Ladder systems and projections in Banach spaces

(joint work with Jesús Ferrer and Piotr Koszmider)

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Outline

1. Background
2. Small non-separable spaces of continuous functions
3. Almost disjoint families
4. Main results
    - Ladder systems and stationary sets

Motivation

Theorem (Sobczyk, 1941)

In a separable Banach space, every copy of c_0 is complemented.

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Separable complementation properties

Definition

A Banach space X has the separable complementation property (SCP) if for every separable set $A \subseteq X$ there is a projection $P \colon X \to X$ such that $A \subseteq \operatorname{im} P$ and $\operatorname{im} P$ is separable.

Fact

SCP $\implies$ every copy of c_0 is complemented.

Example

Weakly compactly generated Banach spaces have the SCP.

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Controlled SCP

Definition (Wójtowicz, Ferrer)

A Banach space X has the controlled SCP if for every countable sets $A \subseteq X$, $B \subseteq X^*$ there exists a projection $P\colon X \to X$ such that

- $\operatorname{im} P$ is separable,
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Skeletons

Definition

A skeleton in a Banach space X is a family $\mathscr{F}$ consisting of closed separable subspaces, satisfying:

1. $X = \bigcup \mathscr{F}$
2. $E, F \in \mathscr{F} \implies (\exists\, G \in \mathscr{F})\ E + F \subseteq G$
3. Given a chain $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$ in $\mathscr{F}$, the space $\operatorname{cl}(\bigcup_{n\in\omega} E_n)$ is in $\mathscr{F}$ (continuity)

Definition

A Banach space has the continuous SCP if it has a skeleton consisting of complemented subspaces.

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WCG Banach spaces have the continuous SCP.

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The smallest non-separable $C(K)$ spaces

- K should be scattered of a very low height.
- K should be "almost" metrizable.

Remark

If K is scattered of height 1 then $C(K) \approx c_0(|K|)$.

Conclusion:

K should be of the form $L \cup \{\infty\}$, where L is first countable, of height 2 and of cardinality $\aleph_1$.

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Almost disjoint families

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A family of sets $\mathscr{A}$ is almost disjoint if

1. Each $A \in \mathscr{A}$ is **countable infinite.**
2. $A \cap B$ is **finite** whenever $A, B \in \mathscr{A}$ are different.

Definition

Let $\mathscr{A}$ be an almost disjoint family with $X = \bigcup \mathscr{A}$. Define $K_{\mathscr{A}}$ to be the Stone space of the Boolean algebra generated by

$$\mathscr{A} \cup [X]^{<\omega} \subseteq \mathscr{P}(X).$$

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Alternative description

$$K_{\mathscr{A}} := X \cup \mathscr{A} \cup \{\infty\}$$

with the topology defined by the following conditions:

- All points of $X = \bigcup \mathscr{A}$ are isolated.
- A basic neighborhood of $A \in \mathscr{A}$ is

  $$U_F(A) := \{A\} \cup (A \setminus F),$$

  where $F \subseteq A$ is finite.
- A basic neighborhood of ∞ is

  $$U_{\mathscr{F}}(\infty) := K \setminus \bigcup_{A \in \mathscr{F}} U_{\emptyset}(A),$$

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Fact

The space $K_{\mathscr{A}}$ is scattered compact of height 3 and $K_{\mathscr{A}} \setminus \{\infty\}$ is first countable.

Fact

A scattered compact K of height 3 such that K'' is a singleton and $K \setminus K''$ is first countable is homeomorphic to $K_{\mathscr{A}}$ for some almost disjoint family $\mathscr{A}$.

Spaces $K_{\mathscr{A}}$ were studied first by Alexandrov and Urysohn in 1929, later by Mrówka and others.

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Theorem (Sobczyk + Whitley + ?)

Let K be a compactification of $\mathbb{N}$. If the canonical copy of c_0 is complemented in $C(K)$ then $K \setminus \mathbb{N}$ carries a strictly positive Radon measure.

Corollary

Let $\mathscr{A}$ be an uncountable almost disjoint family with $\mathbb{N} = \bigcup \mathscr{A}$. Then the canonical copy of c_0 is not complemented in $C(K_{\mathscr{A}})$.

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Main result I

Definition

A compact space K is monolithic if separable subsets of K are second countable.

Fact

Let E be a Banach space with the controlled SCP. Then the dual unit ball B_{E^*} is monolithic.

Theorem

Let $\mathscr{A}$ be an almost disjoint family. Then $C(K_{\mathscr{A}})$ has the controlled SCP if and only if $K_{\mathscr{A}}$ is monolithic.

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Theorem

There exists an almost disjoint family $\mathscr{A} \subseteq [\omega_1]^\omega$ *such that*

1. $|\mathscr{A}| = \aleph_1$,
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Ladder systems

Definition

Fix $S \subseteq \omega_1$ consisting of limit ordinals only. A ladder system based on S is a sequence

$$\mathscr{C} = \{c_\delta\}_{\delta \in S}$$

such that for each $\delta \in S$:

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Stationary sets

Definition

A set $S \subseteq \omega_1$ is stationary if $S \cap C \neq \emptyset$ whenever $C \subseteq \omega_1$ is closed and unbounded.

Theorem (Pol, 1979)

Let $\mathscr{C} = \{c_\delta\}_{\delta \in S}$ be a ladder system, where $S \subseteq \omega_1$ is stationary. Then $K_{\mathscr{C}}$ is not Eberlein compact, yet $C(K_{\mathscr{C}})$ is weakly Lindelöf.

Proposition

Let $\mathscr{C} = \{c_\delta\}_{\delta \in S}$ be a ladder system with S non-stationary. Then $\mathscr{C}$ is equivalent to a disjoint family and $C(K_{\mathscr{C}}) \approx c_0(\omega_1)$.

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Projectional skeletons

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A projectional skeleton in a Banach space X is a pair $(\mathscr{F}, \mathscr{P})$, where $\mathscr{F}$ is a skeleton of separable subspaces of X and $\mathscr{P} = \{P_E\}_{E \in \mathscr{F}}$ is such that for each $E, F \in \mathscr{F}$:

1. P_E is a projection onto E,
2. $E \subseteq F \implies P_E \circ P_F = P_E = P_F \circ P_E$.

Fact

- $\sup_{E \in \mathscr{F}} \|P_E\| < +\infty$.
- *There is a renorming of X such that $\|P_E\| = 1$ for every $E \in \mathscr{F}$, $E \neq 0$.*

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Projectional skeleton $\implies$ *continuous SCP* $\implies$ *SCP.*

Proposition

Every Banach space with the continuous 1-SCP and of density $\aleph_1$ *has a projectional skeleton.*

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SCP $\implies$ $(\exists\, k \in \mathbb{N})$ *k-SCP.*

Theorem (O. Kalenda and W.K., 2012)

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Corollary

Continuous 2-SCP does not imply the existence of a projectional skeleton.

This answers a question from

- O. KALENDA, W. KUBIŚ, *Complementation in spaces of continuous functions on compact lines*, J. Math. Anal. Appl. **386** (2012) 241–257

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Let $\mathscr{A}$ be an almost disjoint family of countable sets with $|\mathscr{A}| = \aleph_1$ and assume $K_{\mathscr{A}}$ is monolithic. Then

- *either $\mathscr{A}$ is equivalent to a disjoint family, or else*
- *there exist $\mathscr{B} \subseteq \mathscr{A}$ and a stationary set $S \subseteq \omega_1$ such that $\mathscr{B}$ is equivalent to a ladder system based on S.*

In other words:

- either $K_{\mathscr{A}}$ is a "standard" Eberlein compact of height 3 and $C(K_{\mathscr{A}}) \approx c_0(\omega_1)$, or else
- or else $C(K_{\mathscr{A}})$ has an isometric copy of $C(K_{\mathscr{C}})$, where $\mathscr{C} = \{c_\delta\}_{\delta \in S}$ is a ladder system and $S \subseteq \omega_1$ is stationary.

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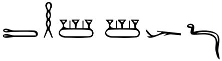
- or else $C(K_{\mathscr{A}})$ has an isometric copy of $C(K_{\mathscr{C}})$, where $\mathscr{C} = \{c_\delta\}_{\delta \in S}$ is a ladder system and $S \subseteq \omega_1$ is stationary.



THE END