Retractive linear orderings

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38th Winter School in Abstract Analysis, Section of Topology
Hejnice, 30 January – 6 February 2010

Some motivations

Preprint:

O. Kalenda, W. Kubiś, *The structure of Valdivia compact lines*, to appear in Topology Appl. (`http://arxiv.org/abs/0811.4144`)

Let $\langle X, < \rangle$ be a line (= linearly ordered set). Then:

($\mathfrak{s}_\omega$)

For every $f \colon \omega \to X$ there exists an infinite set $B \subseteq \omega$ such that $f \restriction B$ is monotone.

Let κ be an uncountable regular cardinal.
We shall say that $\langle X, < \rangle$ has property ($\mathfrak{s}_\kappa$) if, given a function

$$f \colon S \to X$$

with S stationary in κ, there exists a stationary set $T \subseteq S$ such that $f \restriction T$ is monotone.

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Critical uncountable lines

1. ω_1
2. ω_1^{-1}
3. Countryman line
4. (Countryman line)$^{-1}$
5. Uncountable subset of $\mathbb{R}$

Theorem (J. Moore)

PFA implies that every uncountable line contains a copy of one of the above lines.

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A line is scattered if $\mathbb{Q} \not\to X$.

Hausdorff Theorem implies:

Proposition

Every scattered line satisfies $(\mathfrak{s}_\kappa)$ for every regular cardinal $\kappa \geqslant \aleph_0$.

Lemma

Assume $\kappa = \operatorname{cf} \kappa > \aleph_0$, $S \subseteq \kappa$ is stationary, X is a set and $f\colon S \to X$ is a function. Then one of the following possibilities occur:

1. *There exists a stationary set $T_0 \subseteq S$ such that $f \restriction T_0$ is constant.*
2. *There exists a stationary set $T_1 \subseteq S$ such that $f \restriction T_1$ is one-to-one.*

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Kurepa's example

Define

$$Y = \{x \in 2^{\omega_1} : |\operatorname{suppt}(x)| < \aleph_0\} \cup \{1_{c_\alpha} : \alpha \in S\},$$

where $S \subseteq \omega_1$ is stationary and $\{c_\alpha\}_{\alpha \in S}$ is a ladder system. That is, $c_\alpha \approx \omega$ and $\sup c_\alpha = \alpha$ for $\alpha \in S$.

Claim

The line Y fails $(\mathfrak{s}_{\aleph_1})$, but it satisfies

$$(\forall f : \omega_1 \to Y)(\exists T \in [\omega_1]^{\omega_1})\ f \restriction T \text{ is monotone.}$$

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Retractive lines

Definition

A line X is retractive if, given a big enough cardinal θ, for every countable $M \preceq \langle H(\theta), \in \rangle$ such that $X \in M$, there exists an increasing map $r \colon X \to X \cap M$ satisfying

$$f \restriction (X \cap M) = \mathrm{id}_{X \cap M}.$$

Claim

A line X is retractive iff

$$X = \bigcup \mathcal{F},$$

where $\mathcal{F}$ is a family of countable subsets of X satisfying

1. *For every $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ such that $A \cup B \subseteq C$.*
2. *$\bigcup_{n \in \omega} A_n \in \mathcal{F}$ whenever $A_0 \subseteq A_1 \subseteq \dots$ is a sequence in $\mathcal{F}$.*
3. *For every $F \in \mathcal{F}$ the inclusion $F \subseteq X$ is left-invertible, i.e. there is an increasing map $r \colon X \to F$ such that $r \restriction F = \mathrm{id}_F$.*

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Let

$$\mathbb{U}_\kappa = \{x \in \mathbb{Q}^\kappa \colon |\operatorname{suppt}(x)| < \aleph_0\}.$$

Claim

$\mathbb{U}_\kappa$ is retractive.

Proof.

1. Fix $M \preceq H(\theta)$ so that $\kappa \in M$.
2. Let $S = \kappa \cap M$. Then

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Theorem (O. Kalenda & W.K.)

Every retractive line satisfies $(\mathfrak{s}_{\aleph_1})$.

Proof.

1. Let X be a retractive line and fix $y\colon S \to X$ with $S \subseteq \lim(\omega_1)$ stationary.
2. Fix a continuous chain $\{M_\alpha\}_{\alpha<\omega_1}$ of countable elementary substructures of a suitable $H(\theta)$, so that $y \in M_0$.
3. Let $r_\alpha \colon X \to X \cap M_\alpha$ be an increasing retraction.
4. If $\alpha \in S$ then $X \cap M_\alpha = \bigcup_{\xi<\alpha}(X \cap M_\xi)$.
5. $r_\alpha(y_\alpha) \in X \cap M_{\xi(\alpha)}$ with $\xi(\alpha) < \alpha$, whenever $\alpha \in S$.
6. Fodor's pressing-down lemma $\implies \exists\, T \subseteq S$ stationary, such that $r_\alpha(y_\alpha) = v$ for $\alpha \in T$.
7. Let $T^+ = \{\alpha \in T\colon y_\alpha \geqslant v\}$ and $T^- = T \setminus T^+$.

Then $\{y_\alpha\}_{\alpha\in T^+}$ is decreasing and $\{y_\alpha\}_{\alpha\in T^-}$ is increasing. □

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Corollary

For every κ, $\mathbb{U}_\kappa$ satisfies $(\mathfrak{s}_{\aleph_1})$.

Theorem (W.K. 2006)

Every retractive line of size $\leqslant \aleph_1$ embeds into $\mathbb{U}_{\aleph_1}$.

Corollary

For every κ, $\mathbb{U}_\kappa$ *satisfies* $(\mathfrak{s}_{\aleph_1})$.

Theorem (W.K. 2006)

Every retractive line of size $\leqslant \aleph_1$ *embeds into* $\mathbb{U}_{\aleph_1}$.

Exact completions

A line X is exactly κ-complete if every monotone κ-sequence has a limit in X.

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Every retractive line is exactly $\aleph_1$-complete.

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1. Let X be a retractive line and fix a strictly increasing sequence $y\colon \omega_1 \to X$.
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3. Let $r\colon X \to X \cap M$ be an increasing retraction. We claim that $r(y_\delta) = \sup_{\xi<\omega_1} y_\xi$.
4. Clearly $r(y_\delta) \in M$, therefore $y_\xi \leqslant r(y_\delta)$ for every ξ.
5. Suppose $r(y_\delta) \neq \sup_{\xi<\omega_1} y_\xi$ and fix $x \in X \cap M$ with $y_\xi < x < r(y_\delta)$ for every $\xi < \omega_1$. Then $r(y_\delta) \leqslant r(x) = x$, because r is increasing.

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This is a contradiction. □

Exact completions

A line X is exactly κ-complete if every monotone κ-sequence has a limit in X.

Theorem

Every retractive line is exactly $\aleph_1$-complete.

Proof.

1. Let X be a retractive line and fix a strictly increasing sequence $y\colon \omega_1 \to X$.
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Claim

Let $\kappa \geqslant \aleph_0$ be a regular cardinal. For every line X there exists a minimal exactly κ-complete line $c_\kappa(X) \supseteq X$.

$c_\kappa(X)$ is the exact κ-completion of X.

Theorem

Let $\kappa > \aleph_0$ be a regular cardinal and assume X is a line satisfying $(\mathfrak{s}_\kappa)$. Then the exact κ-completion of X satisfies $(\mathfrak{s}_\kappa)$.

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1. Fix $y\colon S \to c_\kappa(X)$, where $S \subseteq \kappa$ is stationary. We may assume that each y_α is the limit of an increasing ω_1-sequence from X.
2. Assume y is one-to-one. Choose inductively a one-to-one sequence $\{x_\alpha\}_{\alpha\in S} \subseteq X$ so that $x_\alpha < y_\alpha$ and

$$\alpha < \beta \wedge y_\alpha < y_\beta \implies y_\alpha < x_\beta.$$

3. $(\mathfrak{s}_{\aleph_1}) \implies \exists\, T \subseteq S$ stationary, such that $\{x_\alpha\}_{\alpha\in T}$ is monotone.
4. If $\{x_\alpha\}_{\alpha\in T}$ is decreasing then so is $y \restriction T$. Assume $\{x_\alpha\}_{\alpha\in T}$ is increasing.
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Main result

Theorem (O. Kalenda & W.K.)

Assume X is an exactly $\aleph_1$-complete line satisfying $(\mathfrak{s}_{\aleph_1})$. If $|X| \leqslant \aleph_1$ then X is retractive.

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$$Y = \{x \in 2^{\omega_1} : |\operatorname{suppt}(x)| < \aleph_0\} \cup \{1_{c_\alpha} : \alpha \in S\}.$$

Claim

Y is exactly $\aleph_1$-complete.

Corollary

Y fails $(\mathfrak{s}_{\aleph_1})$.

Proof.

1. Fix a suitable $M \preceq H(\theta)$ so that $\delta = \omega_1 \cap M \in S$.
2. Then the inclusion $Y \cap M \subseteq Y$ is not left-invertible.

Thus, Y is not retractive. □

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Thus, Y is not retractive. □

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Kurepa trees

Theorem

Assume there no Kurepa trees.
Given a line X of size $\aleph_1$, the following properties are equivalent:

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Natural forcing introducing a Kurepa tree

Let A be an uncountable set.

Define $\mathbb{T}(A)$ to be the set of all pairs $\langle T, f \rangle$, where

1. $T \subseteq 2^{\leqslant \alpha}$ is a countable tree, $\alpha < \omega_1$,
2. every $t \in T$ is below some $s \in T \cap 2^{\alpha}$,
3. $f \colon \operatorname{dom}(f) \to T \cap 2^{\alpha}$ is one-to-one and $\operatorname{dom}(f) \subseteq A$ is countable.

The order of $\mathbb{T}(A)$ is natural:

$$\langle T, f \rangle \leqslant \langle T', f' \rangle \iff$$

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Let G be a $\mathbb{T}(A)$-generic filter over $\mathbb{V}$. Denote by X_G the set of all branches through the generic tree $T_G = \bigcup_{\langle T,f\rangle \in G} T$.
Given $\alpha \in A$ let $x_G(\alpha) = \bigcup_{\langle T,f\rangle \in G} f(\alpha)$.

Lemma

$\mathbb{V}[G] \models X_G = \{x_G(\alpha) \colon \alpha \in A\}$.

Lemma

If $|A| = \aleph_1$ then the line X_G is retractive.

Theorem

Assume $\mathbb{V} \models CH$, fix $\kappa > \aleph_1$ in $\mathbb{V}$ and let G be $\mathbb{T}(\kappa)$-generic. Then in $\mathbb{V}[G]$:

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THE END

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Announcement:

25th *Summer Conference on Topology and its Applications* will be held in Kielce (POLAND), 25 – 30 July 2010.

`http://www.ujk.edu.pl/~topoconf/`