On the Complexity of k-Piecewise Testability and the Depth of Automata

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DLT 2015

Problems

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Problem (k-PiecewiseTestability)

Input: *an automaton (min. DFA or NFA)* $\mathscr{A}$

Output: YES *if and only if* $\mathscr{L}(\mathscr{A})$ *is k-PT*

Quest.: *complexity*

Problem (Bounds on automata of k-PT languages)

Input: $\Sigma = \{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, *and* $k \geq 1$

Quest.: *length of a longest word,* w*, such that*

1. $\mathsf{sub}_k(w) := \{u \in \Sigma^* \mid u \preccurlyeq v, |u| \leq k\}$[1] $= \Sigma^{\leq k}$,
2. *prefixes* $w_1 \neq w_2$ *of* w, $\mathsf{sub}_k(w_1) \neq \mathsf{sub}_k(w_2)$

[1] $abc \preccurlyeq bacabbabca$

Piecewise testable languages (PT)

Definition

A regular language is piecewise testable if it is a finite boolean combination of languages of the form

$$\Sigma^* a_1 \Sigma^* a_2 \Sigma^* \cdots \Sigma^* a_n \Sigma^*$$

where $n \geq 0$ and $a_i \in \Sigma$.

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Example (PT language)

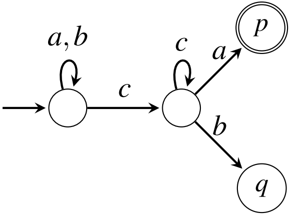
$$\bigcup_{a_1 a_2 \cdots a_n \in L} \Sigma^* a_1 \Sigma^* a_2 \Sigma^* \cdots \Sigma^* a_n \Sigma^*$$

PT recognition

$$\mathfrak{Bool}\left(\Sigma^* a_1 \Sigma^* a_2 \Sigma^* \cdots \Sigma^* a_n \Sigma^*\right)$$

Theorem (min. DFA characterization[2])

1. *Partially ordered* – *acyclic, but with self-loops*
2. *Confluent* – $\forall q \in Q, \forall a,b \in \Sigma, \exists w \in \{a,b\}^*$ *s.t.* $(qa)w = (qb)w$



[2] Version by Klíma + Polák 2013

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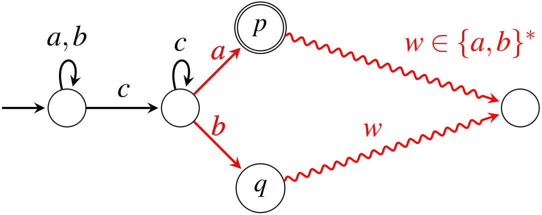

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- Bojańczyk, Segoufin, Straubing 2012
  PT tree languages

Problem 1

k-PiecewiseTestability

$$\mathfrak{Bool}\big(\Sigma^* a_1 \Sigma^* a_2 \Sigma^* \cdots \Sigma^* a_n \Sigma^*\big) \quad \text{with } n \leq k$$

Problem (k-PiecewiseTestability)

Input: *An automaton (min. DFA or NFA)* $\mathscr{A}$

Output: YES *if and only if* $\mathscr{L}(\mathscr{A})$ *is k-PT*

Trivially decidable – finite number of k-PTL over $\Sigma_{\mathscr{A}}$

DFAs

Complexity of k-Piecewise Testability for DFAs

Theorem

The following problem

NAME: K-PIECEWISETESTABILITY
INPUT: *a minimal DFA $\mathscr{A}$*
OUTPUT: YES *if and only if $\mathscr{L}(\mathscr{A})$ is k-PT*

*belongs to co-*NP*.*

0-Piecewise Testability DFAs

$$L(\mathscr{A}) \text{ is } \textcolor{red}{\text{0-PT}} \text{ iff } L(\mathscr{A}) = \begin{cases} \Sigma^* \\ \emptyset \end{cases}$$

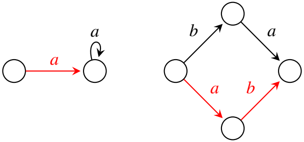
Complexity $O(1)$

1-Piecewise Testability

Theorem

To decide whether a min. DFA recognizes a 1-PT language is in LOGSPACE.

$L(\mathscr{A})$ 1-PT iff the two patterns hold in every state and letter(s)



Syntactic monoids of 1-PTL defined by equations $x = x^2$ and $xy = yx$.[3]

[3] Simon, Blanchet-Sadri

2-Piecewise Testability

Theorem

To decide whether a min. DFA recognizes a 2-PT language is NL*-complete.*

$\mathscr{A}$ min. acyclic and confluent DFA (checked in NL); $L(\mathscr{A})$ 2-PT iff $\forall a \in \Sigma, \forall s \in Q$ s.t. $q_0 w = s$ for a $w \in \Sigma^*$ with $|w|_a \geq 1$, $sba = saba$ $\forall b \in \Sigma \cup \{\varepsilon\}$.

Synt. monoids of 2-PT defined by $xyzx = xyxzx$ and $(xy)^2 = (yx)^2$ (Blanchet-Sadri)

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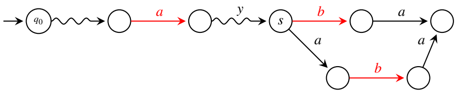

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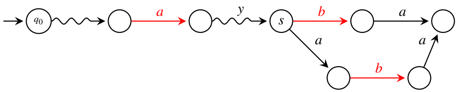

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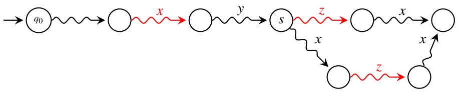

3-Piecewise Testability

Theorem

To decide whether a min. DFA recognizes a 3-PT language is NL*-complete.*

Blachet-Sadri: Equations $(xy)^3 = (yx)^3$, $xzyxvxwy = xzxyxvxwy$ and $ywxvxyzx = ywxvxyxzx$

k-Piecewise Testability

Theorem

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INPUT: *a minimal DFA $\mathscr{A}$*
OUTPUT: YES *if and only if $\mathscr{L}(\mathscr{A})$ is k-PT*

*Complexity: in co-*NP

- $O(1)$ *for* $k = 0$,
- LOGSPACE *for* $k = 1$,
- NL*-complete for* $k = 2, 3$,

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Recently, a co-NP upper bound in terms of separability
Hofman, Martens, "Separability by Short Subsequences and Subwords", ICDT 2015

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- $O(1)$ *for $k = 0$,*
- LOGSPACE *for $k = 1$,*
- NL*-complete for $k = 2, 3$,*
- *co-*NP*-complete for $k \geq 4$.*

Recently, a co-NP upper bound in terms of separability
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Even more recently, co-NP-completeness for $k \geq 4$
Klíma, Kunc, Polák, "Deciding k-piecewise testability", submitted, unaccessible ☺
Thanks to an anonymous reviewer and the authors

NFAs

Complexity of k-PT for NFAs

Theorem

The following problem

NAME: K-PIECEWISETESTABILITYNFA

INPUT: *an NFA* $\mathscr{A}$

OUTPUT: YES *if and only if* $\mathscr{L}(\mathscr{A})$ *is k-PT*

is PSPACE*-complete.*

Problem 2

Bounds on min. DFAs of k-PT languages

Problem (Bounds on automata of k-PT languages)

Input: $\Sigma = \{a_1, a_2, \ldots, a_n\}$, $n \geq 1$, *and* $k \geq 1$

Quest.: *length of a longest word,* w*, s.t.*

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Solution

$$|w| = \binom{k+n}{k} - 1$$

[4] $abc \preccurlyeq bacabbabca$

Consequences

Theorem (Klíma + Polák 2013)

Given a min. DFA recognizing a PT language. If the depth is k, then the language is k-PT.

[5] depth = # states on longest simple path -1; simple path = all states pairwise different

[6] states are $\sim_k$ classes: $u \sim_k v$ iff $sub_k(u) = sub_k(v)$

Consequences

Theorem (Klíma + Polák 2013)

Given a min. DFA recognizing a PT language. If the depth is k, then the language is k-PT.

Opposite does not hold.
Ex.: $(4\ell - 1)$-PTL with the min. DFA of depth $4\ell^2$, for $\ell > 1$.

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Corollary (of Problem 2)

Depth[5] of min. DFA for a k-PTL over an n-letter alphabet is at most $\binom{k+n}{k} - 1$. The bound is tight.

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= depth of the $\sim_k$-canonical DFA[6]

Number of equiv. classes of $\sim_k$ investigated by Karandikar, Kufleitner, Schnoebelen, "On the index of Simon's congruence for piecewise testability", IPL 2015

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Stirling Numbers

For positive integers k and n,

$$\binom{k+n}{k} - 1 = \frac{1}{k!}\sum_{i=1}^{k} \begin{bmatrix} k+1 \\ i+1 \end{bmatrix} n^i,$$

where $\begin{bmatrix} k \\ n \end{bmatrix}$ denotes the Stirling cyclic numbers.

k-PT, NFAs and DFAs

Theorem

For every $k \geq 2$, there exists a language L such that

- L *is* k*-PT*
- L *is not* $(k-1)$*-PT*
- L *is recognized by an NFA of depth* $k-1$*, and*
- L *is recognized by the min. DFA of depth* $2^k - 1$*.*

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- *L is not $(k-1)$-PT*
- *L is recognized by an NFA of depth $k-1$, and*
- *L is recognized by the min. DFA of depth 2^k-1.*

Note

NFA has k states $\rightsquigarrow$ there are NFAs s.t. 2^k states of their min. DFAs form a simple path

Are NFAs better?

Are NFAs more convenient for upper bounds on k?

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Even for 1-PT, the depth of NFA depends on the alphabet.

The language

$$L = \bigcap_{a \in \Sigma} \Sigma^* a \Sigma^*$$

is 1-PT and any NFA requires at least $2^{|\Sigma|}$ states and depth $|\Sigma|$.

Thank you!

Summary of main results

- k-PT of DFAs is in co-NP

	$k=0$	$k=1$	$k=2,3$	$k\geq 4$
Comp.	$O(1)$	LOGSPACE	NL-complete	co-NP-complete[7]

- k-PT for NFAs is PSPACE-complete
- $k,n\geq 1$, the depth of min. DFA of any k-PTL over n letters $\leq \binom{k+n}{k}-1$
- For every $k\geq 2$, there exists L s.t. L is k-PT and not (k−1)-PT, L is recognized by an NFA with k states and depth $k-1$, and the min. DFA for L has depth 2^k-1.

[7] Klíma, Kunc, Polák