_2 + o(1)
$$

$$(q_h, \operatorname{div} \boldsymbol{v}_h)_\Omega = 0 \qquad \forall q_h \in Q_h \ \ni \pi_h$$
$$(\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h), \mathbf{D}\boldsymbol{w}_h)_\Omega - (\pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega = \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w}_h \rangle \qquad \forall \boldsymbol{w}_h \in \boldsymbol{X}_h \ \ni \boldsymbol{v}_h$$

Convergence of discrete solutions

$$\left(\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h) - \overline{\mathbf{S}}, \mathbf{D}\boldsymbol{w}_h\right)_\Omega = (\pi_h - \pi, \operatorname{div} \boldsymbol{w}_h)_\Omega \qquad \forall \boldsymbol{w}_h \in \boldsymbol{X}_h$$
$$C \, \|\mathbf{D}\boldsymbol{v}_h - \mathbf{D}\boldsymbol{v}\|_p \le d(\boldsymbol{v}_h, \boldsymbol{v}) \le \frac{\gamma_0}{\sigma_0} \|\pi_h - \pi\|_2 + o(1)$$

$$\begin{aligned}
\beta \, \|\pi_h - \pi\|_2 &\le \sup_{\boldsymbol{w}_h \in \boldsymbol{X}_h} \frac{(\pi_h - \pi, \operatorname{div} \boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,2}} + (1+\beta) \inf_{q_h \in Q_h} \|q_h - \pi\|_2 \\
&= \frac{(\pi_h - \pi, \operatorname{div} \overline{\boldsymbol{w}}_h)_\Omega}{\|\overline{\boldsymbol{w}}_h\|_{1,2}} + o(1), \qquad \overline{\boldsymbol{w}}_h \rightharpoonup 0 \ \text{ weakly in } \mathbf{W}^{1,2}(\Omega) \\
&\le \|\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h) - \mathbf{S}(\pi, \mathbf{D}\boldsymbol{v})\|_2 + o(1) \\
&\le \sigma_1 d(\boldsymbol{v}_h, \boldsymbol{v}) + \gamma_0 \|\pi_h - \pi\|_2 + o(1) \\
&\le \gamma_0 \left(1 + \frac{\sigma_1}{\sigma_0}\right) \|\pi_h - \pi\|_2 + o(1)
\end{aligned}$$

$$(q, \operatorname{div} \boldsymbol{v})_\Omega = 0 \qquad \forall q \in \mathrm{L}^{p'}(\Omega) \ \ni \pi$$

$$(\mathbf{S}(\pi, \mathbf{D}\boldsymbol{v}), \mathbf{D}\boldsymbol{w})_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w} \rangle \qquad \forall \boldsymbol{w} \in \mathbf{W}^{1,p}_{\mathrm{b.c}}(\Omega) \ \ni \boldsymbol{v}$$

Convergence of discrete solutions

$$\gamma_0 < \beta \frac{\sigma_0}{\sigma_0 + \sigma_1} \quad \Longrightarrow \quad \begin{array}{l} \|\pi_h - \pi\|_2 \to 0 \\ \|\mathbf{D}\boldsymbol{v}_h - \mathbf{D}\boldsymbol{v}\|_p \to 0 \end{array}$$

Vitali's lemma $\Longrightarrow$

$$\int_\Omega \mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h)\mathbf{D}\boldsymbol{w} \to \int_\Omega \mathbf{S}(\pi, \mathbf{D}\boldsymbol{v})\mathbf{D}\boldsymbol{w}$$

$$(q_h, \operatorname{div} \boldsymbol{v}_h)_\Omega = 0 \qquad \forall q_h \in Q_h \ni \pi_h$$
$$(\mathbf{S}(\pi_h, \mathbf{D}\boldsymbol{v}_h), \mathbf{D}\boldsymbol{w}_h)_\Omega - (\pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega = \langle -\boldsymbol{b} + \boldsymbol{f}, \boldsymbol{w}_h \rangle \qquad \forall \boldsymbol{w}_h \in \boldsymbol{X}_h \ni \boldsymbol{v}_h$$

A priori error estimates

$$\mathcal{F}(\mathbf{Q}) := (1 + |\mathbf{Q}|)^{\frac{p-2}{2}} \mathbf{Q}$$
$$d(\boldsymbol{v}, \boldsymbol{w})^2 \sim \|\mathcal{F}(\mathbf{D}\boldsymbol{v}) - \mathcal{F}(\mathbf{D}\boldsymbol{w})\|_2^2$$

$$\begin{aligned}\|\mathbf{D}\boldsymbol{v} - \mathbf{D}\boldsymbol{v}_h\|_p \lesssim \|\mathcal{F}(\mathbf{D}\boldsymbol{v}) - \mathcal{F}(\mathbf{D}\boldsymbol{v}_h)\|_2 \ &\leq \ c \inf_{r_h \in Q_h} \|\pi - r_h\|_{p'} \\ &+ c \inf_{\boldsymbol{u}_h \in \boldsymbol{X}_{h,\mathrm{div}}} \left(\|\mathcal{F}(\mathbf{D}\boldsymbol{v}) - \mathcal{F}(\mathbf{D}\boldsymbol{u}_h)\|_2 + \|\mathbf{D}\boldsymbol{v} - \mathbf{D}\boldsymbol{u}_h\|_p \right)\end{aligned}$$
$$\|\pi - \pi_h\|_{p'} \leq c \|\mathcal{F}(\mathbf{D}\boldsymbol{v}) - \mathcal{F}(\mathbf{D}\boldsymbol{v}_h)\|_2^{2/p'} + c \inf_{r_h \in Q_h} \|\pi - r_h\|_{p'}$$