# Selected aspects of *hp*-FEM in 3D

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# Outline

# Behavior of the solution

- Goal – obtain accurate approximation using as small discrete system, as possible
- Behavior of solution can differ over the domain
  - Smooth
  - Singularity
  - Boundary layer
- Each requires different approach
  - Large higher-order elements
  - Small low-order elements
  - ...
- *hp*-FEM allows us to use optimal elements in each part of the computational domain

# Automatic adaptivity

1. Calculate the solution
2. Calculate the reference solution
3. Estimate the error on each element
4. Sort elements according to the error
5. For elements with big error
   1. Construct list of refinement candidates
   2. Project the reference solution on each candidate
   3. Estimate the error and choose the best candidate
6. Perform refinements and continue, until error is decreased to desired tolerance

# Refinement candidates

- Split the element – *h* adaptivity
- Increase polynomial degree – *p* adaptivity
- Do one of previous or split the element and redistribute polynomial degrees – *hp* adaptivity
  - Best results
  - Exponential convergence
  - Complicated to implement

# Elliptic problem

We solve the problem

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= u_D \quad \text{on } \Gamma_D, \\
\frac{\partial u}{\partial \nu} &= g \quad \text{on } \Gamma_N
\end{aligned}
$$

Weak formulation

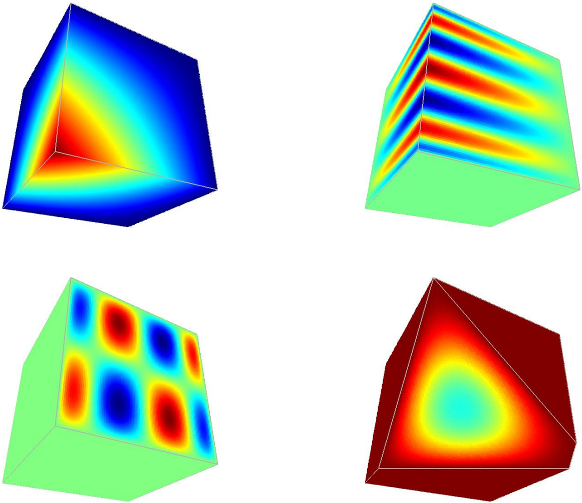
$$(u, v)_\Omega = (f, v)_\Omega + (g, v)_{\Gamma_N}$$

Space for $u$

$$V = \{u \in H1, \quad u = u_D \quad \text{on } \Gamma_D\}$$

Conformity requirements: continuity in vertices, along edges and faces.

# Hexahedral shape functions



# Time-Harmonics Maxwell's Equations

$$\nabla \times (\mu_r^{-1} \nabla \times \mathbf{E}) - k^2 \epsilon_r \mathbf{E} = \mathbf{\Phi},$$

where

- $\mathbf{E} = \mathbf{E}(x)$ electric field intensity
- $\mu_r$, $\epsilon_r$ relative permeability and permittivity
- $k$ wawe number

Boundary conditions:

- Perfect conductor boundary condition

$$\mathbf{E} \times \boldsymbol{\nu} = \mathbf{0} \qquad \text{on } \Gamma_P$$

- Impedance boundary condition

$$\mu_r^{-1}(\nabla \times \mathbf{E}) \times \boldsymbol{\nu} - jk\lambda \mathbf{E_T} = \mathbf{g} \qquad \text{on } \Gamma_I$$

where $\mathbf{E_T} = (\boldsymbol{\nu} \times \mathbf{E}) \times \boldsymbol{\nu}$.

# Weak formulation

Space for **E**

$$V = \{\mathbf{E} \in \mathbf{H}(curl, \Omega); \quad \boldsymbol{\nu} \times \mathbf{E} = \mathbf{0} \quad \text{on } \Gamma_P\}$$

The variational identity

$$a(\mathbf{E}, \mathbf{F}) = l(\mathbf{F}), \quad \text{for all } \mathbf{F} \in V,$$

where

$$\begin{aligned} a(\mathbf{e}, \mathbf{f}) &= (\mu_r^{-1} \nabla \times \mathbf{e}, \nabla \times \mathbf{f})_\Omega - k^2(\epsilon_r \mathbf{e}, \mathbf{f})_\Omega - jk(\lambda \mathbf{e}_T, \mathbf{f}_T)_{\Gamma_I} \\ l(\mathbf{f}) &= (\boldsymbol{\Phi}, \mathbf{f})_\Omega + (\mathbf{g}, \mathbf{f}_\mathbf{T})_{\Gamma_I} \end{aligned}$$

# Conformity requirements

Space

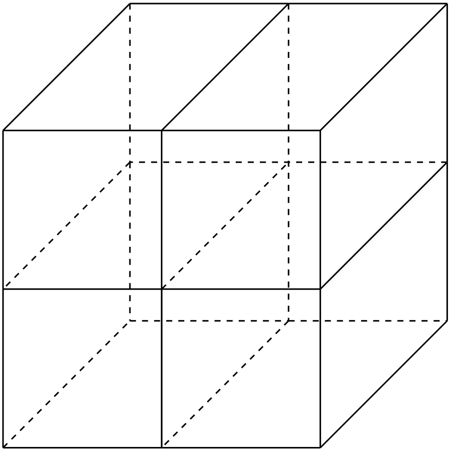
$$\mathbf{H}(curl, \Omega) = \{\mathbf{E} \in (L^2(\Omega))^3; \quad \nabla \times \mathbf{E} \in (L^2(\Omega))^3\}$$

Polygonal domain $\Omega_h$, Finite element mesh

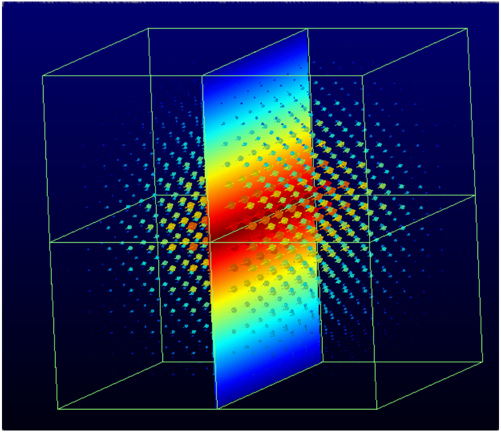
- $\mathbf{E}|_K \in (H^1(K))^3$ for each element $K$
- For each element interface $f = K_1 \cap K_2$ the traces of the tangential components are the same:

$$\boldsymbol{\nu}_f \times \mathbf{E}|_{K_1} = \boldsymbol{\nu}_f \times \mathbf{E}|_{K_2}$$

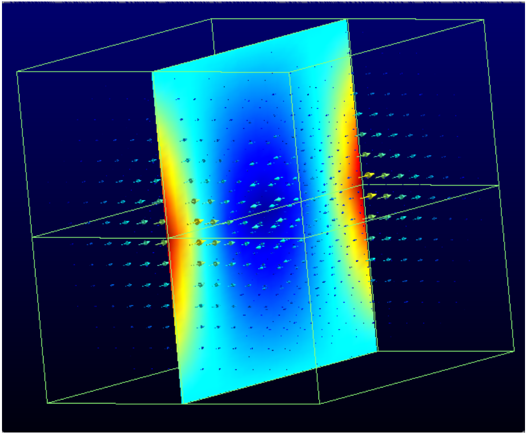
# Hcurl base functions



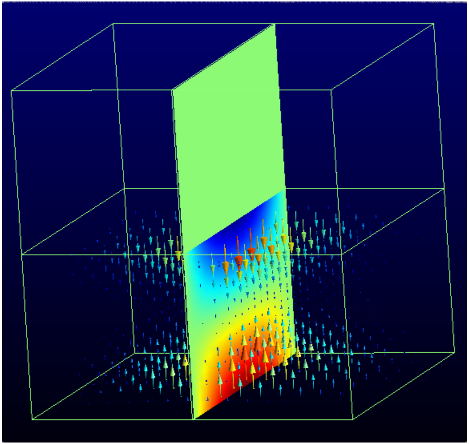
# Hcurl edge base functions



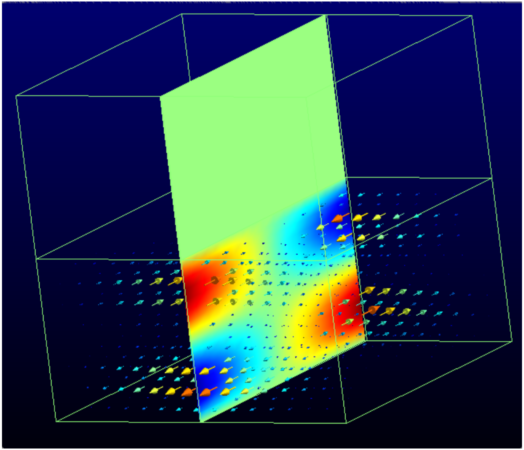
# Hcurl edge base functions



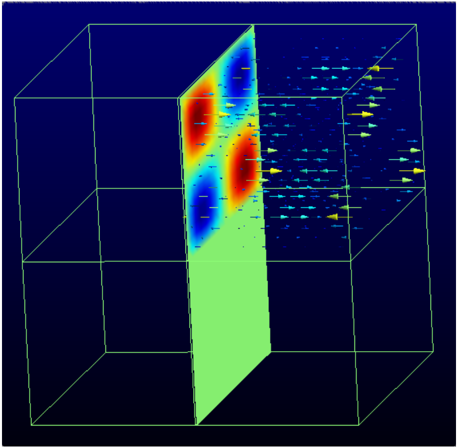
# Hcurl face base functions



# Hcurl face base functions



# Hcurl bubble base functions



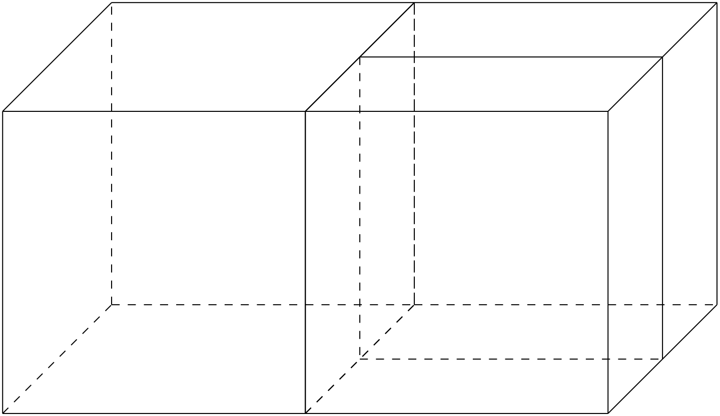
# Regularity of mesh

- When element refined $\longrightarrow$ hanging nodes appear
- In order to keep code simple, we can regularize mesh
  - We are not refining locally
  - We make a lot of unnecessary refinements $\longrightarrow$ size of problem grows with no effect on quality of solution
- Compromise – regularity rules
- Best results – arbitrary level hanging nodes
  - Refining locally
  - No unnecessary refinements
  - Implementation more demanding
  - We use this approach

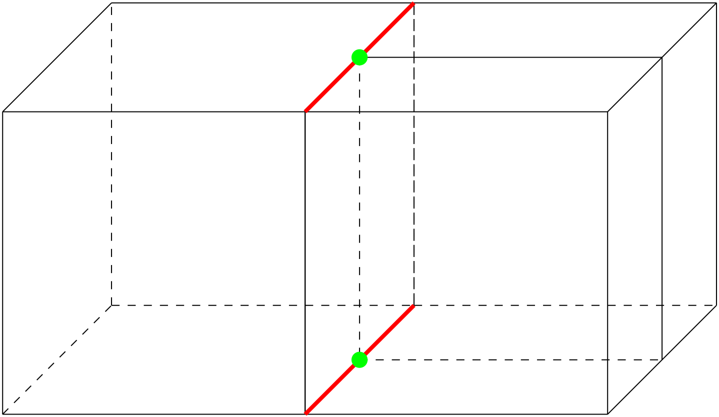
# Constrained approximation

- We have to construct base functions to satisfy conformity conditions
  - H1 space – continuity in vertices and along edges and faces
  - Hcurl space – continuity of tangential components along edges and faces
- In constrained edges and faces no DOFs
- Calculate combnation of shape functions

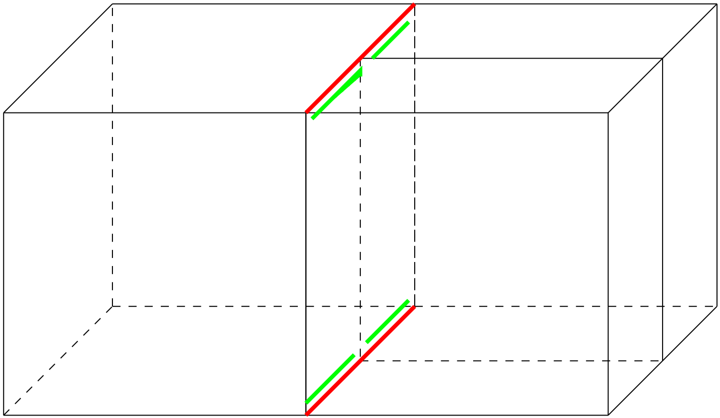
# Model situation



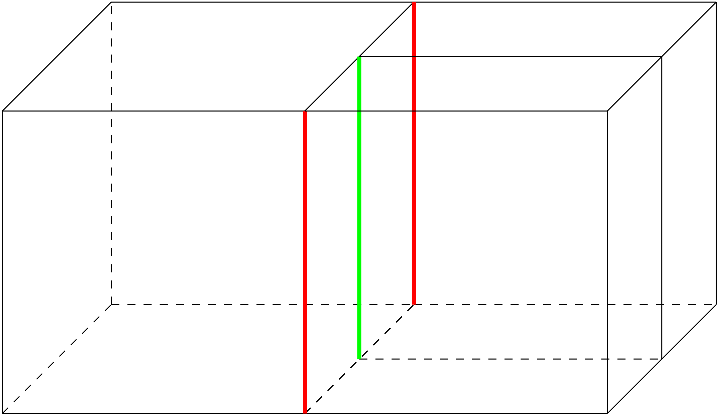
# Edges constraining vertices



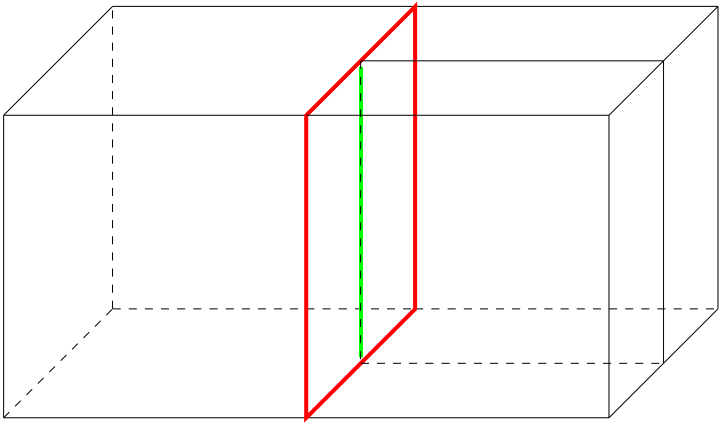
# Edges constraining edges



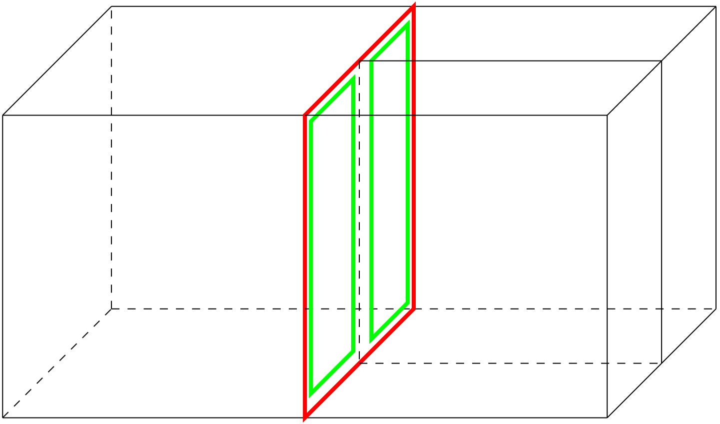
# Edges constraining edges



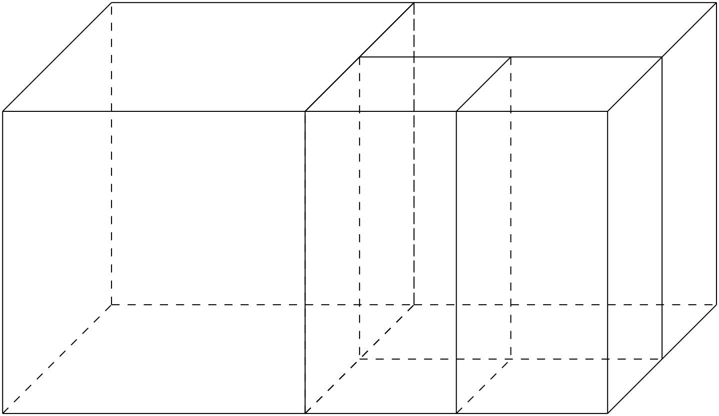
# Face constraining edge



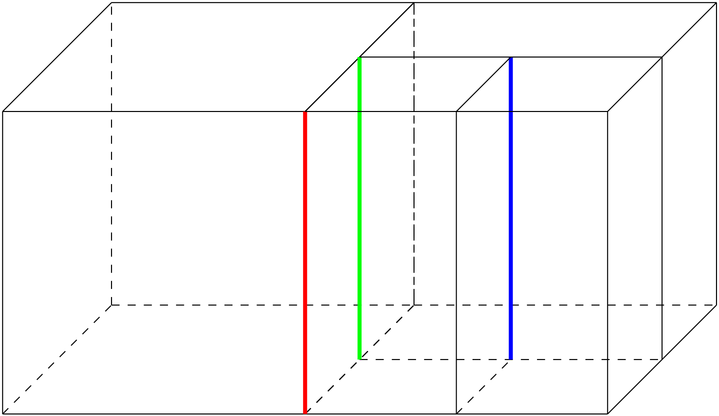
# Face constraining face



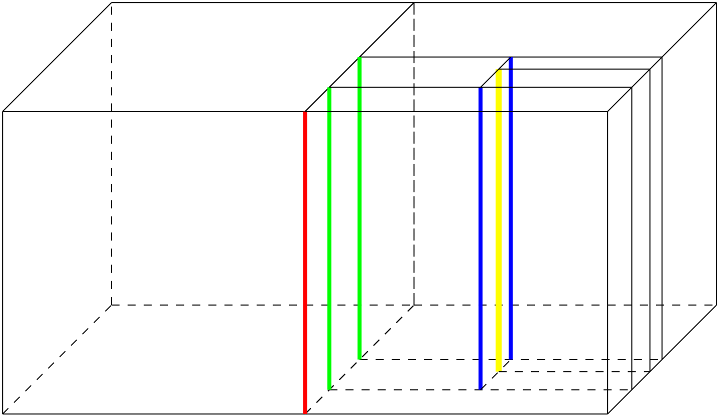
# More complicated situation



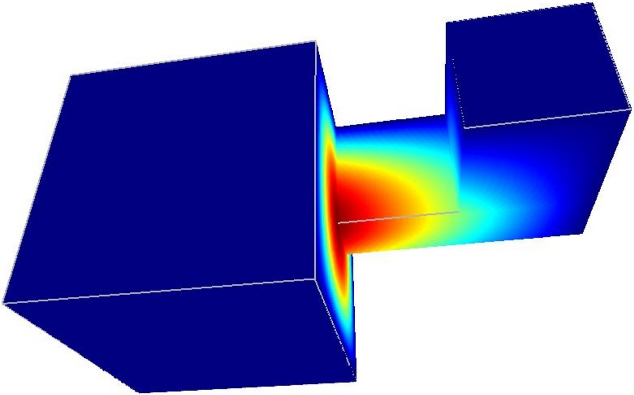
# Edge constrained indirectly



# Complicated situation



# Part of the domain of basis face function



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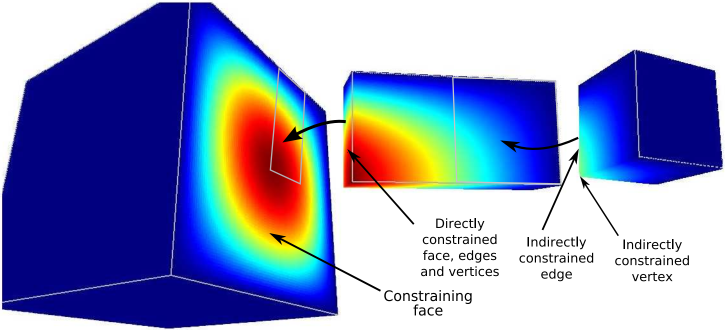


# Electrostatic problem

Distribution of electrostatic potential in the Fichera corner domain $\Omega = (-1,1)^3 \setminus [0,1]^3$
We solve the problem

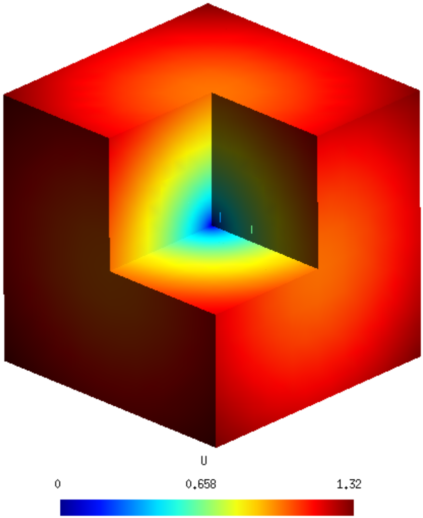
$$-\Delta u = f \quad \text{in } \Omega, \tag{1}$$
$$u = u_D \quad \text{on } \partial\Omega, \tag{2}$$

where $f$ and $u_D$ are chosen to comply with the exact solution

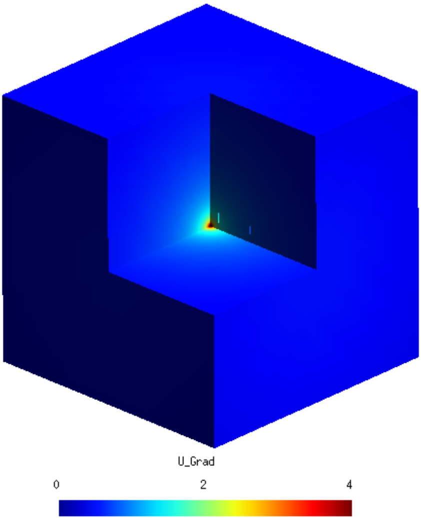
$$u(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)^{1/4}, \tag{3}$$

- The solution is smooth
- The gradient has singularity at $(0,0,0)$

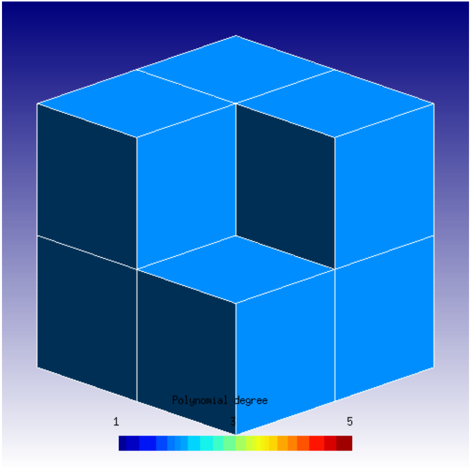
# Exact solution



# Gradient of the exact solution



# *hp*-adaptivity – Fichera corner domain



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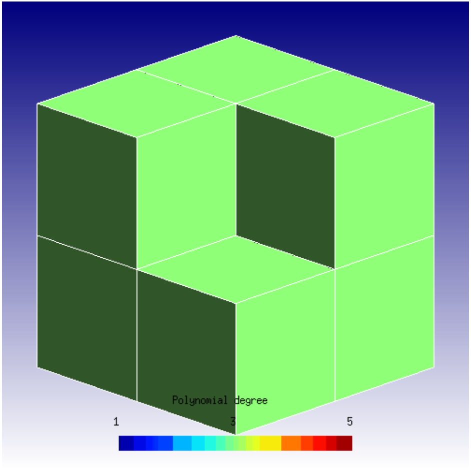

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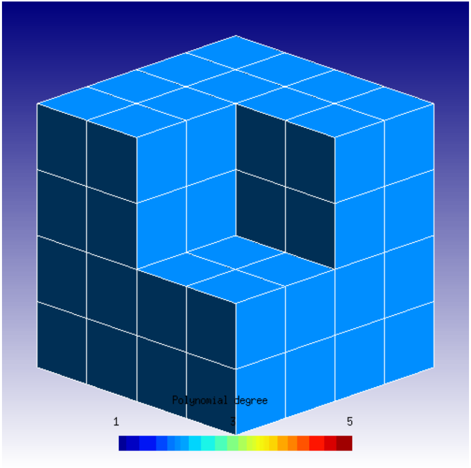

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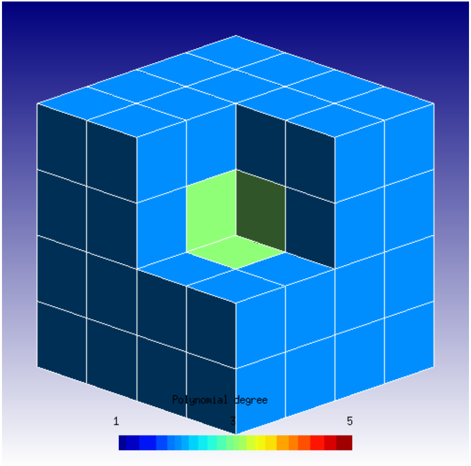

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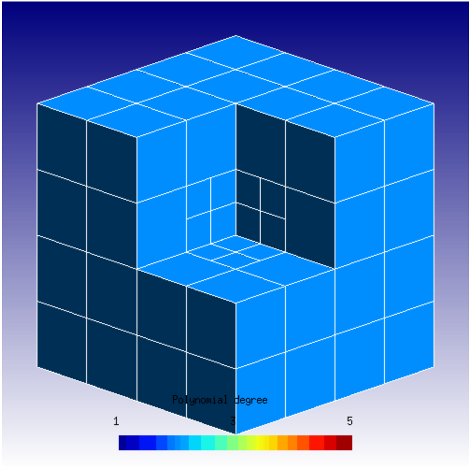

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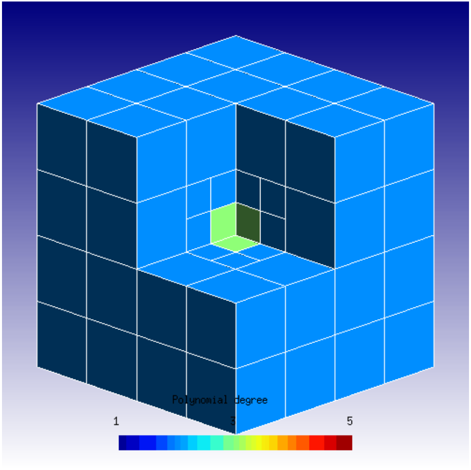

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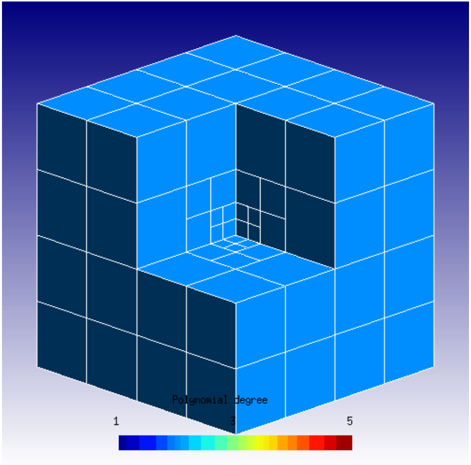

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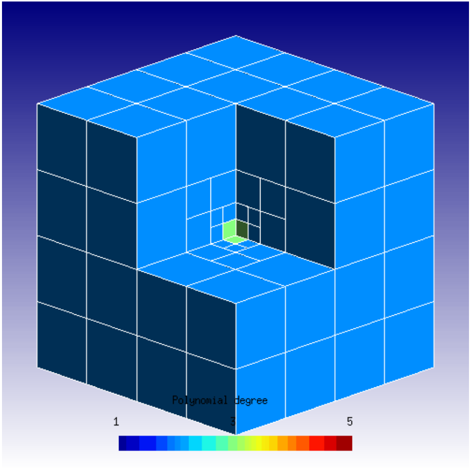

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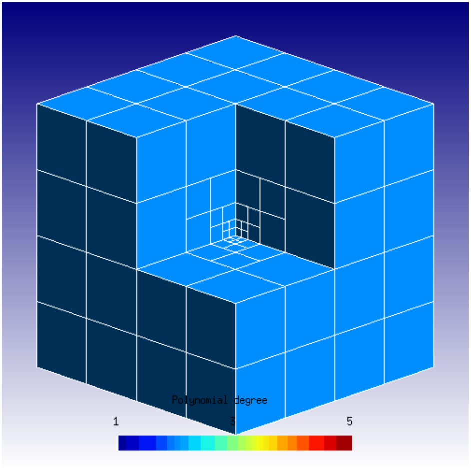

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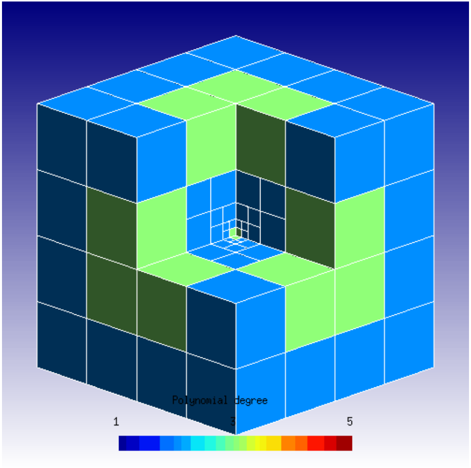

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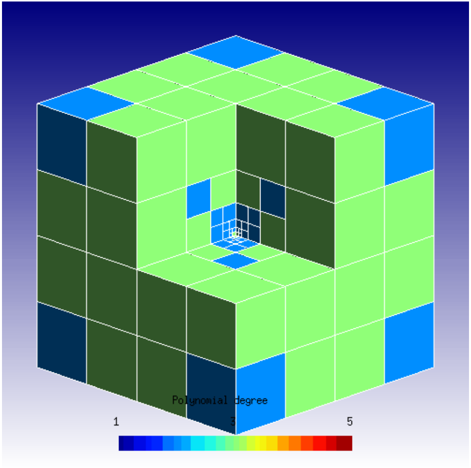

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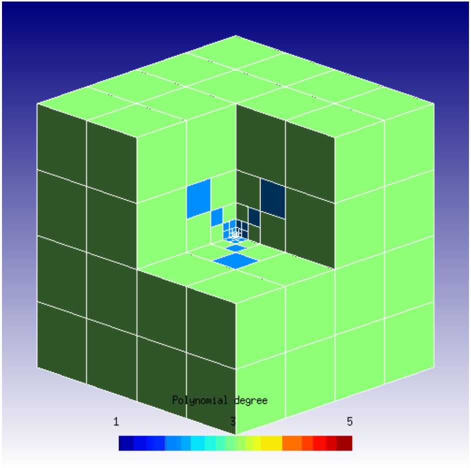

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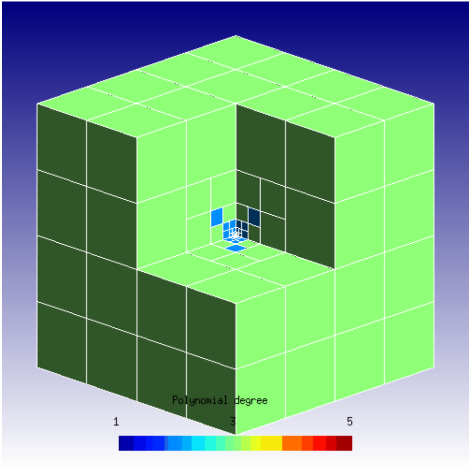

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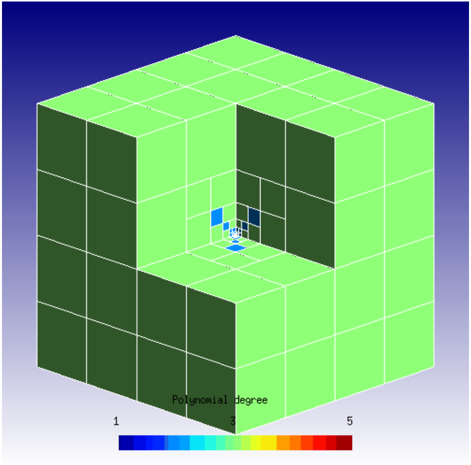

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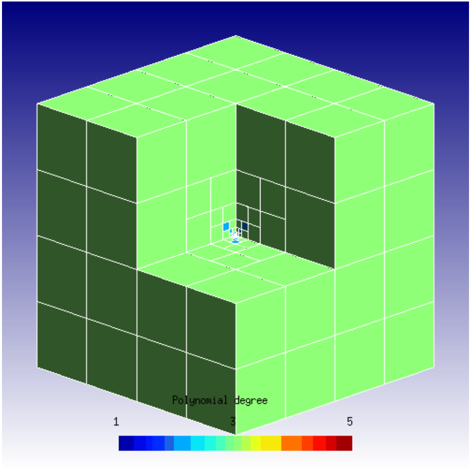

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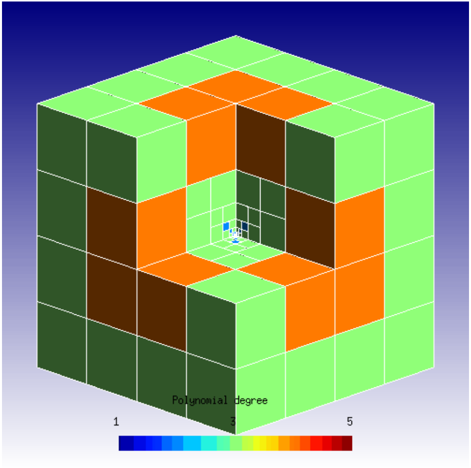

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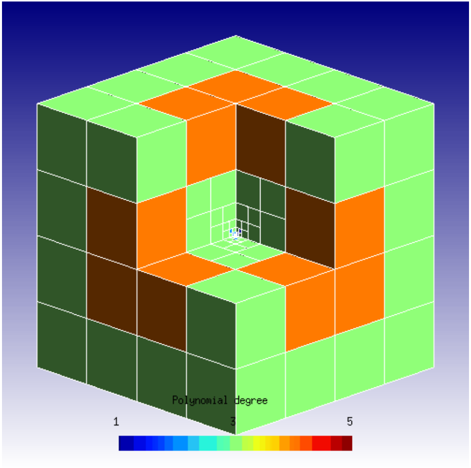

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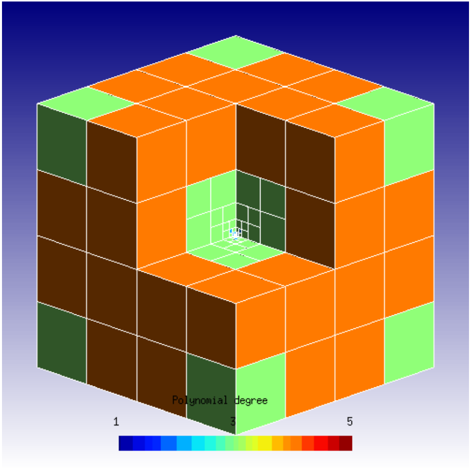

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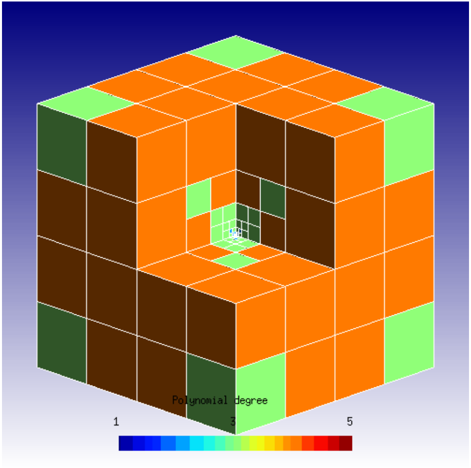

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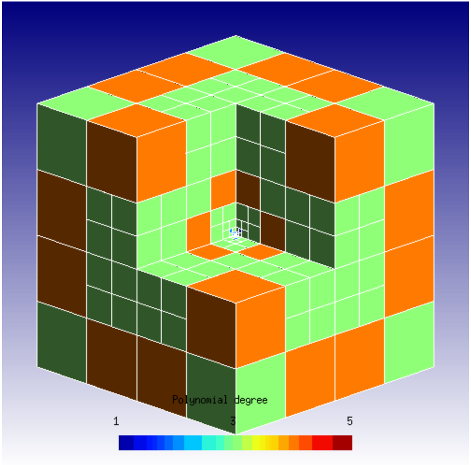

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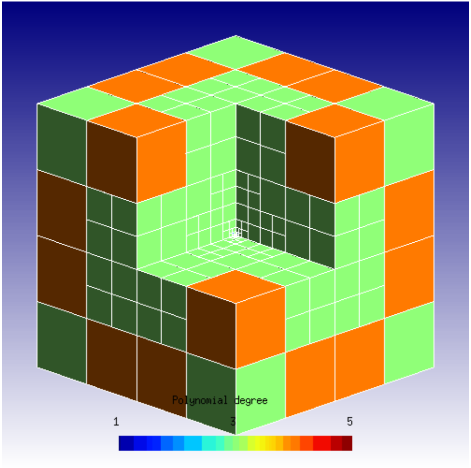

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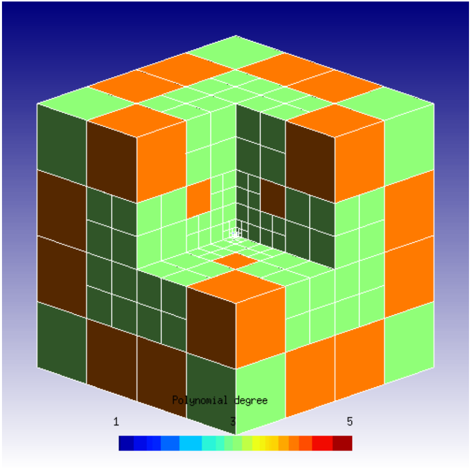
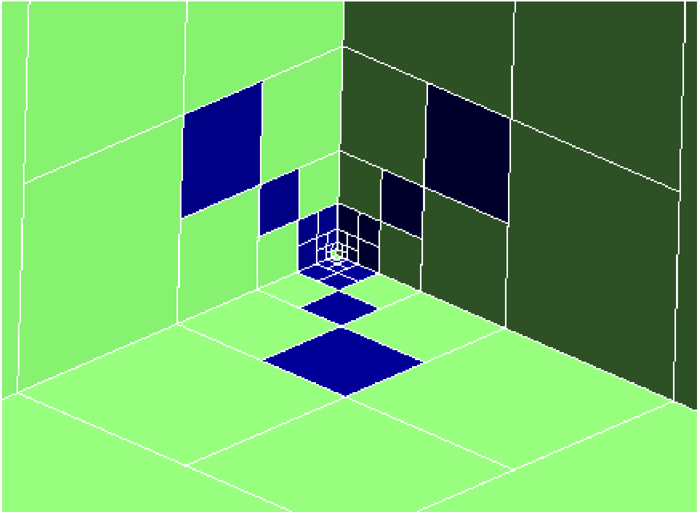


# Detail of the singularity



# Convergence comparison