Numerical approach to a rate-independent model of decohesion in laminated composites

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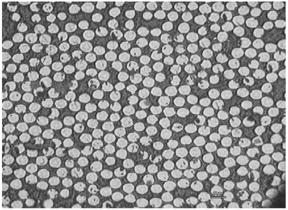
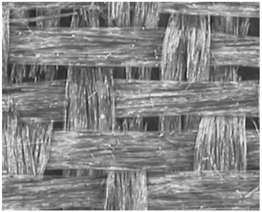
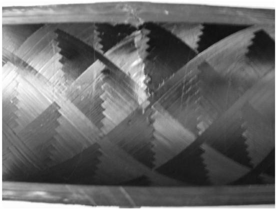
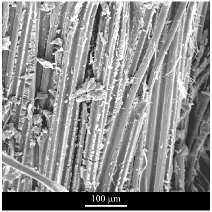
Interfaces in composite materials

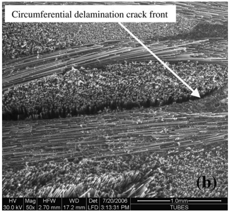
Microscale | ***Mesoscale*** | ***Macroscale***

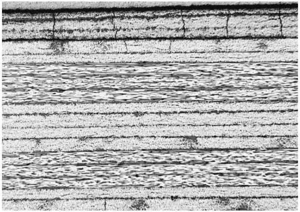












$\approx 10\ \mu$m

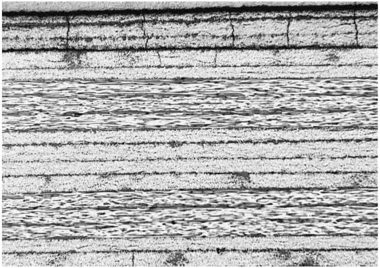
≈ 1 mm

≈ 1 dm

Essential aspect for reliable design of composite structures

Modelling assumptions

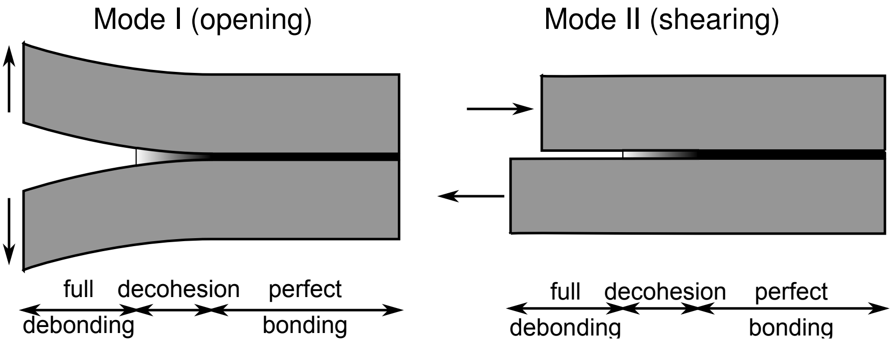
(Macroscopic delamination)



- Energy-based framework
- Decohesion $\equiv$ displacement jumps at interfaces
- Inelastic phenomena concentrated at interfaces
- Rate-independent (quasi-static) approximation
- Frictionless contact conditions
- Small-strain setting

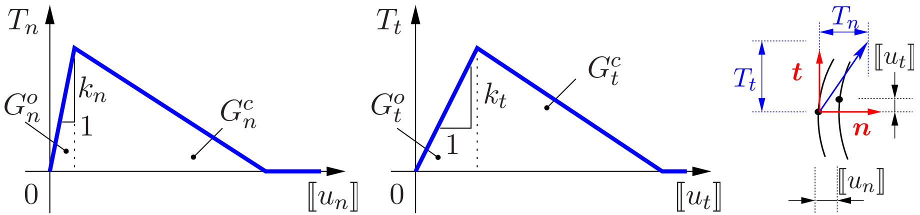
Basic concepts of interfacial damage mechanics

Constitutive characterization



- Initial stiffnesses: k_n, k_t (Nm^{-3})
- Activation energies: G_n^o, G_t^o (Jm^{-2})
- Dissipated energies: G_n^c, G_t^c (Jm^{-2})
- Mode interaction parameters

Basic concepts of interfacial damage mechanics
State variables



- Displacement jump decomposition

$$[\![u_n]\!] = [\![\boldsymbol{u}]\!] \cdot \boldsymbol{n} \geq 0 \qquad [\![u_t]\!] = \|[\![\boldsymbol{u}]\!] - [\![u_n]\!]\boldsymbol{n}\| \tag{1}$$

- Intensity of adhesion

$$T_n = \zeta k_n [\![u_n]\!] \quad T_t = \zeta k_t [\![u_t]\!] \quad 0 \leq \zeta \leq 1$$

Basic concepts of interfacial damage mechanics

Interaction criteria (Valoroso & Champaney, 2006)

- Mode mixity parameter

$$\psi([\![\boldsymbol{u}]\!]) = \sqrt{\frac{k_t}{k_n}}\frac{[\![u_t]\!]}{[\![u_n]\!]} \qquad \text{(mode I} =) \; 0 \leq \psi < +\infty \; (= \text{mode II)}$$

- Mode interaction criteria

$$\begin{aligned} \left(\frac{G^o(\psi)}{(1+\psi^2)G^o_n}\right)^{a_1} + \left(\frac{\psi^2 G^o(\psi)}{(1+\psi^2)G^o_t}\right)^{a_2} &= 1 \\ \left(\frac{G^c(\psi)}{(1+\psi^2)G^c_n}\right)^{b_1} + \left(\frac{\psi^2 G^c(\psi)}{(1+\psi^2)G^c_t}\right)^{b_2} &= 1 \end{aligned}$$

parameters a_1, a_2, b_1, b_2 fitted from experiments or taken as 2

Theoretical aspects

1. No supporting existence theory for mixed-mode description
2. (At least) abstract approximation results

Computational aspects

1. Oscillations of interfacial tractions for almost perfect interfaces ($k_\bullet \to \infty$)
2. Unstable response for brittle interfaces, leading to oscillation of quantity of interests for coarse meshes
3. Efficient and reliable resolution of frictionless contact

Outline

1. Energetic rate-independent systems
   - Energetic solution
   - Incremental energy minimization
   - Existence and approximation results
   - Application to delamination problem
2. Numerical treatment
   - Incremental optimization problems
   - Alternate minimization
   - Backtracking strategy
   - Application of FETI-based methods
3. Examples
4. Summary and outlook

Energetic rate-independent systems

Notation (MIELKE, LEVITAS & THEIL, 2002; MIELKE & ROSSI, 2007)

- Domain $\Omega \subset \mathbb{R}^d$, time interval $\mathbb{I} = [0; T]$
- State variables: $\boldsymbol{q} = (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{Q} = \mathcal{U} \times \mathcal{Z}$
  - Displacements $\boldsymbol{u} : \Omega \to \mathbb{R}^d$
  - Internal variables $\boldsymbol{z} : \Omega \to \mathbb{R}^m$
- Constitutive description
  - Stored energy $E(t, \boldsymbol{q}) : \mathbb{I} \times \mathcal{Q} \to \overline{\mathbb{R}}$
  - Dissipation rate $D\,(\boldsymbol{q}, \dot{\boldsymbol{z}}) : \mathcal{Q} \times \mathcal{Q} \to [0; +\infty]$
    State-dependent
    positively 1-homogeneous $\to$ rate-independence
- Stable sets

$$\mathcal{S}(t) = \{\boldsymbol{q} \in \mathcal{Q} : E(t, \boldsymbol{q}) \leq E(t, \tilde{\boldsymbol{q}}) + D(\boldsymbol{q}, \tilde{\boldsymbol{z}} - \boldsymbol{z}) \text{ for all } \tilde{\boldsymbol{q}} \in \mathcal{Q}\}$$

Energetic solution

Given $\boldsymbol{q}(0) \in \mathcal{Q}$, the process $\boldsymbol{q} : \mathbb{I} \to \mathcal{Q}$ is an *energetic solution* to the rate-independent system $(\mathcal{Q}, E, D)$ if for all $t \in \mathbb{I}$ we have

- **Stability**

$$\boldsymbol{q}(t) \in \mathcal{S}(t)$$

- **Energy inequality**

$$E(t, \boldsymbol{q}(t)) + \int_0^t D(\boldsymbol{q}(s), \dot{\boldsymbol{z}}(s))\,\mathrm{d}s \leq E(0, \boldsymbol{q}(0)) + \int_0^t \partial_s E(s, \boldsymbol{q}(s))\,\mathrm{d}s$$

- Available existence results either assume $D(\boldsymbol{q}, \dot{\boldsymbol{z}}) = D(\boldsymbol{z}, \dot{\boldsymbol{z}})$ or rely on a uniform convexity of $E(t, \cdot)$ (MR, 2007)

- Rothe method

$$\mathcal{P}_\tau = \{0 = t_\tau^0 < t_\tau^1 < \ldots < t_\tau^M = T\} \quad \text{with} \quad \tau = \max_{j=1,\ldots,M} \{t_\tau^j - t_\tau^{j-1}\}$$

Incremental problems

Given $\boldsymbol{q}_\tau^0 = \boldsymbol{q}(0) \in \mathcal{Q}$, solve for $k = 1, \ldots, M$

$$\boldsymbol{q}_\tau^k \in \arg\min_{\boldsymbol{q} \in \mathcal{Q}} E(t_k, \boldsymbol{q}) + D(\boldsymbol{q}_\tau^{k-1}, \boldsymbol{z} - \boldsymbol{z}_\tau^{k-1}) \tag{2}$$

- Used to show convergence as $\tau \to 0$
- Provides convenient basis for numerical treatment (ORTIZ & STAINIER, 1999 —)

Energetic rate-independent systems

Existence results (Mielke and co-workers, 2002—; Kružík & Zeman, in preparation)

Assumptions (Stored energy)

- $E(t, \cdot)$ is weakly lower semicontinuous and coercive

$$E(t, \boldsymbol{q}) < \infty \Rightarrow |\partial_t E(t, \boldsymbol{q})| \leq C_0(C_1 + E(t, \boldsymbol{q}))$$

- $\partial_t E(t, \cdot)$ is weakly continuous for all $t \in \mathbb{I}$ and satisfies

$$|\partial_t E(t_1, \boldsymbol{q}) - \partial_t E(t_2, \boldsymbol{q})| \leq \omega(|t_1 - t_2|)$$

for a non-decreasing $\omega : \mathbb{I} \to \mathbb{R}_0^+$

Energetic rate-independent systems

Existence results (MIELKE and co-workers, 2002—; KRUŽÍK & ZEMAN, in preparation)

Assumptions (Dissipation + "Compatibility")

- For all $\boldsymbol{z} \in \mathcal{Z}$ and $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3 \in \mathcal{Q}$

$$
\begin{aligned}
C_0 \|\boldsymbol{z}_1 - \boldsymbol{z}_2\|_{\mathcal{X}} &\leq D(\boldsymbol{q}_1, \boldsymbol{z}_2 - \boldsymbol{z}_1) \\
|D(\boldsymbol{q}_1, \boldsymbol{z}) - D(\boldsymbol{q}_2, \boldsymbol{z})| &\leq C_1 \|\boldsymbol{q}_2 - \boldsymbol{q}_1\|_{\mathcal{Q}} \|\boldsymbol{z}\|_{\mathcal{Z}} \\
D(\boldsymbol{q}_1, \boldsymbol{z}_3 - \boldsymbol{z}_1) &\leq D(\boldsymbol{q}_1, \boldsymbol{z}_2 - \boldsymbol{z}_1) + D(\boldsymbol{q}_1, \boldsymbol{z}_3 - \boldsymbol{z}_2)
\end{aligned}
$$

  where $\mathcal{Z} \Subset \mathcal{X}$
- For each sequence $\{(t_n, \boldsymbol{q}_n)\}_{n \in \mathbb{N}}$ with $\boldsymbol{q}_n \in \mathcal{S}(t_n)$ and $t_n \to t^*$ and $\boldsymbol{q}_n \rightharpoonup \boldsymbol{q}^*$ and $\sup_n E(t_n, \boldsymbol{q}_n) < \infty$ it holds

$$
\boldsymbol{q}_* \in \mathcal{S}(t^*) \qquad \partial_t E(t^*, \boldsymbol{q}^n) \to \partial_t E(t^*, \boldsymbol{q}^*)
$$

Energetic rate-independent systems

Existence results (Mielke and co-workers, 2002—; Kružík & Zeman, in preparation)

- Solution of the incremental problem exists for all k and τ and is stable
- It satisfies a-posteriori two-sided energy estimates

$$\begin{aligned}\int_{t_\tau^{k-1}}^{t_\tau^k} \partial_s E(s, \boldsymbol{q}_\tau^k)\,\mathrm{d}s &\leq E(t_\tau^k, \boldsymbol{q}_\tau^k) - E(t_\tau^{k-1}, \boldsymbol{q}_\tau^{k-1}) \\ &+ D(\boldsymbol{q}_\tau^{k-1}, \boldsymbol{z}_\tau^k - \boldsymbol{z}_\tau^{k-1}) \leq \int_{t_\tau^{k-1}}^{t_\tau^k} \partial_s E(s, \boldsymbol{q}_\tau^{k-1})\,\mathrm{d}s\end{aligned} \tag{3}$$

- As $\tau \to 0$, there exist $\boldsymbol{z} \in BV(\mathbb{I}; \mathcal{X})$ and $\boldsymbol{u}(t) : \mathbb{I} \to \mathcal{U}$ such that

$$(\boldsymbol{u}(t), \boldsymbol{z}(t)) \in \mathcal{S}(t) \quad \text{and} \quad \widehat{\boldsymbol{z}}_\tau \to \boldsymbol{z} \text{ in } BV(\mathbb{I}; \mathcal{X})$$

and the process satisfies the energy inequality.

Energetic rate-independent systems

Abstract approximation result (MIELKE & ROUBÍČEK, 2009)

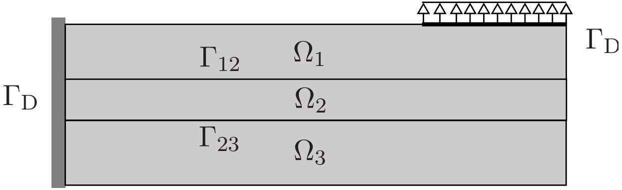
Energetic density of $\mathcal{Q}_h \subset \mathcal{Q}$

For each $(t, \boldsymbol{q}) \in \mathbb{I} \times \mathcal{Q}$, there exists $\{\boldsymbol{q}_h\}$ such that $\boldsymbol{q}_h \in \mathcal{Q}_h$, $\boldsymbol{q}_h \rightarrow \boldsymbol{q}$ and $E(t, \boldsymbol{q}_h) \rightarrow E(t, \boldsymbol{q})$

- Incremental energy minimization yields stable approximate solution $\{\boldsymbol{q}^k_{\tau,h}\}$
- Approximate solutions $\{\boldsymbol{q}^k_{\tau,h}\}$ satisfy the two-sided energy estimate
- If $\mathcal{Q}_h$ satisfies adjusted compatibility condition, the solution converges to an energetic solution as $\tau \rightarrow 0$ and $h \rightarrow 0$

Application to delamination problem

Geometry (Kočvara, Mielke & Roubíček, 2006)



- Body $\Omega \subset \mathbb{R}^d$ with the Lipschitz boundary Γ
- Time-dependent Dirichlet boundary conditions at $\Gamma_{\rm D}$
- Collection of disjoint bodies Ω_α with Lipschitz boundaries Γ_α, $\alpha = 1, 2, \ldots, m$ ($\Gamma_{{\rm D},\alpha} = \Gamma_\alpha \cap \Gamma_{\rm D} \neq \emptyset$)
- Internal boundaries $\Gamma_{\alpha\beta} = \Gamma_\alpha \cap \Gamma_\beta$ $(\alpha > \beta)$ with normal vectors $\boldsymbol{n}_{\alpha\beta}$

Application to delamination problem

State variables and data (Kočvara, Mielke & Roubíček, 2006; Kružík & Zeman)

- Domain displacements $\boldsymbol{u}_\alpha : \Omega_\alpha \to \mathbb{R}^d$
- Internal variable $\zeta_{\alpha\beta} : \Gamma_{\alpha\beta} \to \mathbb{R}$
- Function spaces

$$\mathcal{U} = \prod_{\alpha=1}^{m} \left\{ \boldsymbol{u}_\alpha \in W^{1,2}(\Omega_\alpha; \mathbb{R}^d), \boldsymbol{u}_\alpha|_{\Gamma_{\mathrm{D}\alpha}} = \mathbf{0} \right\}$$

$$\mathcal{Z} = \prod_{\alpha=1}^{m} \prod_{\beta=\alpha+1}^{m} \left\{ \zeta_{\alpha\beta} \in W^{1,2}(\Gamma_{\alpha\beta}) \right\}$$

$$\mathcal{X} = \prod_{\alpha=1}^{m} \prod_{\beta=\alpha+1}^{m} \left\{ \zeta_{\alpha\beta} \in L^{1}(\Gamma_{\alpha\beta}) \right\}$$

- Hard-device loading $\boldsymbol{w}_\mathrm{D} \in C^1(\mathbb{I}; W^{1/2,2}(\Gamma_\mathrm{D}; \mathbb{R}^d))$

Application to delamination problem

Stored energy (KOČVARA, MIELKE & ROUBÍČEK, 2006; KRUŽÍK & ZEMAN)

- Stored energy

$$E(t,\boldsymbol{q}) = \sum_{\alpha=1}^{m} E_\alpha(t,\varepsilon(\boldsymbol{u}_\alpha)) + \sum_{\alpha=1}^{m}\sum_{\beta=\alpha+1}^{m} E_{\alpha\beta}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta, \zeta_{\alpha\beta})$$

- Bulk energy

$$E_\alpha(t,\varepsilon) = \frac{1}{2}\int_{\Omega_\alpha} \boldsymbol{C}_\alpha\left(\varepsilon + \varepsilon(\boldsymbol{u}_{\mathrm{D},\alpha}(t))\right) : \left(\varepsilon + \varepsilon(\boldsymbol{u}_{\mathrm{D},\alpha}(t))\right)\,\mathrm{d}\Omega$$

- Interfacial energy from (1)

$$E_{\alpha\beta}([\![\boldsymbol{u}]\!],\zeta) = \begin{cases} \displaystyle\int_{\Gamma_{\alpha\beta}} \frac{\zeta}{2}\boldsymbol{k}_{\alpha\beta}[\![\boldsymbol{u}]\!]\cdot[\![\boldsymbol{u}]\!] + f_{\alpha\beta}(\psi([\![\boldsymbol{u}]\!]),\zeta) + \frac{\kappa}{2}(\zeta')^2\,\mathrm{d}\Gamma \\ \text{for } [\![\boldsymbol{u}]\!]_n \geq 0 \text{ and } 0 \leq \zeta \leq 1 \text{ a.e. on } \Gamma_{\alpha\beta} \\ +\infty \text{ otherwise} \end{cases}$$

Application to delamination problem

Dissipation function (KOČVARA, MIELKE & ROUBÍČEK, 2006; KRUŽÍK & ZEMAN)

- Dissipation rate

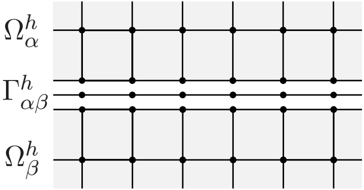
$$D(\boldsymbol{q}, \dot{\boldsymbol{z}}) = \sum_{\alpha=1}^{m} \sum_{\beta=\alpha+1}^{m} D_{\alpha\beta}(\boldsymbol{u}_\alpha - \boldsymbol{u}_\beta, \zeta_{\alpha\beta}, \dot{\zeta}_{\alpha\beta})$$

- Interfacial dissipation

$$D_{\alpha\beta}([\![\boldsymbol{u}]\!], \zeta, \dot{\zeta}) = \begin{cases} -\int_{\Gamma_{\alpha\beta}} G^c_{\alpha\beta}(\psi([\![\boldsymbol{u}]\!]))\dot{\zeta}\, \mathrm{d}\Gamma & \text{for } \dot{\zeta} \leq 0 \\ +\infty & \text{otherwise} \end{cases}$$

- (KMR, 2006) $\Rightarrow$ Existence of energetic solution
- (MR, 2009) $\Rightarrow$ Approximation results for state-independent dissipation and for discretization with $\mathcal{U}$ with P_1 elements and $\mathcal{Z}$ with P_0 elements

- General strategy
  - Reduce numerics to interfaces → FETI-family methods (DOSTÁL, 2009)



- Discrete state variables
  - Nodal displacements: u_α
  - Interfacial displacement jumps: $[\![\mathsf{u}_{\alpha\beta}]\!]$
  - Adhesion intensity variables: $\mathsf{z}_{\alpha\beta}$

$$
\begin{aligned}
(\mathsf{u}, [\![\mathsf{u}]\!]) &\in \mathbb{K}^{\mathsf{u}} = \{(\mathsf{v}, [\![\mathsf{v}]\!]) : \mathsf{B}_{\mathrm{d}}\mathsf{v} = 0, \mathsf{B}_{\mathrm{e}}\mathsf{v} = [\![\mathsf{v}]\!], \mathsf{B}_{\mathrm{i}}[\![\mathsf{v}]\!] \geq 0\} \\
\mathsf{z} &\in \mathbb{K}^{\mathsf{z}} = \{\mathsf{y} : 0 \leq \mathsf{y} \leq 1\}
\end{aligned}
$$

Numerical treatment

Discrete energy functions

- Energy stored in domains

$$E_{\Omega,h}(t,\mathsf{u}) = \frac{1}{2}\left(\mathsf{u}+\mathsf{u}_{\mathrm{D}}(t)\right)^{\mathsf{T}}\mathsf{K}\left(\mathsf{u}+\mathsf{u}_{\mathrm{D}}(t)\right) \qquad \mathsf{K} = \begin{bmatrix} \mathsf{K}_1 & & \\ & \ddots & \\ & & \mathsf{K}_m \end{bmatrix}$$

- Energy stored at interfaces

$$E_{\Gamma,h}([\![\mathsf{u}]\!],\mathsf{z}) = \frac{1}{2}[\![\mathsf{u}]\!]^{\mathsf{T}}\mathsf{k}(\mathsf{z})[\![\mathsf{u}]\!] + f_h(\psi([\![\mathsf{u}]\!]),\mathsf{z})$$

$f_h(\psi,\cdot)$ convex in z

- Interfacial dissipation: $\mathsf{z} \leq 0$

$$D_h([\![\mathsf{u}]\!],\mathsf{z}) = -\mathsf{g}(\psi([\![\mathsf{u}]\!]))^{\mathsf{T}}\mathsf{z}$$

Numerical treatment

Incremental optimization problems

- Incremental energy (2)

$$\begin{aligned} I_{k,h}(\mathsf{z}, \mathsf{u}, [\![\mathsf{u}]\!]) &= \frac{1}{2}\mathsf{u}^{\mathsf{T}}\mathsf{K}\mathsf{u} + \mathsf{u}^{\mathsf{T}}\mathsf{K}\mathsf{u}_{\mathrm{D}}(t_k) + \frac{1}{2}[\![\mathsf{u}]\!]^{\mathsf{T}}\mathsf{k}(\mathsf{z})[\![\mathsf{u}]\!] + f_h(\psi([\![\mathsf{u}]\!]), \mathsf{z}) \\ &- \mathsf{z}^{\mathsf{T}}\mathsf{g}(\psi([\![\mathsf{u}]\!]_{k-1})) \end{aligned}$$

Incremental optimization problems

Given $(\mathsf{z}_0, \mathsf{u}_0, [\![\mathsf{u}]\!]_0) \in \mathbb{K}^{\mathsf{z}} \times \mathbb{K}^{\mathsf{u}}$ solve for $k = 1, \ldots, M$

$$(\mathsf{z}_k, \mathsf{u}_k, [\![\mathsf{u}]\!]_k) = \arg \min_{0 \leq \mathsf{z} \leq \mathsf{z}_{k-1}} \min_{(\mathsf{u}, [\![\mathsf{u}]\!]) \in \mathbb{K}^{\mathsf{u}}} I_{k,h}(\mathsf{z}, \mathsf{u}, [\![\mathsf{u}]\!])$$

- Large-scale, sparse, separately convex and constrained problem

Numerical treatment

Incremental minimization (BOURDIN, FRANCFORT & MARIGO, 2000; BOURDIN, 2007)

- Alternate minimization algorithm ($\equiv$ staggered scheme)

Alternate minimization algorithm

1: Set $j = 0$, $\mathsf{z}^{(0)} = \mathsf{z}_{k-1}$, $\mathsf{u}^{(0)} = \mathsf{u}_{k-1}$, $[\![\mathsf{u}]\!]^{(0)} = [\![\mathsf{u}]\!]_{k-1}$
2: **repeat**
3: Set $j = j + 1$
4: Solve $\left(\mathsf{u}^{(j)}, [\![\mathsf{u}]\!]^{(j)}\right) = \arg \min_{(\mathsf{u}, [\![\mathsf{u}]\!]) \in \mathbb{K}^{\mathsf{u}}} I_{k,h}(\mathsf{z}^{(j-1)}, \mathsf{u}, [\![\mathsf{u}]\!])$
5: Solve $\mathsf{z}^{(j)} = \arg \min_{0 \leq \mathsf{z} \leq \mathsf{z}_{k-1}} I_{k,h}(\mathsf{z}, \mathsf{u}^{(j)}, [\![\mathsf{u}]\!]^{(j)})$
6: **until** $\|\mathsf{z}^{(j)} - \mathsf{z}^{(j-1)}\|_\infty \leq \delta$
7: Set $\mathsf{u}_k = \mathsf{u}^{(j)}$, $[\![\mathsf{u}]\!]_k = [\![\mathsf{u}]\!]^{(j)}$, $\mathsf{z}_k = \mathsf{z}^{(j)}$

- Step 5 can be performed element-by-element at interface
- Step 4 well-suited for FETI-based solvers

Numerical treatment

Backtracking strategy (MIELKE, ROUBÍČEK & ZEMAN, 2010; BENEŠOVÁ, 2009–)

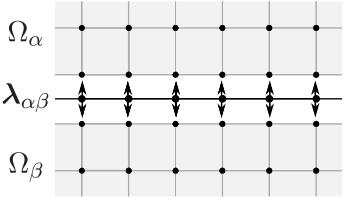
- Convergence to a *critical point* of the objective function
- Not good enough – theory relies on the *global* minimization

Backtracking algorithm

Set $k = 1$, $\mathsf{z}_{-1} = 1, \mathsf{z}_0 = 1$, $\mathsf{z}^{(0)} = 1$
repeat
  Determine z_k using the alternate minimization algorithm
  for time t_k and initial value $\mathsf{z}^{(0)}$.
  Set $\mathsf{z}^{(0)} = \mathsf{z}_k$
  if two-sided estimate (3) is satisfied with tolerance η
    Set $k = k + 1$
  else
    Set $k = k - 1$
  end
until $k \leq M$

Numerical treatment

FETI formulation (FARHAT & ROUX, 1991; KRUIS & BITTNAR, 2008; DOSTÁL; 2009)



- Lagrange function

$$L(\mathsf{u}, [\![\mathsf{u}]\!], \lambda) = \frac{1}{2}\mathsf{u}^{\mathsf{T}}\mathsf{K}\mathsf{u} + \frac{1}{2}[\![\mathsf{u}]\!]^{\mathsf{T}}\mathsf{k}[\![\mathsf{u}]\!] - \mathsf{d}^{\mathsf{T}}\mathsf{u} + \lambda^{\mathsf{T}}\left(\mathsf{B}_{\mathrm{e}}\mathsf{u} - \mathsf{u}\right)$$

- Saddle-point problem

$$(\mathsf{u}, [\![\mathsf{u}]\!], \lambda) = \arg\max_{\lambda}\min_{\mathsf{u}}\min_{\mathsf{B}_{\mathrm{i}}[\![\mathsf{u}]\!]\geq 0} L(\mathsf{u}, [\![\mathsf{u}]\!], \lambda)$$

- The “unresolved” box constraint $\mathsf{B}_{\mathrm{i}}[\![\mathsf{u}]\!] \geq 0$ will be enforced by a heuristic active set strategy

Numerical treatment

Dual problem (FARHAT & ROUX, 1991; KRUIS & BITTNAR, 2008; DOSTÁL; 2009)

- Optimality conditions

$$[\![u]\!] = k^{-1}\lambda \qquad u = K^{+}\left(d - B_e^T\lambda\right) + R\alpha$$

$$R^T\left(d - B_e^T\lambda\right) = 0 \qquad B_e u - [\![u]\!] = 0$$

- Eliminate primary variables u and $[\![u]\!]$ $(R^T d = 0)$

$$\begin{bmatrix} B_e K^{+} B_e^T + k^{-1} & -B_e R \\ -R^T B_e^T & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \alpha \end{bmatrix} = \begin{bmatrix} B_e K^{+} d \\ 0 \end{bmatrix}$$

- Problem reduced to interfaces ✓
- Efficiently solvable using Projected Conjugate Gradient solvers
- Perfect bonding $k^{-1} \to 0$ ✓

Numerical treatment

Active set strategy (GRUBER & ZEMAN, in preparation)

1. Contact detection algorithm
   - Tag interfacial nodes as primal ($\mathcal{P}$) or dual ($\mathcal{D}$)

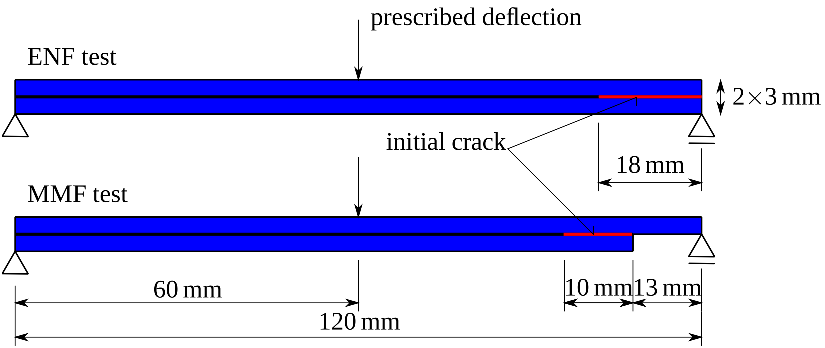
$$[\![\mathsf{u}]\!]_e = \mathsf{k}_e^{-1}\lambda_e$$

   - For $\mathcal{D}$: determine contact nodes ($\mathcal{C}$) from λ
   - For $\mathcal{P}$: determine contact nodes ($\mathcal{C}$) from $[\![\mathsf{u}]\!]$
   - For $\mathcal{C}$: enforce $[\![u]\!]_n = 0$ by setting compliance to zero
   - For $\mathcal{P} - \mathcal{C}$: enforce $\lambda = 0$ by static condensation
2. Solve dual system by standard Projected Conjugate Gradients
3. Check convergence of Lagrange multipliers λ

- Usually converges in 3–4 iterations ✓

Examples

Notched flexure tests (VALOROSO & CHAMPANEY, 2006)



Mesh size h	# of primal DOFs	# of dual DOFs
1 mm	$1,936$	$242\ (12\%)$
0.75 mm	$3,220$	$322\ (10\%)$
0.5 mm	$6,748$	$482\ (7\%)$

- $M = 40$ time steps

Examples

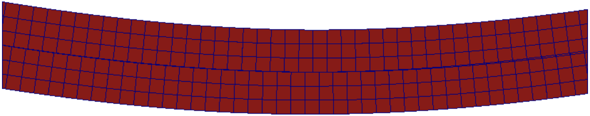
Material data

- Domains: $E = 75$ GPa, $\nu = 0.3$
- Interfaces: Piecewise linear interfacial law

Material parameter	*Ductile*	*Brittle*
$k_n = k_t$ (GNm^{-3})	125	125×10^6
$G_n^o = G_t^o$ (Jm^{-2})	100	0.01
$G_n^c = G_t^c$ (Jm^{-2})	250	25
Interaction parameters a_i, b_i	2	2

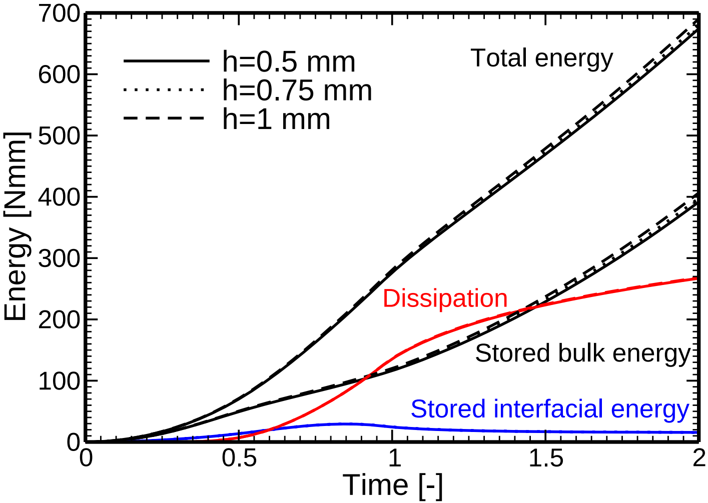
- Algorithmic settings: $\delta = 10^{-5}$, $\eta = 10^{-3}$
- In-house experimental MATLAB implementation

ENF test(displacements are magnified $5\times$)



Examples

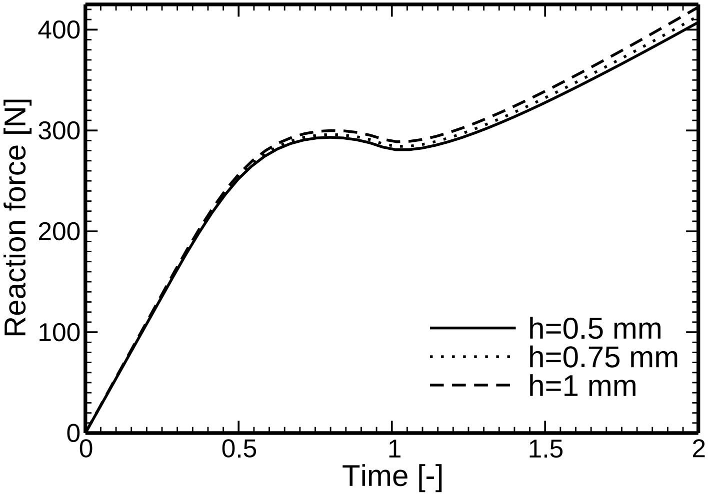
Energetics of ENF test, $h \to 0$, ductile interface, 60–300 s



(No back-tracking activated)

Examples

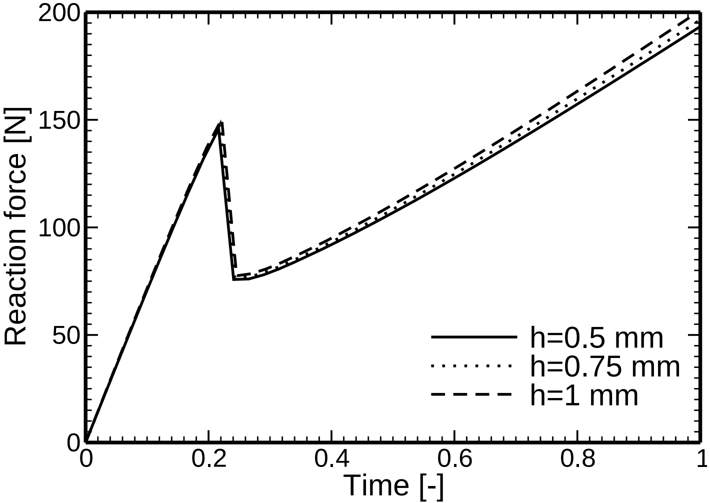
Force-displacement curves of ENF test, $h \to 0$, ductile interface, 60–300 s



(No back-tracking activated)

Examples

Force-displacement curves of ENF test, $h \to 0$, brittle interface, 60–300 s



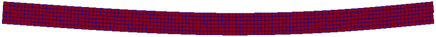
(Back-tracking inactive)

Examples

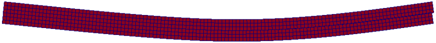
ENF test, ductile interface, snapshots of delamination evolution

$t = 0.33$

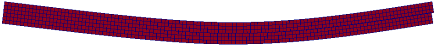
$t = 0.50$



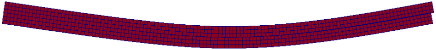
$t = 0.66$



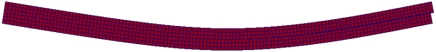
$t = 1.00$



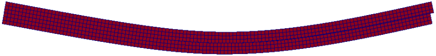
$t = 1.17$



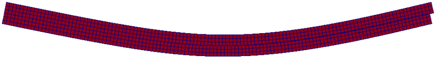
$t = 1.33$



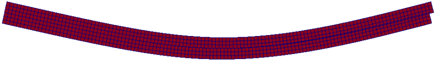
$t = 1.50$



$t = 1.67$



$t = 1.83$

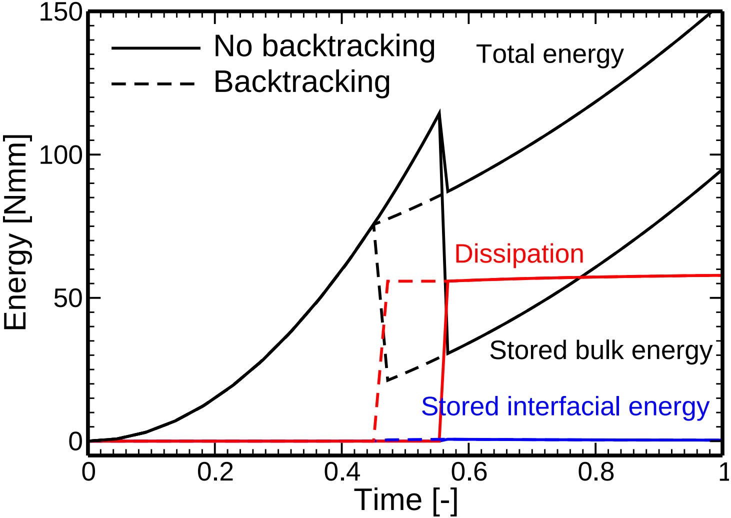


$t = 2$

Displacement are scaled $5\times$

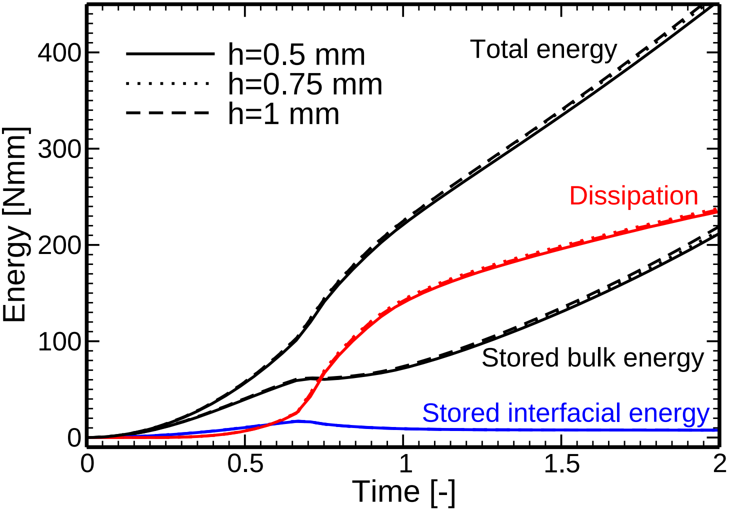
Examples

Energetics of modified ENF test, $h = 0.1$ mm, brittle interface



Examples

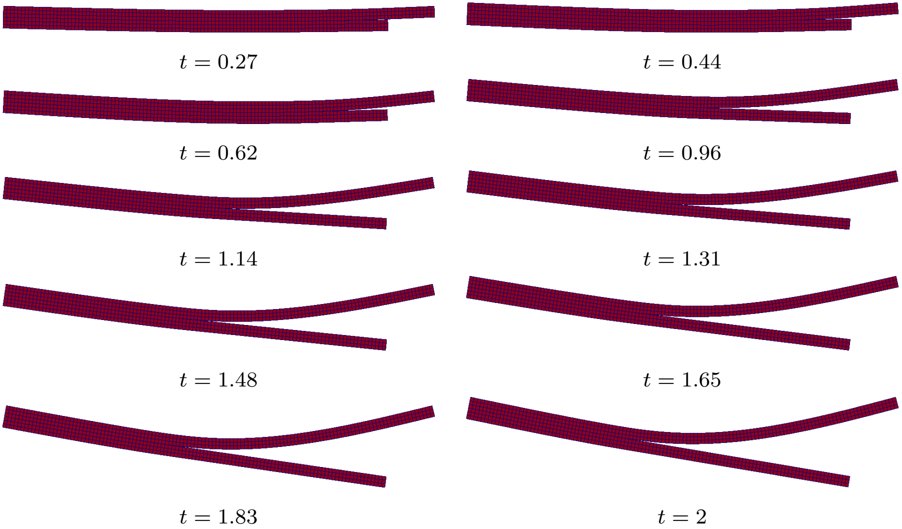
Energetics of MMF test, $h \to 0$, ductile interface



(No back-tracking activated)

Examples

Snapshots of MMF test

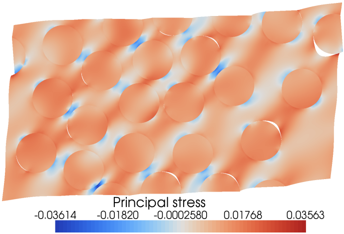


Displacement are scaled $5\times$

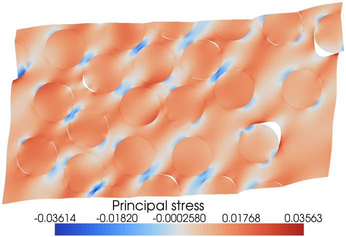
Extensions

Debonding in fibre-reinforced composites

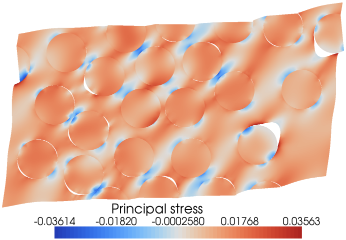
$t = 0.4$



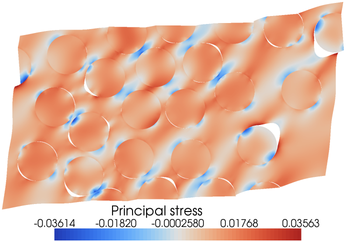
$t = 0.6$



$t = 0.8$

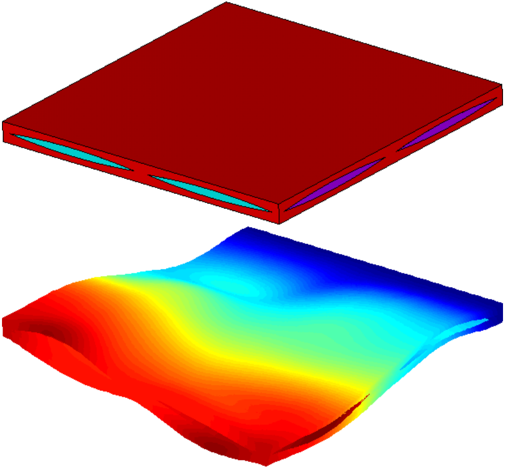


$t = 1$



Extensions

Woven composites, preliminary results (courtesy of T. KOZUBEK, TU Ostrava)



- Analyzed using MATSOL library

Summary and outlook

- Energetic framework
  - is well-suited for analysis of inelastic solid mechanics problems
  - provides background for efficient numerical implementation
  - addresses the problems of *stability* of solution when the response is non-unique
- FETI-based algorithm is well-suited for analysis in composite materials, even in sequential mode
- Future extensions
  - Abstract approximation results, homogenization
  - More efficient duality solvers
  - Application to woven composite materials

Acknowledgements

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