Discontinuous Galerkin method for the simulation of 3D viscous compressible flows

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Outline

Introduction

- **Our aim**: efficient, robust and accurate numerical scheme for a simulation of 3D compressible flows (aerodynamics),
- system of the compressible Navier–Stokes equations,
- DGM – discontinuous Galerkin method,
  DGFE method with SIPG,NIPG and IIPG variant,
- BDF – backward difference formula,
- semi–implict scheme – higher order in space and time coordinates.

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System of the Navier-Stokes equations

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \cdot \vec{f}(\mathbf{w}) = \nabla \cdot \vec{R}(\mathbf{w}, \nabla \mathbf{w}) \quad \text{in } \Omega \times (0, T),$$

$$\mathbf{w} = (\rho, \rho v_1, \rho v_2, \rho v_3, E)^{\mathrm{T}},$$

$$\mathbf{f}_s(\mathbf{w}) = (\rho v_s, \rho v_s v_1 + p\delta_{s1}, \rho v_s v_2 + p\delta_{s2}, \rho v_s v_3 + p\delta_{s3}, (E+p)\, v_s)^{\mathrm{T}}, \quad s = 1, 2, 3,$$

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Properties of the inviscid/viscous fluxes

- inviscid fluxes $\mathbf{f}_s$:

$$\mathbf{f}_s(\mathbf{w}) \quad = \quad \mathbf{A}_s(\mathbf{w})\mathbf{w}, \; s = 1, 2, 3,$$

$$\mathbf{P} = \sum_{s=1}^{3} n_s \mathbf{A}_s(\mathbf{w}) = T \mathbf{\Lambda} T^{-1} = \mathbf{P}^{+}(\mathbf{w}, \vec{n}) + \mathbf{P}^{-}(\mathbf{w}, \vec{n})$$

- viscous fluxes $\mathbf{R}_s$:

$$\mathbf{R}_s(\mathbf{w}, \nabla\varphi) = \sum_{k=1}^{3} \mathbf{K}_{s,k}(\mathbf{w}) \frac{\partial \varphi}{\partial x_k}, \qquad s = 1, 2, 3.$$

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Discretization — Notations

- DGFE method – piecewise polynomial discontinuous approximation,
- $\mathcal{T}_h$ – triangulation of Ω (tetrahedra, pyramids or hexahedra),
- $\mathbf{S}_h \equiv (S_h)^5, \quad S_h \equiv \{v; v|_K \in P^s(K) \ \forall K \in \mathcal{T}_h\},$
- traces:
  $v\big|_{\Gamma(R)} =$ trace of $v|_{K_R}$ on Γ, $\quad v\big|_{\Gamma(L)} =$ trace of $v|_{K_L}$ on Γ,
- jump: $[v]_\Gamma = v\big|_{\Gamma(L)} - v\big|_{\Gamma(R)},$
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space semidiscretization (1)

- diffusion form: for $\mathbf{w}, \varphi \in \mathbf{S}_h$:

$$\begin{aligned}
\bar{\mathbf{a}}_h(\mathbf{w}, \varphi) &= \sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^{3} R_s(\mathbf{w}, \nabla \mathbf{w}) \frac{\partial \varphi}{\partial x_s} \, \mathrm{d}x \\
&- \sum_{\Gamma \in \mathcal{F}_h^{ID}} \int_\Gamma \sum_{s=1}^{3} (\langle \sum_{k=1}^{3} \mathbf{K}_{s,k}(\mathbf{w}) \frac{\partial \mathbf{w}}{\partial x_k} \rangle n_s) \cdot [\varphi] \, \mathrm{d}S \\
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space semidiscretization (2)

- convection form: for $\mathbf{w}, \varphi \in \mathbf{S}_h$:

$$\bar{\mathbf{b}}_h(\mathbf{w}, \varphi) = -\sum_{K \in \mathcal{T}_h} \int_K \vec{f}(\mathbf{w}) \cdot \nabla \varphi \, \mathrm{d}x + \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma H\left(\mathbf{w}|_\Gamma^{(L)}, \mathbf{w}|_\Gamma^{(R)}, \vec{n}_\Gamma\right) [\varphi]_\Gamma \, \mathrm{d}S,$$

H is the Vijayasundaram numerical flux,

$$H\left(\mathbf{w}|_\Gamma^{(L)}, \mathbf{w}|_\Gamma^{(R)}, \vec{n}_\Gamma\right) = \mathbf{P}^+\left(\langle \mathbf{w}_h \rangle, \vec{n}\right) \mathbf{w}_h|_\Gamma^{(L)} + \mathbf{P}^-\left(\langle \mathbf{w}_h \rangle, \vec{n}\right) \mathbf{w}_h|_\Gamma^{(R)}$$

- stabilization form: for $\mathbf{w}, \varphi \in \mathbf{S}_h$:

$$J_h^\sigma(\mathbf{w}, \varphi) = \sum_{\Gamma \in \mathcal{F}_h^{ID}} \int_\Gamma \sigma [\mathbf{w}] [\varphi] \, \mathrm{d}S - \sum_{\Gamma \in \mathcal{F}_h^D} \int_\Gamma \sigma \, \mathbf{w}_B(t) \, \varphi \, \mathrm{d}S$$

$\sigma|_\Gamma = (|\Gamma| \, Re)^{-1}$,

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Semidiscrete problem

- method of lines for the N.S. equation
- approximate solution $\mathbf{w}_h(t) \in \mathbf{S}_h$ satisfies the identity:

$$\left(\frac{\partial \mathbf{w}_h(t)}{\partial t}, \varphi_h\right) + \bar{\mathbf{a}}_h(\mathbf{w}_h(t), \varphi_h) + \bar{\mathbf{b}}_h(\mathbf{w}_h(t), \varphi_h) + J_h(\mathbf{w}_h, \varphi_h) = 0 \quad \forall \varphi_h \in \mathbf{S}_h \; \forall t \in (0, T),$$

- system of ODE with initial condition,
- Runge–Kutta methods → high time step restriction,
- full implicit scheme → system of nonlinear algebraic equations.

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Linearization of the inviscid fluxes

- inviscid terms: $\mathbf{b}_h(\tilde{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h)$, $\tilde{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h \in \mathbf{S}_h$

$$\mathbf{b}_h(\tilde{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h) = -\sum_{K \in \mathcal{T}_h} \int_K \sum_{s=1}^{3} \mathbf{A}_s(\tilde{\mathbf{w}}_h)\mathbf{w}_h \cdot \frac{\partial \varphi_h}{\partial x_s} \, \mathrm{d}\mathbf{x}$$
$$+ \sum_{\Gamma \in \mathcal{F}_h} \int_\Gamma \left(\mathbf{P}^+ \left(\langle \tilde{\mathbf{w}}_h \rangle, \vec{n} \right) \mathbf{w}_h|_\Gamma^{(L)} + \mathbf{P}^- \left(\langle \tilde{\mathbf{w}}_h \rangle, \vec{n} \right) \mathbf{w}_h|_\Gamma^{(R)} \right) \cdot \varphi_h \mathrm{d}S,$$

- linear with respect to $\mathbf{w}_h, \varphi_h$
- consistent with $\bar{\mathbf{b}}_h$:

$$\bar{\mathbf{b}}_h(\mathbf{w}_h, \varphi_h) = \mathbf{b}_h(\mathbf{w}_h, \mathbf{w}_h, \varphi_h) \ \forall \ \mathbf{w}_h, \varphi_h \in \mathbf{S}_h.$$

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Linearization of the viscous fluxes

- viscous terms: $\mathbf{a}_h(\tilde{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h)$, $\tilde{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h \in \mathbf{S}_h$,

$$
\begin{aligned}
a_h(\tilde{\mathbf{w}}_h, \mathbf{w}_h, \varphi_h) &= \sum_{K\in\mathcal{T}_h} \int_K \sum_{s=1}^{3} \Big(\sum_{k=1}^{3} (\mathbf{K}_{s,k}(\tilde{\mathbf{w}}_h) \frac{\partial \mathbf{w}_h}{\partial x_k}) n_s \Big) \cdot \frac{\partial \varphi_h}{\partial x_s} \, \mathrm{d}x \\
&- \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \sum_{s=1}^{3} (\langle \sum_{k=1}^{3} \mathbf{K}_{s,k}(\tilde{\mathbf{w}}_h) \frac{\partial \mathbf{w}_h}{\partial x_k} \rangle n_s) \cdot [\varphi_h] \, \mathrm{d}S \\
&- \Theta \sum_{\Gamma\in\mathcal{F}_h^{ID}} \int_\Gamma \sum_{s=1}^{3} \Big(\langle \sum_{k=1}^{3} \mathbf{K}_{s,k}(\tilde{\mathbf{w}}_h) \frac{\partial \varphi_h}{\partial x_k} \rangle n_s \Big) \cdot [\mathbf{w}_h] \, \mathrm{d}S \\
&+ \Theta \sum_{\Gamma\in\mathcal{F}_h^{D}} \int_\Gamma \sum_{s=1}^{3} \Big(\sum_{k=1}^{3} (\mathbf{K}_{s,k}(\tilde{\mathbf{w}}_h) \frac{\partial \varphi_h}{\partial x_k}) n_s \Big) \cdot [\mathbf{w}_B] \, \mathrm{d}S
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Linearization of the viscous fluxes

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Full space-time discrete problem

- semi–implicit scheme: nonlinear parts of $\mathbf{a}_h$, $\mathbf{b}_h$ are treated explicitly and linear parts implicitly,
- BDF – backward difference formula,
- $(0,T) \to t_0 < t_1 < \cdots < t_r$, $\tau_k \equiv t_{k+1} - t_k$,
- $\mathbf{w}_h(t_k) \approx \mathbf{w}_h^k \in \mathbf{S}_h, \quad k = 0, \dots, r:$

$$\frac{1}{\tau_k}\left(\sum_{l=0}^{n} \alpha_l \mathbf{w}_h^{k+1-l}, \varphi_h\right) + \mathbf{a}_h\left(\sum_{l=1}^{n} \beta_l \mathbf{w}_h^{k+1-l}, \mathbf{w}_h^{k+1}, \varphi_h\right)$$

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$\mathbf{w}_h^0$ is S_h–approximation of $\mathbf{w}^0$,

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Library structure

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- CELL, FACE, EDGE
  - entity where to apply discretization
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    3D : Tetrahedra, Prism, Pyramids, ...
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language of Testcase

Simulator.SubSystem.SpaceMethod = DiscontGalerkinSolver

Simulator.SubSystem.DiscontGalerkinSolver.Builder = DG

Simulator.SubSystem.DiscontGalerkinSolver.Data.VolumeIntegratorQuadrature = DGGaussLegendre

Simulator.SubSystem.DiscontGalerkinSolver.Data.VolumeIntegratorOrder = P3

NACA

- benchmark of unsteady transonic flow around the NACA0012 airfoil,
- $M = 0.85, \alpha = 0.0$, $Re = 10000$,
- simulation for $t \in (0, 80)$,
- grid with 3 206 triangles, P1 approximation, ABDF 2^{nd} order
- flow regime with periodic propagation

Naca-video

(Loading NACA)

Conclusion

- combination of DGFE method (SIPG, NIPG, IIPG) and BDF,
- semi–implicit scheme: high order of approximation with respect to space and time coordinates,
- linear system of algebraic equations at each time step,
- implement this method to COOLFluid,
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