Banach spaces and compact lines

Wiesław Kubiś

Czech Academy of Sciences, Prague
and
Jan Kochanowski University in Kielce

http://www.math.cas.cz/~kubis/

Spring Conference on Banach spaces
Paseky nad Jizerou, 13 – 19 April 2008

***

A compact line is a compact space whose topology is induced by a linear order.

Banach spaces and compact lines

Wiesław Kubiś

Czech Academy of Sciences, Prague
and
Jan Kochanowski University in Kielce

http://www.math.cas.cz/~kubis/

Spring Conference on Banach spaces
Paseky nad Jizerou, 13 – 19 April 2008

***

A compact line is a compact space whose topology is induced by a linear order.

Selected results

Theorem (Nakhmanson 1988)

Assume K is a compact line and $\mathcal{C}_p(K)$ is Lindelöf. Then K is metrizable.

Theorem (Haydon, Jayne, Namioka, Rogers 2000)

For every compact line K, the space $\mathcal{C}(K)$ has a Kadec renorming.

Selected results

Theorem (Nakhmanson 1988)

Assume K is a compact line and $\mathcal{C}_p(K)$ is Lindelöf. Then K is metrizable.

Theorem (Haydon, Jayne, Namioka, Rogers 2000)

For every compact line K, the space $\mathcal{C}(K)$ has a Kadec renorming.

Selected results

Theorem (Nakhmanson 1988)

Assume K is a compact line and $\mathcal{C}_p(K)$ is Lindelöf. Then K is metrizable.

Theorem (Haydon, Jayne, Namioka, Rogers 2000)

For every compact line K, the space $\mathcal{C}(K)$ has a Kadec renorming.

Increasing functions

Let X, Y be linearly ordered sets. A function $f \colon X \to Y$ is increasing if $x \leqslant y \implies f(x) \leqslant f(y)$ for every $x, y \in X$.

Claim

For every compact line K, increasing functions form a linearly dense subset of $\mathcal{C}(K)$.

Claim

Given a nontrivial interval $[a, b]$ in a compact line K, there is an increasing continuous function $f \colon K \to \mathbb{R}$ such that $f(a) = 0$ and $f(b) = 1$.

Increasing functions

Let X, Y be linearly ordered sets. A function $f\colon X \to Y$ is increasing if $x \leqslant y \implies f(x) \leqslant f(y)$ for every $x, y \in X$.

Claim

For every compact line K, increasing functions form a linearly dense subset of $\mathcal{C}(K)$.

Claim

Given a nontrivial interval $[a, b]$ in a compact line K, there is an increasing continuous function $f\colon K \to \mathbb{R}$ such that $f(a) = 0$ and $f(b) = 1$.

Increasing functions

Let X, Y be linearly ordered sets. A function $f\colon X \to Y$ is increasing if $x \leqslant y \implies f(x) \leqslant f(y)$ for every $x, y \in X$.

Claim

For every compact line K, increasing functions form a linearly dense subset of $\mathcal{C}(K)$.

Claim

Given a nontrivial interval $[a, b]$ in a compact line K, there is an increasing continuous function $f\colon K \to \mathbb{R}$ such that $f(a) = 0$ and $f(b) = 1$.

Increasing functions

Let X, Y be linearly ordered sets. A function $f\colon X \to Y$ is increasing if $x \leqslant y \implies f(x) \leqslant f(y)$ for every $x, y \in X$.

Claim

For every compact line K, increasing functions form a linearly dense subset of $\mathcal{C}(K)$.

Claim

Given a nontrivial interval $[a, b]$ in a compact line K, there is an increasing continuous function $f\colon K \to \mathbb{R}$ such that $f(a) = 0$ and $f(b) = 1$.

Lemma

Let K be a compact line and let $f \in \mathcal{C}(K)$. Then there exists a closed separable subspace $X \subseteq K$ such that f is constant on every interval whose interior is disjoint from X.

Lemma

Let K be a compact line, let $X \subseteq K$ be closed. Then there exists a regular extension operator $T: \mathcal{C}(X) \to \mathcal{C}(K)$.

Theorem

Assume K is a compact line in which all separable subsets are metrizable. Then $\mathcal{C}(K)$ has the separable complementation property.

Lemma

Let K be a compact line and let $f \in \mathcal{C}(K)$. Then there exists a closed separable subspace $X \subseteq K$ such that f is constant on every interval whose interior is disjoint from X.

Lemma

Let K be a compact line, let $X \subseteq K$ be closed. Then there exists a regular extension operator $T : \mathcal{C}(X) \to \mathcal{C}(K)$.

Theorem

Assume K is a compact line in which all separable subsets are metrizable. Then $\mathcal{C}(K)$ has the separable complementation property.

Lemma

Let K be a compact line and let $f \in \mathcal{C}(K)$. Then there exists a closed separable subspace $X \subseteq K$ such that f is constant on every interval whose interior is disjoint from X.

Lemma

Let K be a compact line, let $X \subseteq K$ be closed. Then there exists a regular extension operator $T\colon \mathcal{C}(X) \to \mathcal{C}(K)$.

Theorem

Assume K is a compact line in which all separable subsets are metrizable. Then $\mathcal{C}(K)$ has the separable complementation property.

Theorem (Corson 1961)

Let K be a double arrow line, i.e. $K = \mathbb{K}(A)$, where $A \subseteq \mathbb{R}$ is uncountable. Then $\mathcal{C}(K)$ fails the SCP.

Given a linearly ordered set X, define

$$\mathbb{K}(X) = \Big\{ p \in \{0,1\}^X : p \text{ is increasing} \Big\}.$$

Given a compact 0-dimensional line K, define

$$\mathbb{X}(K) = \Big\{ p \in \mathcal{C}(K, \{0,1\}) : p \text{ is increasing}, \ p(0_K) = 0, \ p(1_K) = 1 \Big\}.$$

Claim

After suitable identifications, $\mathbb{X}(\mathbb{K}(X)) = X$ and $\mathbb{K}(\mathbb{X}(K)) = K$.

Theorem (Corson 1961)

Let K be a double arrow line, i.e. $K = \mathbb{K}(A)$, where $A \subseteq \mathbb{R}$ is uncountable. Then $\mathcal{C}(K)$ fails the SCP.

Given a linearly ordered set X, define

$$\mathbb{K}(X) = \left\{ p \in \{0,1\}^X : p \text{ is increasing} \right\}.$$

Given a compact 0-dimensional line K, define

$$\mathbb{X}(K) = \left\{ p \in \mathcal{C}(K, \{0,1\}) : p \text{ is increasing}, \ p(0_K) = 0, \ p(1_K) = 1 \right\}.$$

Claim

After suitable identifications, $\mathbb{X}(\mathbb{K}(X)) = X$ and $\mathbb{K}(\mathbb{X}(K)) = K$.

Theorem (Corson 1961)

Let K be a double arrow line, i.e. $K = \mathbb{K}(A)$, where $A \subseteq \mathbb{R}$ is uncountable. Then $\mathcal{C}(K)$ fails the SCP.

Given a linearly ordered set X, define

$$\mathbb{K}(X) = \Big\{p \in \{0,1\}^X : p \text{ is increasing}\Big\}.$$

Given a compact 0-dimensional line K, define

$$\mathbb{X}(K) = \Big\{p \in \mathcal{C}(K,\{0,1\}) : p \text{ is increasing, } p(0_K) = 0,\ p(1_K) = 1\Big\}.$$

Claim

After suitable identifications, $\mathbb{X}(\mathbb{K}(X)) = X$ and $\mathbb{K}(\mathbb{X}(K)) = K$.

Theorem (Corson 1961)

Let K be a double arrow line, i.e. $K = \mathbb{K}(A)$, where $A \subseteq \mathbb{R}$ is uncountable. Then $\mathcal{C}(K)$ fails the SCP.

Given a linearly ordered set X, define

$$\mathbb{K}(X) = \left\{ p \in \{0,1\}^X : p \text{ is increasing} \right\}.$$

Given a compact 0-dimensional line K, define

$$\mathbb{X}(K) = \left\{ p \in \mathcal{C}(K, \{0,1\}) : p \text{ is increasing, } p(0_K) = 0,\ p(1_K) = 1 \right\}.$$

Claim

After suitable identifications, $\mathbb{X}(\mathbb{K}(X)) = X$ and $\mathbb{K}(\mathbb{X}(K)) = K$.

Theorem (Corson 1961)

Let K be a double arrow line, i.e. $K = \mathbb{K}(A)$, where $A \subseteq \mathbb{R}$ is uncountable. Then $\mathcal{C}(K)$ fails the SCP.

Given a linearly ordered set X, define

$$\mathbb{K}(X) = \left\{ p \in \{0,1\}^X : p \text{ is increasing} \right\}.$$

Given a compact 0-dimensional line K, define

$$\mathbb{X}(K) = \left\{ p \in \mathcal{C}(K, \{0,1\}) : p \text{ is increasing}, \ p(0_K) = 0, \ p(1_K) = 1 \right\}.$$

Claim

After suitable identifications, $\mathbb{X}(\mathbb{K}(X)) = X$ and $\mathbb{K}(\mathbb{X}(K)) = K$.

Markushevich bases

A Markushevich basis in a Banach space X is a bi-orthogonal system $\mathfrak{m} = \{\langle x_\alpha, y_\alpha \rangle\}_{\alpha \in \kappa}$ such that

- $\{x_\alpha : \alpha \in \kappa\}$ is linearly dense in X,
- $\{y_\alpha : \alpha \in \kappa\}$ is total.

- $\mathfrak{m}$ is countably norming if the space

$$\{y \in X^* : |\{\alpha : y(x_\alpha) \neq 0\}| \leqslant \aleph_0\}$$

  is norming.

A Banach space with a countably norming Markushevich basis is called a Plichko space.

Markushevich bases

A Markushevich basis in a Banach space X is a bi-orthogonal system $\mathfrak{m} = \{\langle x_\alpha, y_\alpha \rangle\}_{\alpha \in \kappa}$ such that

- $\{x_\alpha : \alpha \in \kappa\}$ is linearly dense in X,
- $\{y_\alpha : \alpha \in \kappa\}$ is total.
- $\mathfrak{m}$ is countably norming if the space

$$\{y \in X^* : |\{\alpha : y(x_\alpha) \neq 0\}| \leqslant \aleph_0\}$$

is norming.

A Banach space with a countably norming Markushevich basis is called a Plichko space.

Markushevich bases

A Markushevich basis in a Banach space X is a bi-orthogonal system $\mathfrak{m} = \{\langle x_\alpha, y_\alpha \rangle\}_{\alpha \in \kappa}$ such that

- $\{x_\alpha : \alpha \in \kappa\}$ is linearly dense in X,
- $\{y_\alpha : \alpha \in \kappa\}$ is total.

- $\mathfrak{m}$ is countably norming if the space

$$\{y \in X^* : |\{\alpha : y(x_\alpha) \neq 0\}| \leqslant \aleph_0\}$$

  is norming.

A Banach space with a countably norming Markushevich basis is called a Plichko space.

Markushevich bases

A Markushevich basis in a Banach space X is a bi-orthogonal system $\mathfrak{m} = \{\langle x_\alpha, y_\alpha \rangle\}_{\alpha \in \kappa}$ such that

- $\{x_\alpha \colon \alpha \in \kappa\}$ is linearly dense in X,
- $\{y_\alpha \colon \alpha \in \kappa\}$ is total.

- $\mathfrak{m}$ is countably norming if the space

$$\{y \in X^* \colon |\{\alpha \colon y(x_\alpha) \neq 0\}| \leqslant \aleph_0\}$$

  is norming.

A Banach space with a countably norming Markushevich basis is called a Plichko space.

- WCG $\implies$ WLD $\implies$ Plichko $\implies$ SCP.
- If K is a compact line and $\mathcal{C}(K)$ is WLD then $w(K) \leqslant \aleph_0$.

Theorem (Kalenda 2002)

$\mathcal{C}(\omega_2 + 1)$ *is not a Plichko space.*

Remark

$\mathcal{C}(\omega_1 + 1)$ is Plichko.

Conjecture

$\mathcal{C}(\omega_2 + 1)$ is not embeddable into any Plichko space.

- WCG $\implies$ WLD $\implies$ Plichko $\implies$ SCP.
- If K is a compact line and $\mathcal{C}(K)$ is WLD then $\mathrm{w}(K) \leqslant \aleph_0$.

Theorem (Kalenda 2002)

$\mathcal{C}(\omega_2 + 1)$ is not a Plichko space.

Remark

$\mathcal{C}(\omega_1 + 1)$ is Plichko.

Conjecture

$\mathcal{C}(\omega_2 + 1)$ is not embeddable into any Plichko space.

- WCG $\implies$ WLD $\implies$ Plichko $\implies$ SCP.
- If K is a compact line and $\mathcal{C}(K)$ is WLD then $w(K) \leqslant \aleph_0$.

Theorem (Kalenda 2002)

$\mathcal{C}(\omega_2 + 1)$ is not a Plichko space.

Remark

$\mathcal{C}(\omega_1 + 1)$ is Plichko.

Conjecture

$\mathcal{C}(\omega_2 + 1)$ is not embeddable into any Plichko space.

- WCG $\implies$ WLD $\implies$ Plichko $\implies$ SCP.
- If K is a compact line and $\mathcal{C}(K)$ is WLD then $\mathrm{w}(K) \leqslant \aleph_0$.

Theorem (Kalenda 2002)

$\mathcal{C}(\omega_2 + 1)$ is not a Plichko space.

Remark

$\mathcal{C}(\omega_1 + 1)$ is Plichko.

Conjecture

$\mathcal{C}(\omega_2 + 1)$ is not embeddable into any Plichko space.

- WCG $\implies$ WLD $\implies$ Plichko $\implies$ SCP.
- If K is a compact line and $\mathcal{C}(K)$ is WLD then $\mathrm{w}(K) \leqslant \aleph_0$.

Theorem (Kalenda 2002)

$\mathcal{C}(\omega_2 + 1)$ is not a Plichko space.

Remark

$\mathcal{C}(\omega_1 + 1)$ is Plichko.

Conjecture

$\mathcal{C}(\omega_2 + 1)$ is not embeddable into any Plichko space.

Theorem (O. Kalenda & W.K.)

Assume K is a first countable compact line. If $\mathcal{C}(K)$ embeds into a Plichko space then K is metrizable.

Theorem

Assume K is a compact line of character $\leqslant \aleph_1$. If $\mathcal{C}(K)$ embeds into a Plichko space then $\mathrm{w}(K) \leqslant \aleph_1$.

Theorem (O. Kalenda & W.K.)

Assume K is a first countable compact line. If $\mathcal{C}(K)$ embeds into a Plichko space then K is metrizable.

Theorem

Assume K is a compact line of character $\leqslant \aleph_1$. If $\mathcal{C}(K)$ embeds into a Plichko space then $\mathrm{w}(K) \leqslant \aleph_1$.

Generalized PRIs

A generalized PRI in a Banach space X is a sequence of projections $\{P_\alpha\}_{\alpha<\kappa}$ satisfying

- $X = \operatorname{cl}\left(\bigcup_{\alpha<\kappa} P_\alpha X\right)$,
- $P_\xi \circ P_\eta = P_\eta \circ P_\xi = P_{\min\{\xi,\eta\}}$ for every $\xi, \eta < \kappa$,
- $P_\delta X = \operatorname{cl}\left(\bigcup_{\xi<\delta} P_\xi X\right)$ for every limit ordinal $\delta < \kappa$,
- $\operatorname{dens}(P\alpha X) < |\alpha| + \aleph_0$ for every $\alpha < \kappa$.

Generalized PRIs

A generalized PRI in a Banach space X is a sequence of projections $\{P_\alpha\}_{\alpha<\kappa}$ satisfying

- $X = \mathrm{cl}\left(\bigcup_{\alpha<\kappa} P_\alpha X\right)$,
- $P_\xi \circ P_\eta = P_\eta \circ P_\xi = P_{\min\{\xi,\eta\}}$ for every $\xi, \eta < \kappa$,
- $P_\delta X = \mathrm{cl}\left(\bigcup_{\xi<\delta} P_\xi X\right)$ for every limit ordinal $\delta < \kappa$,
- $\mathrm{dens}(P\alpha X) < |\alpha| + \aleph_0$ for every $\alpha < \kappa$.

Generalized PRIs

A generalized PRI in a Banach space X is a sequence of projections $\{P_\alpha\}_{\alpha<\kappa}$ satisfying

- $X = \mathrm{cl}\left(\bigcup_{\alpha<\kappa} P_\alpha X\right)$,
- $P_\xi \circ P_\eta = P_\eta \circ P_\xi = P_{\min\{\xi,\eta\}}$ for every $\xi, \eta < \kappa$,
- $P_\delta X = \mathrm{cl}\left(\bigcup_{\xi<\delta} P_\xi X\right)$ for every limit ordinal $\delta < \kappa$,
- $\mathrm{dens}(P\alpha X) < |\alpha| + \aleph_0$ for every $\alpha < \kappa$.

Generalized PRIs

A generalized PRI in a Banach space X is a sequence of projections $\{P_\alpha\}_{\alpha<\kappa}$ satisfying

- $X = \text{cl}\left(\bigcup_{\alpha<\kappa} P_\alpha X\right)$,
- $P_\xi \circ P_\eta = P_\eta \circ P_\xi = P_{\min\{\xi,\eta\}}$ for every $\xi, \eta < \kappa$,
- $P_\delta X = \text{cl}\left(\bigcup_{\xi<\delta} P_\xi X\right)$ for every limit ordinal $\delta < \kappa$,
- $\text{dens}(P\alpha X) < |\alpha| + \aleph_0$ for every $\alpha < \kappa$.

Generalized PRIs

A generalized PRI in a Banach space X is a sequence of projections $\{P_\alpha\}_{\alpha<\kappa}$ satisfying

- $X = \text{cl}\left(\bigcup_{\alpha<\kappa} P_\alpha X\right)$,
- $P_\xi \circ P_\eta = P_\eta \circ P_\xi = P_{\min\{\xi,\eta\}}$ for every $\xi, \eta < \kappa$,
- $P_\delta X = \text{cl}\left(\bigcup_{\xi<\delta} P_\xi X\right)$ for every limit ordinal $\delta < \kappa$,
- $\text{dens}(P\alpha X) < |\alpha| + \aleph_0$ for every $\alpha < \kappa$.

Generalized PRIs

A generalized PRI in a Banach space X is a sequence of projections $\{P_\alpha\}_{\alpha<\kappa}$ satisfying

- $X = \mathrm{cl}\left(\bigcup_{\alpha<\kappa} P_\alpha X\right)$,
- $P_\xi \circ P_\eta = P_\eta \circ P_\xi = P_{\min\{\xi,\eta\}}$ for every $\xi, \eta < \kappa$,
- $P_\delta X = \mathrm{cl}\left(\bigcup_{\xi<\delta} P_\xi X\right)$ for every limit ordinal $\delta < \kappa$,
- $\mathrm{dens}(P\alpha X) < |\alpha| + \aleph_0$ for every $\alpha < \kappa$.

A characterization of Plichko spaces

Theorem

Let X be a Banach space of density $\aleph_1$. TFAE:

1. *X has a countably norming Markushevich basis.*
2. *X has a generalized PRI.*
3. *$X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a continuous chain of separable subspaces such that, after some renorming,*
   - *for each α, the space X_α is 1-complemented in $X_{\alpha+1}$.*

Proposition

A Banach space of density $\aleph_1$ has the SCP iff $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a chain of complemented separable subspaces.

A characterization of Plichko spaces

Theorem

Let X be a Banach space of density $\aleph_1$. TFAE:

1. *X has a countably norming Markushevich basis.*
2. *X has a generalized PRI.*
3. *$X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a continuous chain of separable subspaces such that, after some renorming,*
   - *for each α, the space X_α is 1-complemented in $X_{\alpha+1}$.*

Proposition

A Banach space of density $\aleph_1$ has the SCP iff $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a chain of complemented separable subspaces.

A characterization of Plichko spaces

Theorem

Let X be a Banach space of density $\aleph_1$. TFAE:

1. *X has a countably norming Markushevich basis.*
2. *X has a generalized PRI.*
3. *$X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a continuous chain of separable subspaces such that, after some renorming,*
   - *for each α, the space X_α is 1-complemented in $X_{\alpha+1}$.*

Proposition

A Banach space of density $\aleph_1$ has the SCP iff $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a chain of complemented separable subspaces.

A characterization of Plichko spaces

Theorem

Let X be a Banach space of density $\aleph_1$. TFAE:

1. *X has a countably norming Markushevich basis.*
2. *X has a generalized PRI.*
3. *$X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a continuous chain of separable subspaces such that, after some renorming,*
    - *for each α, the space X_α is 1-complemented in $X_{\alpha+1}$.*

Proposition

A Banach space of density $\aleph_1$ has the SCP iff $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a chain of complemented separable subspaces.

A characterization of Plichko spaces

Theorem

Let X be a Banach space of density $\aleph_1$. TFAE:

1. *X has a countably norming Markushevich basis.*
2. *X has a generalized PRI.*
3. *$X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a continuous chain of separable subspaces such that, after some renorming,*
    - *for each α, the space X_α is 1-complemented in $X_{\alpha+1}$.*

Proposition

A Banach space of density $\aleph_1$ has the SCP iff $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a chain of complemented separable subspaces.

A characterization of Plichko spaces

Theorem

Let X be a Banach space of density $\aleph_1$. TFAE:

1. *X has a countably norming Markushevich basis.*
2. *X has a generalized PRI.*
3. *$X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a continuous chain of separable subspaces such that, after some renorming,*
    - *for each α, the space X_α is 1-complemented in $X_{\alpha+1}$.*

Proposition

A Banach space of density $\aleph_1$ has the SCP iff $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a chain of complemented separable subspaces.

A characterization of Plichko spaces

Theorem

Let X be a Banach space of density $\aleph_1$. TFAE:

1. *X has a countably norming Markushevich basis.*
2. *X has a generalized PRI.*
3. *$X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a continuous chain of separable subspaces such that, after some renorming,*
   - *for each α, the space X_α is 1-complemented in $X_{\alpha+1}$.*

Proposition

A Banach space of density $\aleph_1$ has the SCP iff $X = \bigcup_{\alpha<\omega_1} X_\alpha$, where $\{X_\alpha\}_{\alpha<\omega_1}$ is a chain of complemented separable subspaces.

Theorem (W.K. 2006)

There exist a compact line K and an increasing quotient $f\colon K \to L$ such that

1. *$\mathcal{C}(K)$ is a Plichko space.*
2. *$\mathcal{C}(L)$ is not a Plichko space.*
3. *$\mathcal{C}(L)$ has the SCP.*

- $K = \mathbb{K}(Q)$, where

$$Q = \{x \in \mathbb{Q}^{\omega_1} : \ \mathrm{suppt}(x) \text{ is finite } \}.$$

- L is the "connectification" of K.

Theorem (W.K. 2006)

There exist a compact line K and an increasing quotient $f\colon K \to L$ such that

1. $\mathcal{C}(K)$ *is a Plichko space.*
2. $\mathcal{C}(L)$ *is not a Plichko space.*
3. $\mathcal{C}(L)$ *has the SCP.*

- $K = \mathbb{K}(Q)$, where

$$Q = \{x \in \mathbb{Q}^{\omega_1} : \text{ suppt}(x) \text{ is finite } \}.$$

- L is the "connectification" of K.

Theorem (W.K. 2006)

There exist a compact line K and an increasing quotient $f\colon K \to L$ such that

1. $\mathcal{C}(K)$ *is a Plichko space.*
2. $\mathcal{C}(L)$ *is not a Plichko space.*
3. $\mathcal{C}(L)$ *has the SCP.*

- $K = \mathbb{K}(Q)$, where

$$Q = \{x \in \mathbb{Q}^{\omega_1} : \text{ suppt}(x) \text{ is finite }\}.$$

- L is the "connectification" of K.

Theorem (W.K. 2006)

There exist a compact line K and an increasing quotient $f\colon K \to L$ such that

1. $\mathcal{C}(K)$ *is a Plichko space.*
2. $\mathcal{C}(L)$ *is not a Plichko space.*
3. $\mathcal{C}(L)$ *has the SCP.*

- $K = \mathbb{K}(Q)$, where

$$Q = \{x \in \mathbb{Q}^{\omega_1}\colon \operatorname{suppt}(x) \text{ is finite }\}.$$

- L is the "connectification" of K.

Theorem (W.K. 2006)

There exist a compact line K and an increasing quotient $f : K \to L$ such that

1. $\mathcal{C}(K)$ *is a Plichko space.*
2. $\mathcal{C}(L)$ *is not a Plichko space.*
3. $\mathcal{C}(L)$ *has the SCP.*

- $K = \mathbb{K}(Q)$, where

$$Q = \{x \in \mathbb{Q}^{\omega_1} : \text{ suppt}(x) \text{ is finite } \}.$$

- L is the "connectification" of K.

Theorem (W.K. 2006)

There exist a compact line K and an increasing quotient $f\colon K \to L$ such that

1. $\mathcal{C}(K)$ *is a Plichko space.*
2. $\mathcal{C}(L)$ *is not a Plichko space.*
3. $\mathcal{C}(L)$ *has the SCP.*

- $K = \mathbb{K}(Q)$, where

$$Q = \{x \in \mathbb{Q}^{\omega_1} : \text{ suppt}(x) \text{ is finite } \}.$$

- L is the "connectification" of K.

A Valdivia compact is a space homeomorphic to $K \subseteq [0,1]^\kappa$ such that

$$K = \mathrm{cl}(K \cap \Sigma(\kappa)).$$

Proposition

- *If K is a Valdivia compact then $\mathcal{C}(K)$ is a Plichko space.*
- *If X is a Plichko space then $\overline{\mathrm{B}}_{X^*}$ is Valdivia compact.*

Theorem (W.K. 2005)

Valdivia compact lines have weight $\leqslant \aleph_1$.

Theorem (W.K. 2005)

The class of nonempty Valdivia compact lines has a universal order pre-image.

A Valdivia compact is a space homeomorphic to $K \subseteq [0,1]^\kappa$ such that

$$K = \mathrm{cl}(K \cap \Sigma(\kappa)).$$

Proposition

- *If K is a Valdivia compact then $\mathcal{C}(K)$ is a Plichko space.*
- *If X is a Plichko space then $\overline{\mathrm{B}}_{X^*}$ is Valdivia compact.*

Theorem (W.K. 2005)

Valdivia compact lines have weight $\leqslant \aleph_1$.

Theorem (W.K. 2005)

The class of nonempty Valdivia compact lines has a universal order pre-image.

A Valdivia compact is a space homeomorphic to $K \subseteq [0,1]^\kappa$ such that

$$K = \mathrm{cl}(K \cap \Sigma(\kappa)).$$

Proposition

- *If K is a Valdivia compact then $\mathcal{C}(K)$ is a Plichko space.*
- *If X is a Plichko space then $\overline{\mathrm{B}}_{X^*}$ is Valdivia compact.*

Theorem (W.K. 2005)

Valdivia compact lines have weight $\leqslant \aleph_1$.

Theorem (W.K. 2005)

The class of nonempty Valdivia compact lines has a universal order pre-image.

A Valdivia compact is a space homeomorphic to $K \subseteq [0,1]^\kappa$ such that

$$K = \mathrm{cl}(K \cap \Sigma(\kappa)).$$

Proposition

- *If K is a Valdivia compact then $\mathcal{C}(K)$ is a Plichko space.*
- *If X is a Plichko space then $\overline{\mathrm{B}}_{X^*}$ is Valdivia compact.*

Theorem (W.K. 2005)

Valdivia compact lines have weight $\leqslant \aleph_1$.

Theorem (W.K. 2005)

The class of nonempty Valdivia compact lines has a universal order pre-image.

Theorem (O. Kalenda & W.K.)

Let X be a linearly ordered set. Then $\mathbb{K}(X)$ is Valdivia compact iff

(1) $|X| \leqslant \aleph_1$.

(2) *Every bounded monotone ω_1-sequence is convergent in X.*

(3) *For every $f\colon S \to X$ defined on a stationary set $S \subseteq \omega_1$ there exists a stationary set $T \subseteq S$ such that $f \restriction T$ is monotone.*

Theorem (O. Kalenda & W.K.)

Let X be a linearly ordered set. Then $\mathbb{K}(X)$ is Valdivia compact iff

(1) $|X| \leqslant \aleph_1$.

(2) *Every bounded monotone ω_1-sequence is convergent in X.*

(3) *For every $f\colon S \to X$ defined on a stationary set $S \subseteq \omega_1$ there exists a stationary set $T \subseteq S$ such that $f \restriction T$ is monotone.*

Theorem (O. Kalenda & W.K.)

Let X be a linearly ordered set. Then $\mathbb{K}(X)$ is Valdivia compact iff

(1) $|X| \leqslant \aleph_1$.

(2) *Every bounded monotone ω_1-sequence is convergent in X.*

(3) *For every $f\colon S \to X$ defined on a stationary set $S \subseteq \omega_1$ there exists a stationary set $T \subseteq S$ such that $f \restriction T$ is monotone.*

Theorem (O. Kalenda & W.K.)

Let X be a linearly ordered set. Then $\mathbb{K}(X)$ is Valdivia compact iff

(1) $|X| \leqslant \aleph_1$.

(2) *Every bounded monotone ω_1-sequence is convergent in X.*

(3) *For every $f\colon S \to X$ defined on a stationary set $S \subseteq \omega_1$ there exists a stationary set $T \subseteq S$ such that $f \restriction T$ is monotone.*

Theorem (O. Kalenda & W.K.)

Let X be a linearly ordered set. Then $\mathbb{K}(X)$ is Valdivia compact iff

1. $|X| \leqslant \aleph_1$.
2. *Every bounded monotone ω_1-sequence is convergent in X.*
3. *For every $f \colon S \to X$ defined on a stationary set $S \subseteq \omega_1$ there exists a stationary set $T \subseteq S$ such that $f \restriction T$ is monotone.*

- Let $Q = \{x \in \mathbb{Q}^{\omega_1} : \text{suppt}(x) \text{ is finite } \}$.
- Fix a stationary set $S \subseteq \omega_1$ consisting of limit ordinals.
- Given $\delta \in S$, choose $C_\delta \nearrow \delta$.
- Let $Y = \{1_{C_\delta} : \delta \in S\}$.
- Finally, let $X_S = Q \cup Y$.

Claim

X_S satisfies conditions (1), (2) and

($2\frac{1}{2}$) *For every function $f : \omega_1 \to X_S$ there exists an uncountable $T \subseteq \omega_1$ such that $f \restriction T$ is monotone.*

Question

Is $\mathcal{C}(\mathbb{K}(X_S))$ a Plichko space?

- Let $Q = \{x \in \mathbb{Q}^{\omega_1} : \text{ suppt}(x) \text{ is finite } \}$.
- Fix a stationary set $S \subseteq \omega_1$ consisting of limit ordinals.
- Given $\delta \in S$, choose $C_\delta \nearrow \delta$.
- Let $Y = \{1_{C_\delta} : \delta \in S\}$.
- Finally, let $X_S = Q \cup Y$.

Claim

X_S satisfies conditions (1), (2) and

($2\frac{1}{2}$) *For every function $f : \omega_1 \to X_S$ there exists an uncountable $T \subseteq \omega_1$ such that $f \restriction T$ is monotone.*

Question

Is $\mathcal{C}(\mathbb{K}(X_S))$ a Plichko space?

- Let $Q = \{x \in \mathbb{Q}^{\omega_1} : \text{ suppt}(x) \text{ is finite } \}$.
- Fix a stationary set $S \subseteq \omega_1$ consisting of limit ordinals.
- Given $\delta \in S$, choose $C_\delta \nearrow \delta$.
- Let $Y = \{1_{C_\delta} : \delta \in S\}$.
- Finally, let $X_S = Q \cup Y$.

Claim

X_S satisfies conditions (1), (2) and

($2\frac{1}{2}$) *For every function $f : \omega_1 \to X_S$ there exists an uncountable $T \subseteq \omega_1$ such that $f \restriction T$ is monotone.*

Question

Is $\mathcal{C}(\mathbb{K}(X_S))$ a Plichko space?

- Let $Q = \{x \in \mathbb{Q}^{\omega_1} : \text{ suppt}(x) \text{ is finite }\}$.
- Fix a stationary set $S \subseteq \omega_1$ consisting of limit ordinals.
- Given $\delta \in S$, choose $C_\delta \nearrow \delta$.
- Let $Y = \{1_{C_\delta} : \delta \in S\}$.
- Finally, let $X_S = Q \cup Y$.

Claim

X_S satisfies conditions (1), (2) and

($2\frac{1}{2}$) *For every function $f : \omega_1 \to X_S$ there exists an uncountable $T \subseteq \omega_1$ such that $f \restriction T$ is monotone.*

Question

Is $\mathcal{C}(\mathbb{K}(X_S))$ a Plichko space?

- Let $Q = \{x \in \mathbb{Q}^{\omega_1} : \text{ suppt}(x) \text{ is finite } \}$.
- Fix a stationary set $S \subseteq \omega_1$ consisting of limit ordinals.
- Given $\delta \in S$, choose $C_\delta \nearrow \delta$.
- Let $Y = \{1_{C_\delta} : \delta \in S\}$.
- Finally, let $X_S = Q \cup Y$.

Claim

X_S satisfies conditions (1), (2) and

$(2\frac{1}{2})$ *For every function $f : \omega_1 \to X_S$ there exists an uncountable $T \subseteq \omega_1$ such that $f \restriction T$ is monotone.*

Question

Is $\mathcal{C}(\mathbb{K}(X_S))$ a Plichko space?

- Let $Q = \{x \in \mathbb{Q}^{\omega_1} : \text{ suppt}(x) \text{ is finite } \}$.
- Fix a stationary set $S \subseteq \omega_1$ consisting of limit ordinals.
- Given $\delta \in S$, choose $C_\delta \nearrow \delta$.
- Let $Y = \{1_{C_\delta} : \delta \in S\}$.
- Finally, let $X_S = Q \cup Y$.

Claim

X_S satisfies conditions (1), (2) and

($2\frac{1}{2}$) *For every function $f : \omega_1 \to X_S$ there exists an uncountable $T \subseteq \omega_1$ such that $f \restriction T$ is monotone.*

Question

Is $\mathcal{C}(\mathbb{K}(X_S))$ a Plichko space?

- Let $Q = \{x \in \mathbb{Q}^{\omega_1} : \text{ suppt}(x) \text{ is finite } \}$.
- Fix a stationary set $S \subseteq \omega_1$ consisting of limit ordinals.
- Given $\delta \in S$, choose $C_\delta \nearrow \delta$.
- Let $Y = \{1_{C_\delta} : \delta \in S\}$.
- Finally, let $X_S = Q \cup Y$.

Claim

X_S satisfies conditions (1), (2) and

($2\frac{1}{2}$) *For every function $f : \omega_1 \to X_S$ there exists an uncountable $T \subseteq \omega_1$ such that $f \restriction T$ is monotone.*

Question

Is $\mathcal{C}(\mathbb{K}(X_S))$ a Plichko space?

- Let $Q = \{x \in \mathbb{Q}^{\omega_1} : \text{ suppt}(x) \text{ is finite } \}$.
- Fix a stationary set $S \subseteq \omega_1$ consisting of limit ordinals.
- Given $\delta \in S$, choose $C_\delta \nearrow \delta$.
- Let $Y = \{1_{C_\delta} : \delta \in S\}$.
- Finally, let $X_S = Q \cup Y$.

Claim

X_S satisfies conditions (1), (2) and

$(2\frac{1}{2})$ *For every function $f : \omega_1 \to X_S$ there exists an uncountable $T \subseteq \omega_1$ such that $f \restriction T$ is monotone.*

Question

Is $\mathcal{C}(\mathbb{K}(X_S))$ a Plichko space?

This page intentionally left blank.

Conference

Set theory, Topology and Banach Spaces
7 – 11 July 2008
Kielce, Poland
`http://www.pu.kielce.pl/~topoconf`