Discontinuous Galerkin method for convection-diffusion problems

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Outline

Introduction

- **Our aim**: efficient, accurate and robust numerical scheme for the simulation of viscous compressible flows,
- **Model problem**:
  scalar nonstationary convection–diffusion equation with nonlinear convection and nonlinear diffusion,
- discontinuous Galerkin finite element method (DGFEM) with NIPG, SIPG or IIPG variant,
- error estimates of DGFEM for nonlinear nonstationary convection–diffusion problems

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Scalar convection-diffusion equation

- Let $\Omega \subset \boldsymbol{R}^2$, $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, $Q_T \equiv \Omega \times (0, T)$, we seek $u : Q_T \to \boldsymbol{R}$ such that

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{f}(u) - \nabla \cdot (\mathbb{K}(u)\nabla u) = g \quad \text{in } Q_T, \tag{1}$$

$$u = u_D \quad \text{on } \partial\Omega_D,\ t \in (0, T), \tag{2}$$

$$\mathbb{K}(u)\nabla(u) \cdot \vec{n} = g_N \quad \text{on } \partial\Omega_N,\ t \in (0, T), \tag{3}$$

$$u(x, 0) = u^0(x), \quad x \in \Omega, \tag{4}$$

where: $\vec{f} = (f_1, f_2)$, $f_s \in C^1(\boldsymbol{R}), s = 1, 2,$
$\mathbb{K}(u)$ are matrices 2x2.

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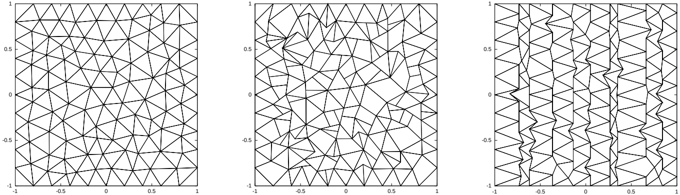
Triangulations



- let $\mathcal{T}_h, \; h > 0$ be a partition of $\overline{\Omega}$
- $\mathcal{T}_h = \{K\}_{K \in \mathcal{T}_h}$, K are polygons (convex, nonconvex),
- let $\mathcal{F}_h = \{\Gamma\}_{\Gamma \in \mathcal{F}_h}$ be a set of all faces of $\mathcal{T}_h$,
- we distinguish
  - inner faces $\mathcal{F}_h^I$,
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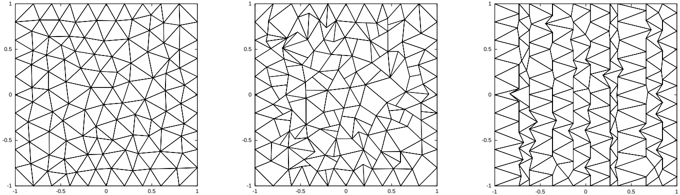

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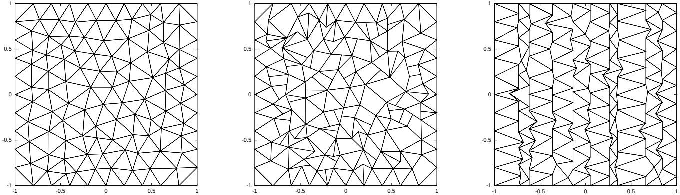

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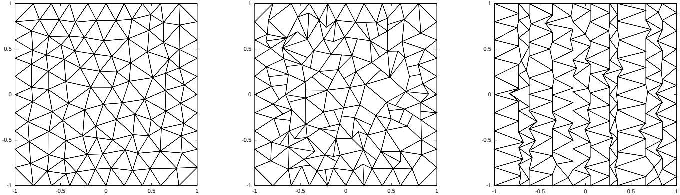

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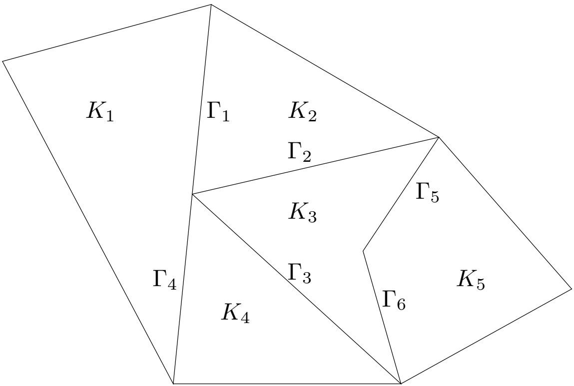
Notation



Spaces of discontinuous functions

- let $s \geq 1$ denote the Sobolev index,
- let $p \geq 1$ polynomial degree,
- over $\mathcal{T}_h$ we define:
  - *broken Sobolev space*

$$H^s(\Omega, \mathcal{T}_h) = \{v; v|_K \in H^s(K)\ \forall\, K \in \mathcal{T}_h\}$$

  - the space of piecewise polynomial functions

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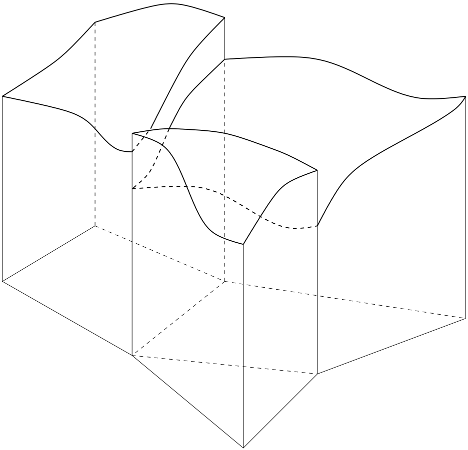
Example of a function from $S_{hp} \subset H^s(\Omega, \mathcal{T}_h)$



Broken Sobolev spaces, cont.

for $H^s(\Omega, \mathcal{T}_h)$ we define

- the seminorm

$$|v|_{H^k(\Omega,\mathcal{T}_h)} \equiv \left(\sum_{K\in\mathcal{T}_h} |v|^2_{H^k(K)} \right)^{1/2}.$$

- for $u \in H^1(\Omega, \mathcal{T}_h)$
  - $\langle v \rangle_\Gamma$ = mean value of v over face Γ,
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Space discretization

- let u be a strong (regular) solution,
- we multiply (1) by $v \in H^2(\Omega, \mathcal{T}_h)$,
- integrate over each $K \in \mathcal{T}_h$,
- apply Green's theorem,
- sum over all $K \in \mathcal{T}_h$,
- we include additional terms vanishing for regular solution,
- we obtain the identity

$$\left(\frac{\partial u}{\partial t}(t), v\right) + a_h(u(t), v) + b_h(u(t), v) + J_h^\sigma(u(t), v) = \ell_h(v)(t) \quad \forall v \in H^2(\Omega, \mathcal{T}_h)\ \forall t \in (0, T), \quad (5)$$

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Diffusive form

- diffusion term: $-\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot(\mathbb{K}(u)\nabla u)\, v\,\mathrm{d}x$,

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\begin{aligned}
a_h(u,v) &= \sum_{K\in\mathcal{T}_h}\int_K \mathbb{K}(u)\nabla u\cdot\nabla v\,\mathrm{d}x \\
&\quad - \sum_{\Gamma\in\mathcal{F}_h^{ID}}\int_\Gamma \langle\mathbb{K}(u)\nabla u\rangle\cdot\vec{n}[v]\mathrm{d}S \\
&\quad + \eta\sum_{\Gamma\in\mathcal{F}_h^{ID}}\int_\Gamma \langle\mathbb{K}(u)\nabla v\rangle\cdot\vec{n}[u]\mathrm{d}S,
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- $\eta=-1$ SIPG formulation,
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Convective form

- convective term ("finite volume approach"):

$$\sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\vec{f}(u)\,v\,\mathrm{d}x$$

$$= -\sum_{K\in\mathcal{T}_h}\int_K \vec{f}(u)\cdot\nabla v\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\vec{f}(u)\cdot\vec{n}\,v\,\mathrm{d}S.$$

$$\vec{f}(u)\cdot\vec{n}|_\Gamma \approx H\left(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \vec{n}_\Gamma\right), \quad \Gamma\in\mathcal{F}_h,$$

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Definition of forms, cont.

Interior and boundary penalty

$$J_h^\sigma(u,v) = \sum_{\Gamma \in \mathcal{F}_h^I} \int_\Gamma \sigma [u]\,[v]\,\mathrm{d}S + \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \sigma\, u\, v\,\mathrm{d}S,$$

$$\sigma_\Gamma = \frac{C_W}{d(\Gamma)}, \quad d(\Gamma) \equiv \min(d(K_\Gamma^{(L)}), d(K_\Gamma^{(R)})), \quad d(K) \equiv \frac{h_K}{p_K^2}$$

Right-hand-side

$$\ell_h(v)(t) = \int_\Omega g(t)\, v\,\mathrm{d}x + \sum_{\Gamma \in \mathcal{F}_h^N} \int_\Gamma g_N(t)\, v\,\mathrm{d}S + \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \left(\eta \mathbb{K}(u) \nabla v \cdot \vec{n}\, u_D(t) + \sigma\, u_D(t)\, v\right) \mathrm{d}S$$

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$$\ell_h(v)(t) = \int_\Omega g(t)\, v\,\mathrm{d}x + \sum_{\Gamma\in\mathcal{F}_h^N}\int_\Gamma g_N(t)\, v\,\mathrm{d}S$$

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Definition of forms, cont.

Interior and boundary penalty

$$J_h^\sigma(u,v) \;=\; \sum_{\Gamma\in\mathcal{F}_h^I}\int_\Gamma \sigma[u]\,[v]\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}_h^D}\int_\Gamma \sigma\, u\, v\,\mathrm{d}S,$$

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Semi-discrete variant

- For $u(t,x) \in C^1(0,T;H^2(\Omega))$, we have identity

$$\left(\frac{\partial u}{\partial t}(t), v\right) + a_h(u(t),v) + b_h(u(t),v) + J_h^\sigma(u(t),v) = \ell_h(v)(t), \quad v \in H^2(\Omega,\mathcal{T}_h),\ t \in (0,T), \tag{6}$$

- (6) makes sense also for $u \in H^2(\Omega,\mathcal{T}_h)$.
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Definition

We say that u_h is a DGFE solution iff

a) $u_h \in C^1(0, T; S_{hp})$,

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c) $u_h(0) = u_h^0$,

- system of ODEs,
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Interior penalty (1)

- penalty form

$$J_h^\sigma(u,v) = \sum_{\Gamma\in\mathcal{F}_h^{ID}} \frac{C_W}{d(\Gamma)} \int_\Gamma [u]\,[v]\,\mathrm{d}S,$$

- $J_h^\sigma(u,v)$ "replace" inter-element continuity,
- $J_h^\sigma(u,v)$ ensures the coercivity, i.e, $\exists c > 0$

$$a_h(v,v)+J_h^\sigma(v,v) \geq c\,|||v|||^2, \quad |||v|||^2 \equiv |v|^2_{H^1(\Omega,\mathcal{T}_h)} + J_h^\sigma(v,v),$$

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Interior penalty (2) - choice of C_W

- NIPG: $C_W > 0$ is sufficient since

$$a_h(v, v) \geq c_1 |v|^2_{H^1(\Omega, \mathcal{T}_h)},$$

- SIPG: $C_W \geq \widetilde{C_W}$,
  - linear diffusion: [Dolejší, Feistauer, NFAO 2005],
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Error estimates for space semi-discretization - assumptions

- non-linear diffusion: $-\nabla \cdot (\mathbb{K}(u) \cdot \nabla u)$,
  where $\mathbb{K}(u) = \{k_{ij}(u)\}_{i,j=1}^{2}$ satisfy:
  - $k_{ij}(u) : R \to R$, such that $|k_{ij}(u)| < C_U < \infty, \ i,j = 1,2$,
  - $k_{ij}(u)$ is Lipschitz continuous for $i,j = 1,2$,
  - $\xi^T \mathbb{K}(u) \xi \geq C_E \|\xi\|^2, \ C_E > 0, \xi \in R^2$
- u is sufficient regular:
  - $u \in L^2(0,T;H^{s+1}), \ \partial u/\partial t \in L^2(0,T;H^s), \ s \geq 1$,
  - $\|\nabla u(t)\|_{L^\infty(\Omega)} \leq C_D$ for a. a. $t \in (0,T)$
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error estimates

- sub-optimal in the L^2-norm, i.e., $O(h^{\mu-1})$,
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$$\frac{\partial u}{\partial t} + \sum_{s=1}^{2} u \frac{\partial u}{\partial x_s} - \varepsilon \Delta u = g \quad \text{in} \quad Q_T = [-1, 1]^2 \times (0, T)$$

- nonlinear $f_1(u) = f_2(u) = u^2/2$ and linear $\mathbb{K}(u) = \varepsilon \mathbb{I}$,
- numerical flux:

$$H\left(u|_\Gamma^{(L)}, u|_\Gamma^{(R)}, \vec{n}_\Gamma\right) = \begin{cases} \sum_{s=1}^{2} f_s(u|_\Gamma^{(L)}) n_s, \text{if } A > 0 \\ \sum_{s=1}^{2} f_s(u|_\Gamma^{(R)}) n_s, \text{if } A \le 0 \end{cases},$$

where $A = \sum_{s=1}^{2} f_s'(\langle u \rangle) n_s$,

- exact solution:

$$u(x, y, t) = (1 - x^2)^2 (1 - y^2)^2 \left(1 - \frac{e^{-t}}{2}\right)$$

- mesh with "hanging nodes", P_1 approximation, SIPG variant

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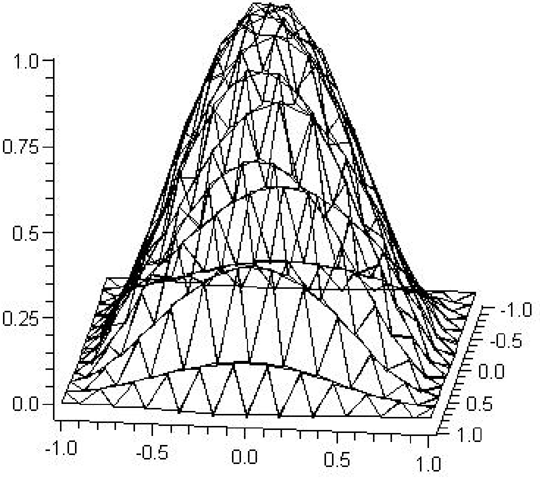
Numerical example (2)

- experimental orders of convergence (EOC)
- EOC is optimal in L^2-norm, i.e. $O(h^2)$ for P_1 approximation

			$t = 4.0$		$t \to \infty$	
l	$\#\mathcal{T}_{h_l}$	h_l	e_h	α_l	e_h	α_l
1	136	2.795E-01	1.6599E-02	-	7.0934E-02	-
2	253	2.033E-01	8.3203E-03	2.169	3.0605E-02	2.640
3	528	1.398E-01	3.8102E-03	2.084	1.1299E-02	2.659
4	1081	9.772E-02	1.8194E-03	2.037	5.7693E-03	1.852
5	2080	6.988E-02	9.1509E-04	2.081	3.0657E-03	1.915
6	4095	4.969E-02	4.7598E-04	1.917	1.4538E-03	2.188
	$\overline{\alpha}$			2.059		2.214

Numerical example (3)

- steady-state solution



Navier-Stokes equations

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^{2} \frac{\partial}{\partial x_s} \mathbf{f}_s(\mathbf{w}) = \sum_{s=1}^{2} \frac{\partial}{\partial x_s} \left(\sum_{k=1}^{2} \mathbf{K}_{sk}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_k} \right), \quad (7)$$

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- $\mathbf{w} : \Omega \times (0, T) \to R^4$,
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DGFEM for the Navier-Stokes equations

Space semi-discretization

- inviscid terms: finite volume approach
- viscous terms: SIPG, NIPG, IIPG techniques
- interior and boundary penalty: heuristic choice of C_W.

Other aspects

- semi-implicit time discretization,
- unconditionally stable higher order scheme,
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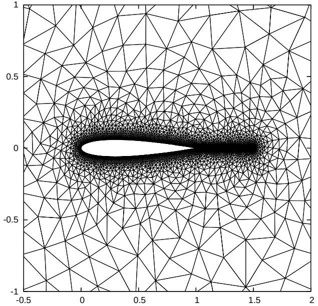
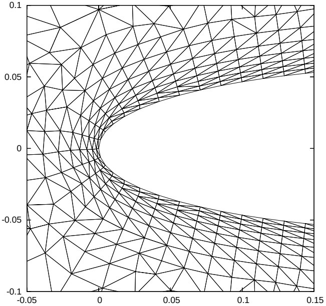
- steady non-symmetric laminar flow around the NACA0012 ($M = 0.5, \alpha = 2.0°, Re = 5000$)
- adaptive refined mesh





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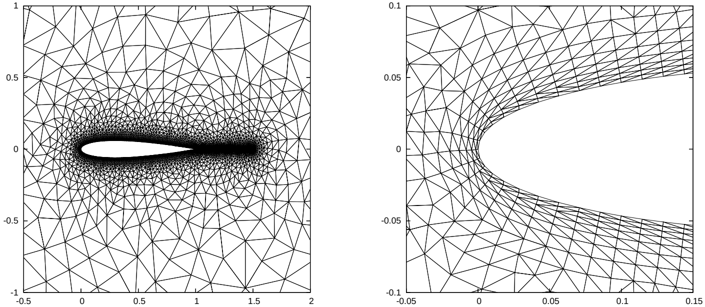

NACA 0012 profile – steady flow(2)

- semi-implicit scheme, $P_1 - P_3$ approximation, adaptive BDF scheme
- SIPG, IIPG, NIPG variant of DGFEM
- drag and lift coefficients (comparison with DLR, VKI)

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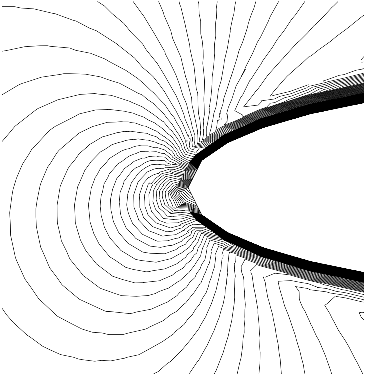
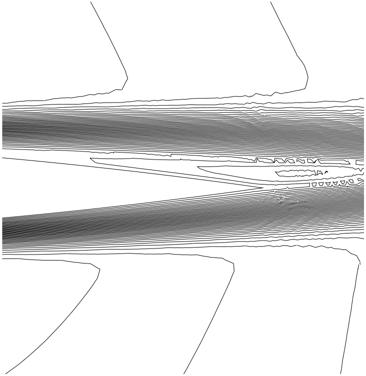
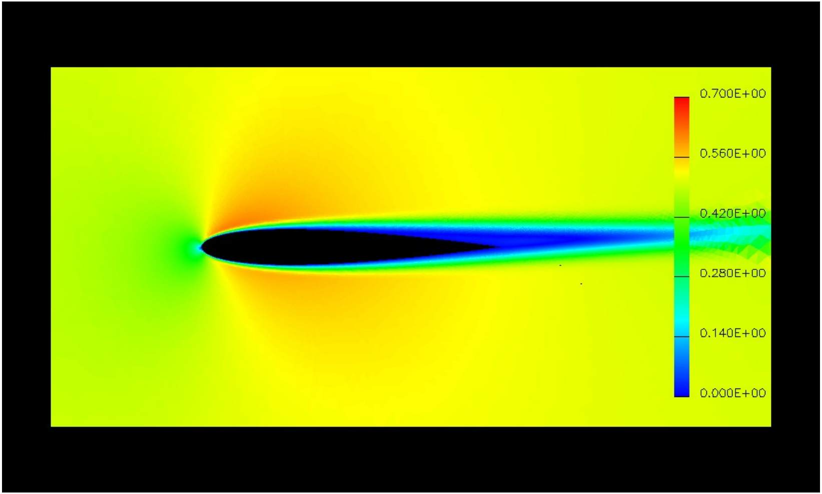
NACA 0012 profile – steady flow(3)

- Mach number isolines

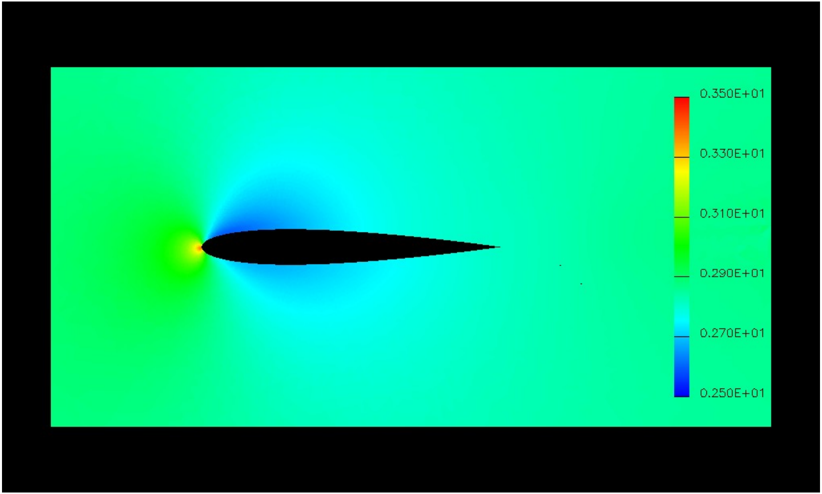




Mach number distribution, $t \longrightarrow \infty$



Pressure distribution, $t \longrightarrow \infty$



Thank you for your attention