The Box Method and Some Error Estimation

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Abstract

The box method is one of mathematical methods used to numerical solutions of thermal conduction. This article focuses its attention on a use of the box method for solution of certain type of partial differential equation. We consider heat conduction problem described by elliptic partial differential equation of second order with the Newton boundary conditions on rectangular domain. The article contains description of numerical solution procedure of heat problem and estimation of box method error. The solution of practical problem is presented as well.

1. Introduction

This paper deals with heat stationary conduction problem. Our objective is to solve classical formulation of the problem

$$(1)\quad -\frac{\partial}{\partial x_1}\left(a_{11}\frac{\partial u}{\partial x_1}\right)-\frac{\partial}{\partial x_2}\left(a_{22}\frac{\partial u}{\partial x_2}\right)+cu=f$$

in a rectangular domain $\Omega \subset \mathbf{R}^2$ with Newton's boundary condition

$$(2)\quad \alpha u+\frac{\partial}{\partial n_A}u=g\,.$$

Derivative with respect conormal in (2) is defined by relation

$$(3)\quad \frac{\partial u}{\partial n_A}=a_{11}\frac{\partial u}{\partial x_1}n_1+a_{22}\frac{\partial u}{\partial x_2}n_2$$

and $n=(n_1,n_2)$ denotes unit outward normal to $\partial\Omega$. Unknown function u denotes warming of material and we suppose $u\in C^2\left(\overline{\Omega}\right)$, functions $a_{11}, a_{22}, c, f \in C^1\left(\overline{\Omega}\right)$ and $\alpha, g\in C\left(\partial\Omega\right)$, $\alpha(s)\geq 0$ on $\partial\Omega$. The coefficients a_{11} and a_{22} describe the heat conduction character of the physical medium.

We will describe numerical solution procedure of heat conduction problem model by box method and error estimation of this method in following paragraphs.

2. Use of box method

We construct triangulation τ on closure of rectangular Ω $(X_1\leq x_1\leq X_2, Y_1\leq x_2\leq Y_2)$ by similar way as we use finite element method. We construct regular rectangular mesh with increments $h_1=\dfrac{X_2-X_1}{p}$ and $h_2=\dfrac{Y_2-Y_1}{q}$ in the x_1-axis and x_2-axis direction, respectively, where p and q denotes the number of segments, to which the region is divided in the x_1-axis and x_2-axis direction, respectively. The general node point has coordinates $V_{rs} = [X_1 + rh_1, Y_1 + sh_2]$, where $r \in \{0, 1, ..., p\}$, $s \in \{0, 1, ..., q\}$. The rectangles with vertexes defined in points of mesh created elements of triangulation τ. We construct special case of dual mesh to triangulation τ published in [4], p. 215. Points $T_i, 1 \le i \le 4,$ are midpoints of abscissas defined by mesh point V_{rs} and adjacent mesh points. Then points $S_i, 1 \le i \le 4,$ are intersection points of mentioned abscissas. The rectangle belongs to node V_{rs} and given by vertexes S_1, S_2, S_3 and S_4 created element b_{rs} of dual mesh to τ (see Figure 1).

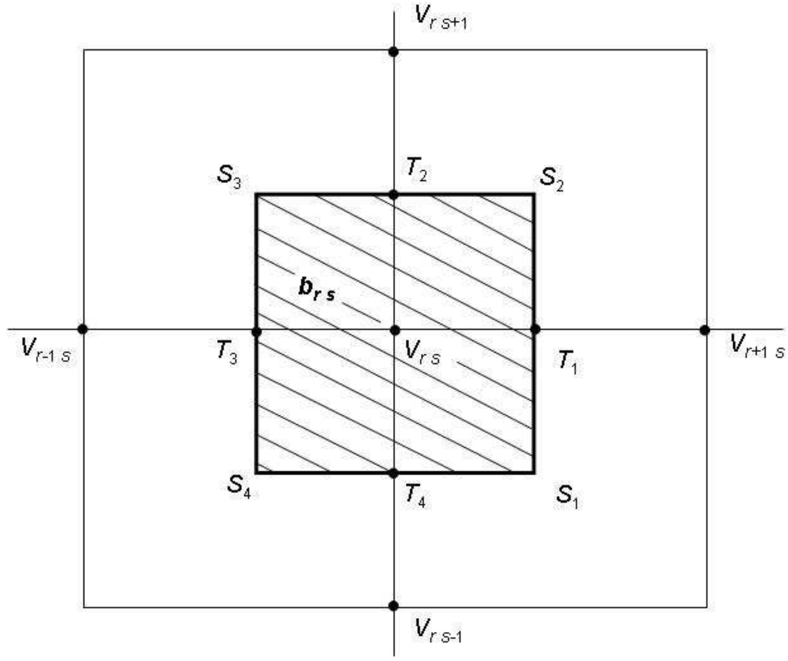

Figure 1 – Element b_{rs} of dual mesh belongs to node V_{rs}

If the node V_{rs} lies on the boundary of Ω, the element b_{rs} is by corresponding way modified (see Figure 2 and Figure 3).

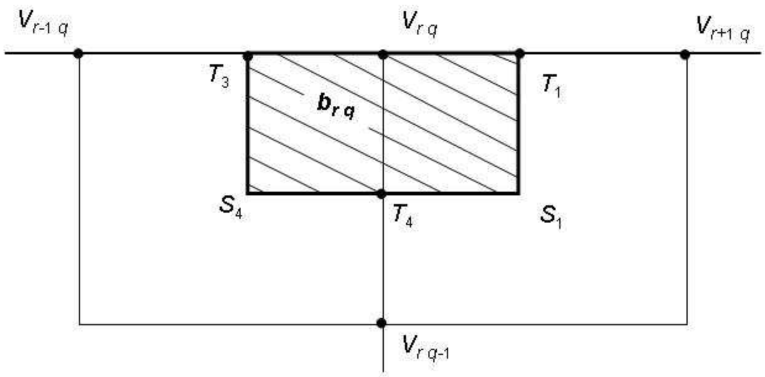

Figure 2 – Element b_{rq} of dual mesh belongs to boundary node V_{rq}

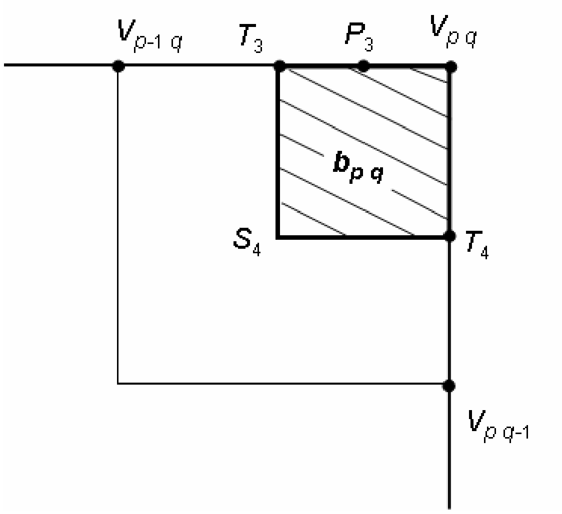

Figure 3 – Element b_{pq} of dual mesh belongs to "corner" boundary node V_{pq}

The elements b_{rs} are characterized by two aspects: $\overline{\Omega} = \bigcup_{0\le r\le p,\, 0\le s\le q} b_{rs}$ and $\operatorname{int} b_{rs} \cap \operatorname{int} b_{kl} = \emptyset$ for $V_{rs} \ne V_{kl}$.

We can transfer member $c\,u$ to the right hand side in equation (1) and integrate left and right hand side over element b_{rs} .We then get

$$(4)\quad \int_{b_{rs}} \left[-\frac{\partial}{\partial x_1}\left(a_{11}\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(a_{22}\frac{\partial u}{\partial x_2}\right)\right] dx = \int_{b_{rs}} \left(f - cu \right) dx\ .$$

Using now Green's formula on left hand of relation (4), we find that

$$\int_{b_{rs}} \left[-\frac{\partial}{\partial x_1}\left(a_{11}\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(a_{22}\frac{\partial u}{\partial x_2}\right)\right] dx = -\int_{b_{rs}} \frac{\partial}{\partial x_1}\left(a_{11}\frac{\partial u}{\partial x_1}\right) dx - \int_{b_{rs}} \frac{\partial}{\partial x_2}\left(a_{22}\frac{\partial u}{\partial x_2}\right) dx =$$

$$= -\int_{\partial b_{rs}} a_{11}\frac{\partial u}{\partial x_1} n_1\, ds - \int_{\partial b_{rs}} a_{22}\frac{\partial u}{\partial x_2} n_2\, ds.$$

Then relation (4) can be modified in the form of

$$(5)\quad -\int_{\partial b_{rs}} a_{11}\frac{\partial u}{\partial x_1} n_1\, ds - \int_{\partial b_{rs}} a_{22}\frac{\partial u}{\partial x_2} n_2\, ds = \int_{b_{rs}} \left(f - cu \right) dx\ .$$

The left hand side of equations (6) describes the quantity of heat supplied from or delivered to boundary of element b_{rs}, the right hand side express waste heat arising in the element b_{rs}. Because element b_{rs} is rectangle, in case of internal element we can make an approximation

(6) $a_{11}(T_1)\dfrac{\partial u(T_1)}{\partial x_1}n_1 \approx a_{11}(T_1)\dfrac{u(V_{r+1\,s})-u(V_{rs})}{h_1}$.

With respect to supposed smooth of function u, we make the $O(h_1^2)$-order error mistake in approximation (6). Similar approximations are possible to carry out for points T_2, T_3 and T_4.

We focus now on boundary element , for example element b_{rq} in Figure 2. Using relation (2) and (3) we obtain

(7) $a_{22}\left(V_{rq}\right)\dfrac{\partial u}{\partial x_2}(V_{rq})n_2 = g(V_{rq}) - \alpha\,(V_{rq})\,u(V_{rq})$.

In case of boundary "corner" element (see Figure 3), we can make an approximation of the value $u(P_3)$ from values $u(V_{p-1q})$ and $u(V_{pq})$ with using Lagrange's interpolation polynomial of first degree. Because we suppose $u \in C^2(\overline{\Omega})$, error order of approximation is $O(h_1^2)$ (see [3], p.64) and the value $a_{22}(P_3)\dfrac{\partial u}{\partial x_2}(P_3)\eta_2$ is possible to approximate through the use of relation (2) .

Now we target the approximation of integrals in relation (5). However, first we introduce auxiliary thesis.

Lemma.

If function $v \in C^1_{<a,b>}$, then $\left|\displaystyle\int_a^b v(x)\,dx - v\left(\frac{a+b}{2}\right)\cdot\left(b-a\right)\right| \le M\left(b-a\right)^2$, where $M \in \mathbf{R}$.

Proof.

Let us consider Lagrange's interpolation polynomial of degree equal to 0 with interpolate point $\dfrac{a+b}{2}$. Using now Lagrange's theorem about error of Lagrange's interpolation polynomial (see [3], p. 63), for arbitrary x exists value $\xi \in <\min(x, \frac{a+b}{2}), \max(x, \frac{a+b}{2})>$, that is true

$v(x) = P_0\left(\dfrac{a+b}{2}\right) + v'(\xi)\left(x - \dfrac{a+b}{2}\right)$.

Then $\displaystyle\int_a^b v(x)\,dx = \int_a^b v\left(\frac{a+b}{2}\right)dx + \int_a^b v'(\xi)\left(x-\frac{a+b}{2}\right)dx$ and we can use first mean value theorem for integration

$$\int_a^b v(x)\,dx - \int_a^b v\left(\frac{a+b}{2}\right)dx = \int_a^b v'(\xi)\left(x-\frac{a+b}{2}\right)dx = \int_a^{\frac{a+b}{2}} v'(\xi)\left(x-\frac{a+b}{2}\right)dx +$$

$$+\int_{\frac{a+b}{2}}^{b} v'(\xi)\left(x-\frac{a+b}{2}\right)dx = v'(\eta_1)\int_{a}^{\frac{a+b}{2}}\left(x-\frac{a+b}{2}\right)dx + v'(\eta_2)\int_{\frac{a+b}{2}}^{b}\left(x-\frac{a+b}{2}\right)dx =$$

$$= v'(\eta_1)\frac{1}{2}\left[\left(x-\frac{a+b}{2}\right)^2\right]_a^{\frac{a+b}{2}} + v'(\eta_2)\frac{1}{2}\left[\left(x-\frac{a+b}{2}\right)^2\right]_{\frac{a+b}{2}}^{b} = v'(\eta_1)\frac{1}{2}\left(-\left(a-\frac{a+b}{2}\right)^2\right)+$$

$$+v'(\eta_2)\frac{1}{2}\left(b-\frac{a+b}{2}\right)^2 = v'(\eta_1)\frac{1}{2}\left(-\frac{(a-b)^2}{4}\right)+v'(\eta_2)\frac{1}{2}\frac{(b-a)^2}{4} = -v'(\eta_1)\frac{1}{8}(b-a)^2 +$$

$$+v'(\eta_2)\frac{1}{8}(b-a)^2 = (v'(\eta_2)-v'(\eta_1))\frac{1}{8}(b-a)^2,$$

where $\eta_1, \eta_2 \in <a, b>$ and generally $\eta_1 \neq \eta_2$.
Hence we obtain

$$\left|\int_a^b v(x)\,dx - \int_a^b v\left(\frac{a+b}{2}\right)dx\right| = \left|(v'(\eta_2)-v'(\eta_1))\frac{1}{8}(b-a)^2\right| = |v'(\eta_1)-v'(\eta_2)|\frac{1}{8}(b-a)^2 \leq$$

$$\leq M(b-a)^2,\ \text{kde}\ \ M\in\mathbf{R}\ \ \text{a}\ \ M\geq 0,\ \text{neboť}\ \ v\in C^1_{<a,b>}.$$

Let us set $h = \max(h_1, h_2)$. From lemma implied, error of the midpoint rule applications to approximation of integrals in equation (5) is $O(h^2)$ order for every internal element. In boundary elements, we use midpoint rule and rectangle rule too. Analogously it is possible to prove that error of the rectangle rule is in this case $O(h^2)$ (see [1], p.178).
By application of the above mentioned approximations for every element b_{rs} we obtain system of linear algebraic equations. Approximation error is order $O(h^2)$.
The general questions of box method estimation error are solved in [2].

3. Practical numerical example

We will solve now a real-live technical problem finding the warming in aluminium oil transformer screening by using above mentioned box method. Transformer screening is considered in the form of a thin-walled cylinder and the temperature field is supposed to be rotationally symmetric. Hence the warming problem can be solved in screening cross section on two dimensional closed rectangular domain Ω. Then the problem is defined by equation

$$(7)\quad \frac{\partial}{\partial x_1}\left(\lambda_1 x_1 \frac{\partial u}{\partial x_1}\right) + x_1\frac{\partial}{\partial x_2}\left(\lambda_2\frac{\partial u}{\partial x_2}\right) = -x_1\delta^2(x_2)\rho(1+\alpha_T u)$$

with the Newton boundary condition

(8) $\lambda_1 \dfrac{\partial u}{\partial x_1}\eta_1 + \lambda_2 \dfrac{\partial u}{\partial x_2}\eta_2 + \alpha u = \alpha k(x_2 - Y_1)$

on rectangular Ω. Here solution u denotes screening warming, real values λ_1 and λ_2 stand for heat conductivities of the material in the x_1-axis and x_2-axis directions, respectively; $\delta(x_2)$ denotes the density of eddy currents, ρ is the specific resistance of the material, and α_T is the factor for the dependence of a specific resistance on temperature. In boundary condition (8) the constant α means heat transfer coefficient on the boundary of domain, real constant k allows to express the variable temperature of oil in the vicinity of screening in the x_2-axis direction. The problem has the following parameters:
$X_1 = 0.86$ m, $X_2 = 0.868$ m, $Y_1 = 0.8864$ m, $Y_2 = 2.51$ m, $\lambda_1 = \lambda_2 = 220$ W/mK, $\rho = 0.3\times10^{-7}\,\Omega$ m, $\alpha_T = 0.00409\,\mathrm{K}^{-1}$, $\alpha = 50\,\mathrm{W/m^2K}$, $k = 10$ K/m, the current density $\delta(x_2)$ is given by means of 19 values between $0.2498\times10^5\,\mathrm{Am}^{-2}$ and $0.3508\times10^7\,\mathrm{Am}^{-2}$, the current density at the other node points is computed by means of linear interpolation. Table 1 lists approximate values of warming in chosen nodes computed using the box method.

x_2 [m]	$X_1 = 0{,}860$ [m]	$x_1 = 0{,}864$ [m]	$X_2 = 0{,}868$ [m]
$Y_2 = 2{,}51$	29,444	29,446	29,444
2,30705	24,423	24,423	24,423
2,10410	22,133	22,133	22,133
1,90115	20,682	20,682	20,682
1,69820	19,466	19,467	19,466
1,49525	18,569	18,571	18,569
1,29230	17,567	17,569	17,567
1,08935	15,535	15,537	15,535
$Y_1 = 0{,}8864$	12,809	12,811	12,809

Table 1: The values of screening warming in K for selected nodes at $h_1 = 0.004$ m and $h_2 = 0.025369$ m.

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