On poset Boolean algebras

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Abstract

Let $\langle P, \leq \rangle$ be a partially ordered set. The poset Boolean algebra of P, denoted $F(P)$, is defined as follows: The set of generators of $F(P)$ is $\{x_p : p \in P\}$, and the set of relations is $\{x_p \cdot x_q = x_p : p \leq q\}$. We say that a Boolean algebra B is well-generated, if B has a sublattice G such that G generates B and $\langle G, \leq^B \restriction G \rangle$ is well-founded. A well-generated algebra is superatomic.

Theorem 1. *Let $\langle P, \leq \rangle$ be a partially ordered set. The following are equivalent.* (i) *P does not contain an infinite set of pairwise incomparable elements, and P does not contain a subset isomorphic to the chain of rational numbers.* (ii) *$F(P)$ is superatomic.* (iii) *$F(P)$ is well-generated.*

The equivalence (i) $\Leftrightarrow$ (ii) is due to M. Pouzet [P]. A partially ordered set W is well-ordered, if W does not contain a strictly decreasing infinite sequence, and W does not contain an infinite set of pairwise incomparable elements.

Theorem 2. *Let $F(P)$ be a superatomic poset algebra. Then there are a well-ordered set W and a subalgebra B of $F(W)$, such that $F(P)$ is a homomorphic image of B.*

This is similar but weaker than the fact that every interval algebra of a scattered chain is embeddable in an ordinal algebra ([AB1]). Remember that an interval algebra is a special case of a poset algebra.

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1 Introduction

In this paper we investigate poset Boolean algebras and especially superatomic poset Boolean algebras. The construction of a Boolean algebra from a partially ordered set (poset) is natural. Yet, not much is known about the relationship between the properties of the poset and the properties of the Boolean algebra constructed from it. Two of the three theorems proved in this work are of this kind.

A poset $\langle P, \leq^P \rangle$ is denoted for simplicity by P, and when the context is clear we write $\leq$ rather than $\leq^P$.

For Boolean algebras we use the notations of [Ko]. Thus $+$, $\cdot$, $-$ and $\leq$ denote the join, meet, complementation and partial ordering of a Boolean algebra B. The zero and one of B are denoted by 0 and 1.

Poset algebras are defined as follows. Let $\langle P, \leq \rangle$ be a partially ordered set. The poset Boolean algebra of $\langle P, \leq \rangle$, denoted by $F(\langle P, \leq \rangle)$, is defined by specifying a set of generators together with a set of relations on them. The set of generators for $F(\langle P, \leq \rangle)$ is $\{x_p : p \in P\}$. The set of relations is $\{x_p \cdot x_q = x_p : p, q \in P \text{ and } p \leq q\}$ union with the set of all identities of the variety of Boolean algebras. We shall need a more formal definition of $F(\langle P, \leq \rangle)$.

Definition 1.1. Let $Y = \{y_p : p \in P\}$ be a set of distinct variables and $T(Y)$ be the set of all terms in the language $\{+, \cdot, -, 0, 1\}$ with variables from Y. Let

$$R_P = \{y_p \cdot y_q = y_p : p, q \in P \text{ and } p \leq q\}.$$

Define an equivalence relation $\sim$ on $T(Y)$: $t \sim u$ if u can be obtained from t by a sequence of substitutions using the identities of the variety of Boolean algebras and the relations of R_P. For $t \in T(Y)$ denote $[t] = t/\sim$ and let $T[Y] = \{[t] : t \in T(Y)\}$. Define $[t] + [u] = [t + u]$, $[t] \cdot [u] = [t \cdot u]$ etc..

It is well-known from universal algebra that such a construction yields a Boolean algebra. The resulting Boolean algebra $\langle T[Y], +, \cdot, -, 0, 1 \rangle$ is called the *poset algebra* of P. We denote the set of equivalence classes $T[Y]$ by $F(P)$ and $[y_p]$ by x_p.

We shall use the following well-known property of $T[Y]$ which is a special case of a more general theorem in universal algebra.

Theorem 1.2. *Let B be a Boolean algebra and $g : Y \to B$. Suppose that $\{g(y) : y \in Y\}$ satisfies the relations of R_P. That is, for every $p \leq q$ in P, $g(y_p) \cdot g(y_q) = g(y_p)$. Then there is a homomorphism $\tilde{g} : T[Y] \to B$ such that for every $y \in Y$, $\tilde{g}([y]) = g(y)$.*

There is an alternative description of $F(P)$ using topology. A subset R of P is a final segment if for every $p \in R$ and $q \in P$: if $p \leq q$ then $q \in R$. Let $Fs(P)$ be the set of final segments of P. Viewing a final segment as its characteristic function, it is obvious that $Fs(P)$ is a closed subspace of $\{0,1\}^P$. Let $\widehat{F}(P)$ be the Boolean algebra of clopen subsets of $Fs(P)$. Consider the map $x_p \mapsto \{R \in Fs(P) \colon p \in R\}$. In Theorem 2.3 it will be shown that this map extends to an isomorphism between $F(P)$ and $\widehat{F}(P)$. Thus we have a description of $F(P)$ as an algebra of subsets of $Fs(P)$.

Poset algebras can be viewed in yet another way. A *topological lattice* is a structure of the form $\langle X, \tau, \vee, \wedge \rangle$ where $\langle X, \tau \rangle$ is a topological space, $\langle X, \vee, \wedge \rangle$ is a lattice and $\vee$ and $\wedge$ are continuous. In $Fs(P)$ define $x \vee y = x \cup y$ and $x \wedge y = x \cap y$. This makes $Fs(P)$ into a topological lattice which we denote by $L(P)$. Obviously, $L(P)$ is a distributive lattice.

Our first main theorem (Theorem 2.6) is that every Hausdorff compact 0-dimensional topological distributive lattice is isomorphic to $L(P)$ for some poset P.

We next define the notions used in the statement of the main theorems of Sections 3 and 4.

A poset P is *well-founded*, if P has no strictly decreasing infinite sequence.

Let B be a Boolean algebra. A member $a \in B$ is called an *atom* of B, if a is a minimal element of $B - \{0\}$. A Boolean algebra B is *superatomic*, if every subalgebra of B has an atom. A Boolean algebra B is called a *well-generated algebra*, if B has a sublattice G (that is, a subset closed under $+$ and $\cdot$) such that:

(1) G generates B;

(2) $\langle G, \leq^B \restriction G \rangle$ is well-founded.

Every well-generated algebra is superatomic. This is proved in [BR] Proposition 2.7(b). However, the proof is easy, and can be found by the reader.

We now turn to partial orderings. A subset of a poset P consisting of pairwise incomparable elements is called an *antichain* of P.

A poset P is *narrow*, if P does not contain infinite antichains.

A poset P is *scattered*, if P does not contain a subset isomorphic to the chain of rational numbers.

We can now state the second main theorem. It will be proved in Section 3.

Theorem 1.3. *Let P be a partially ordered set. Then the following properties are equivalent.*

(1) *P is narrow and scattered.*

(2) *$F(P)$ is superatomic.*

(3) *$F(P)$ is well-generated.*

The implication (1) $\Rightarrow$ (3) is the difficult part of Theorem 1.3. M. Pouzet [P] proved the implication (1) $\Rightarrow$ (2) (see R. Fraïssé [F] Section 6.7). But now this fact follows easily from the implication (1) $\Rightarrow$ (3).

The interval algebra of a linearly ordered set $\langle L, < \rangle$ is the subalgebra of $\mathcal{P}(L)$ generated by the set $\{[a, \infty) \colon a \in L\}$ of left closed rays of L. This algebra is denoted by $B(L)$. The interval algebra of L is isomorphic to the poset algebra of L when L has no minimum, and to the poset algebra of L

minus its minimum when L has a minimum. Thus interval algebras are a special case of poset algebras.

In [AB1] Theorem 3.4 it is proved that an interval algebra of a scattered linear ordering is embeddable in the interval algebra of a well-ordering. This result motivates our third main theorem.

A partially ordered set is *well-ordered*, if it is narrow and well-founded.

Theorem 1.4. *Let P be a narrow scattered partially ordered set. Then there is a well-ordered poset W and a subalgebra B of $F(W)$ such that $F(P)$ is a homomorphic image of B.*

Theorems 1.3 and 1.4 use a structure theorem for narrow scattered partially ordered sets, which is similar to the theorem of Hausdorff on the structure of scattered linear orderings. This theorem is due to Abraham. It is proved in [AB2] Theorem 3.4. We next quote this theorem.

Suppose that P is a poset and for every $a \in P$, Q_a is a poset. We define the *lexicographic sum* of the indexed family $\{Q_a : a \in P\}$, and denote it by $\sum\{Q_a : a \in P\}$. The universe of $\sum\{Q_a : a \in P\}$ is $\bigcup\{\{a\} \times Q_a : a \in P\}$. The partial ordering on $\sum\{Q_a : a \in P\}$ is defined as follows: $\langle a, q\rangle \leq \langle b, r\rangle$, if either (i) $a <^P b$ or (ii) $a = b$ and $q \leq^{Q_a} r$.

We say that a poset $\langle P, \leq\rangle$ is *anti well-ordered*, if $\langle P, \geq\rangle$ is well-ordered. Let $\mathcal{W}$ denote the class of posets which are either well-ordered or anti well-ordered.

Suppose that $\leq^1$ and $\leq^2$ are partial orderings of a set P. We say that $\langle P, \leq^2\rangle$ is an *augmentation* of $\langle P, \leq^1\rangle$ iff $\leq^1 \subseteq \leq^2$.

Theorem 1.5. ([AB2] Theorem 3.4)
Let $\mathcal{H}_0$ be the class of all posets whose universe is a singleton. Let $\mathcal{H}$ be the closure of $\mathcal{H}_0$ under lexicographic sums indexed by members of $\mathcal{W}$, and under isomorphism.
Then for every poset P the following are equivalent.
(1) *P is narrow and scattered.*
(2) *P is an augmentation of a member of $\mathcal{H}$.*

Theorem 1.4 follows from Theorem 1.5 and from the following theorem which is of independent interest.

Theorem 1.6. *Let $P \in \mathcal{H}$. Then there is a well-ordered poset W such that $F(P)$ is embeddable in $F(W)$.*

Theorems 1.6 and 1.4 will be proved in Section 4.

We conjecture that Theorem 1.4 cannot be strengthened by requiring that for some well-ordered poset W, $F(P)$ is embeddable in $F(W)$. We thus ask the following question.

Question 1.7. Is there a narrow scattered poset P such that $F(P)$ cannot be embedded in $F(W)$ whenever W is a well-ordered poset?

In [AB1] Theorem 1, it is proved that a superatomic subalgebra of an interval algebra is embeddable in an interval algebra of a scattered linear ordering. We ask whether the analogous fact for partially ordered sets is also true.

Question 1.8. Let B be a superatomic subalgebra of a poset algebra. Does there exist a narrow scattered poset P such that B is embeddable in $F(P)$?

2 Topological descriptions of poset algebras

In this section we show that the poset algebra of P is the algebra of clopen sets of the space of final segments of P. We also characterize poset algebras as the algebras of clopen sets of Hausdorff compact 0-dimensional topological distributive lattices.

Referring to Definition 1.1, let P be a poset, $Y = \{y_p : p \in P\}$, B be a Boolean algebra and $f : Y \to B$. We say that f *respects the relations of* R_P if $f(y_p) \cdot f(y_q) = f(y_p)$ whenever $p \le q$. Note that this is equivalent to the fact that the function $p \mapsto f(y_p)$ is order preserving.

The property of $F(P)$ mentioned in Theorem 1.2 characterizes $F(P)$ up to an isomorphism. This is noted in the following theorem.

Theorem 2.1. *Let P be a poset, B be a Boolean algebra and $g : Y \to B$. Suppose that g and B satisfy the following three properties:*

(1) *g respects the relations of R_P.*

(2) *The image of g generates B.*

(3) *For every Boolean algebra C and a function h from Y to C: if h respects R_P, then there is a homomorphism $\alpha : B \to C$ such that $\alpha \circ g = h$.*

Then $F(P) \cong B$ and there is such an isomorphism ψ satisfying

$$\psi(x_p) = g(y_p), \text{ for every } p \in P.$$

Proof. It is trivial that Clauses (1) - (3) characterize B up to an isomorphism. And by Theorem 1.2, $F(P)$ fulfills these clauses. □

Let $\mathcal{P}(A)$ denote the powerset of A. $\mathcal{P}(A)$ is equipped with its Cantor topology defined as follows. Let σ, τ be finite subsets of A. Denote $U_{\sigma,\tau} = \{x \in \mathcal{P}(A) : \sigma \subseteq x \text{ and } x \cap \tau = \emptyset\}$. The set of all $U_{\sigma,\tau}$'s is a base for the topology of $\mathcal{P}(A)$.

Definition 2.2. A subset R of a poset P is a *final segment* of P if for every $p \in R$ and $q \in P$: if $p \le q$, then $q \in R$. The set of final segments of P is denoted by $Fs(P)$.

For every $p \in P$ let $V_p = \{x \in Fs(P) : p \in x\}$. Let $\widehat{F}(P)$ be the subalgebra of $\mathcal{P}(Fs(P))$ generated by the set $\{V_p : p \in P\}$.

The following theorem provides an alternative (and sometimes more convenient) description of $F(P)$. It will be also used to prove some computational facts about $F(P)$.

Theorem 2.3. *Let P be a poset.*

(a) $Fs(P)$ is closed in $\mathcal{P}(P)$, and $\widehat{F}(P)$ is the set of clopen subsets of $Fs(P)$.

(b) There is an isomorphism φ between $F(P)$ and $\widehat{F}(P)$ such that for every $p \in P$, $\varphi(x_p) = V_p$.

(c) For every $p \neq q$ in P, $x_p \neq x_q$.

Proof. (a) It is trivial that $Fs(P)$ is closed. Denote by $clop(X)$ the set of clopen subsets of a space X. If X is a Hausdorff compact 0-dimensional space, and $F \subseteq X$ is closed, then $clop(F) = \{U \cap F : U \in clop(X)\}$. It is easy to see that a subset $U \subseteq \mathcal{P}(P)$ is clopen iff it is a finite union of sets of the form $U_{\sigma,\tau}$.

It is also easy to see that for every V, $V \in \widehat{F}(P)$ iff there is U of the above form such that $V = U \cap Fs(P)$. It follows that $\widehat{F}(P) = clop(Fs(P))$.

(b) By the definition of $\widehat{F}(P)$ for every $p, q \in P$: if $p \leq q$, then $V_p \subseteq V_q$. So the relation $V_p \cap V_q = V_p$ holds between the generators V_p and V_q of $\widehat{F}(P)$. Recall that $F(P)$ is $T[Y]$ of Definition 1.1 and $x_p = [y_p]$. So by Theorem 1.2, there is a homomorphism $\varphi : F(P) \to \widehat{F}(P)$ such that $\varphi([y_p]) = V_p$.

Since $\{V_p : p \in P\}$ generates $\widehat{F}(P)$, φ is surjective.

Let σ, τ be finite subsets of P. It is easy to see that:

$$\text{If } \bigcap\{V_p : p \in \sigma\} \cap \bigcap\{-V_q : q \in \tau\} = \emptyset, \text{ then for some } p \in \sigma \text{ and } q \in \tau,\ p \leq q. \tag{1}$$

Also, by the definition of the relations of $F(P)$,

$$\text{If } p \leq q, \text{ then in } F(P),\ x_p \cdot -x_q = 0. \tag{2}$$

Now suppose that $a \in F(P)$ and $\varphi(a) = \emptyset$. The element a is the sum of elements of the form $\prod\{x_p : p \in \sigma\} \cdot \prod\{-x_q : q \in \tau\}$. Let us take one of these summands. Then
$\emptyset = \varphi(\prod\{x_p : p \in \sigma\} \cdot \prod\{-x_q : q \in \tau\}) = \bigcap\{V_p : p \in \sigma\} \cap \bigcap\{-V_q : q \in \tau\}$.
By (1), there are $p \in \sigma$ and $q \in \tau$ such that $p \leq q$. By (2),
$\prod\{x_p : p \in \sigma\} \cdot \prod\{-x_q : q \in \tau\} = 0$. It follows that $a = 0$. So φ is injective.

(c) If $p \neq q$, then $V_p \neq V_q$. Since $\varphi(x_p) = V_p$ and $\varphi(x_q) = V_q$, $x_p \neq x_q$. $\square$

Definition 2.4. Let $\langle P, \leq \rangle$ be a poset.

(a) We denote by $\langle P, \leq \rangle^*$ the *inverse ordering* of $\langle P, \leq \rangle$. That is, $\langle P, \leq \rangle^* = \langle P, \geq \rangle$. We abbreviate $\langle P, \leq \rangle^*$, and denote it by P^*.

(b) Let $a \in P$. Then $P^{\geq a}$ denotes the set $\{p \in P : p \geq a\}$. For $\sigma \subseteq P$ let $P^{\geq \sigma} = \{p \in P : \text{there is } a \in \sigma \text{ such that } p \geq a\}$.

(c) Let P and Q be posets and $\alpha : P \to Q$. We say that α is a *homomorphism* from P to Q, if for every $s, t \in P$: if $s \leq t$, then $\alpha(s) \leq \alpha(t)$.

We say that α is an *embedding* of P in Q, if for every $s, t \in P$: $s \leq t$, iff $\alpha(s) \leq \alpha(t)$.

The next proposition collects some easy facts needed in the subsequent sections.

Proposition 2.5. *Let P be a poset and denote the set of generators of $F(P)$ by $X := \{x_p : p \in P\}$.*

(a) We denote the set of generators of $F(P^*)$ by $\{x_p^* : p \in P\}$. Then the function $x_p \mapsto -x_p^*$, $p \in P$ extends to an isomorphism between $F(P)$ and $F(P^*)$. We denote this isomorphism by ψ_P.

(b) Let σ and τ be finite subsets of P. The following are equivalent.
(i) $\prod\{x_p : p \in \sigma\} \cdot \prod\{-x_q : q \in \tau\} = 0$.
(ii) There are $p \in \sigma$ and $q \in \tau$ such that $p \leq q$.

(c) Let $\sigma, \tau \subseteq P$ be such that for every $p \in \sigma$ and $q \in \tau$, p and q are incomparable. Suppose that b and c belong to the subalgebras of $F(P)$ generated respectively by $\{x_r : r \in \sigma\}$ and $\{x_r : r \in \tau\}$. If $b \cdot c = 0$, then $b = 0$ or $c = 0$.

(d) Let Q be a poset and $\alpha : P \to Q$ be a homomorphism. Let $Z := \{z_q : q \in Q\}$ be the set of generators of $F(Q)$. Define $\rho : X \to Z$ as follows: $\rho(x_p) = z_{\alpha(p)}$. Then
(1) ρ extends to a homomorphism ρ_α from $F(P)$ to $F(Q)$.
(2) If α is an embedding of P in Q, then ρ_α is an embedding of $F(Q)$ in $F(P)$.

Proof. (a) The function ψ'_P defined by $x_p \mapsto -x_p^*$, $p \in P$, is an order preserving function from $\{x_p : p \in P\}$ to $F(P^*)$. That is, for every relation $x_p \cdot x_q = x_p$ in the defining set of relations of $F(P)$, the relation $(-x_p^*) \cdot (-x_q^*) = -x_p^*$ holds for the images of the x's under ψ'_P. So ψ'_P extends to a homomorphism ψ_P between $F(P)$ and $F(P^*)$. Since $\mathrm{Rng}(\psi'_P)$ generates $F(P^*)$, ψ_P is surjective.

The homomorphism $\psi_{P^*} : F(P^*) \to F(P)$ is defined similarly. $\psi_{P^*}(x_p^*) = -x_p$. So $\psi_{P^*}(-x_p^*) = x_p$. Hence $(\psi_{P^*} \circ \psi_P) \restriction \{x_p : p \in P\} = Id$. It follows that $\psi_{P^*} \circ \psi_P = Id$. So ψ_P is injective.

(b) Suppose that there are $p \in \sigma$ and $q \in \tau$ such that $p \leq q$. Then in $F(P)$, $x_p \cdot x_q = x_p$. So in $F(P)$, $x_p - x_q = 0$. So in $F(P)$, $\prod\{x_p : p \in \sigma\} \cdot \prod\{-x_q : q \in \tau\} = 0$.

For the second direction we use the isomorphism φ between $F(P)$ and $\widehat{F}(P)$ defined in Theorem 2.3(b). Suppose that for every $p \in \sigma$ and $q \in \tau$, $p \not\leq q$. The set $R := \bigcup\{P^{\geq p} : p \in \sigma\}$ is a final segment of P. For every $p \in \sigma$, $p \in R$; and for every $q \in \tau$, $q \notin R$. So $R \in \bigcap\{V_p : p \in \sigma\} \cap \bigcap\{-V_q : q \in \tau\}$. It follows that

$$\varphi(\prod\{x_p : p \in \sigma\} \cdot \prod\{-x_q : q \in \tau\}) = \bigcap\{V_p : p \in \sigma\} \cap \bigcap\{-V_q : q \in \tau\} \neq \emptyset .$$

So $\prod\{x_p : p \in \sigma\} \cdot \prod\{-x_q : q \in \tau\} \neq 0$.

(c) Part (c) follows easily from Part (b).

(d) (1) A relation between generators of $F(P)$, has the form $x_s \cdot x_t = x_s$, where $s \leq t$ in P. Since α is a homomorphism, $\alpha(s) \leq \alpha(t)$. So

$z_{\alpha(s)} \cdot z_{\alpha(t)} = z_{\alpha(s)}$ holds in $F(Q)$. That is, $\rho(x_s) \cdot \rho(x_t) = \rho(x_s)$. This implies (1).

(2) Let $\sigma, \tau \subseteq P$ be finite. Denote
$x_{\sigma,\tau} = \prod\{x_p : p \in \sigma\} \cdot \prod\{-x_q : q \in \tau\}$. It suffices to show that for every finite $\sigma, \tau \subseteq P$, if $\rho_\alpha(x_{\sigma,\tau}) = 0$, then $x_{\sigma,\tau} = 0$.

Now, $\rho_\alpha(x_{\sigma,\tau}) = \prod\{z_{\alpha(p)} : p \in \sigma\} \cdot \prod\{-z_{\alpha(q)} : q \in \tau\}$. So if $\rho_\alpha(x_{\sigma,\tau}) = 0$, then by Part (b), there are $p \in \sigma$ and $q \in \tau$ such that $\alpha(p) \leq \alpha(q)$. Since α is an embedding, $p \leq q$. So by Part (b), $x_{\sigma,\tau} = 0$. □

Recall that the Cantor topology on $\mathcal{P}(A)$ was defined by the basic clopen sets

$$U_{\sigma,\tau} = \{x \in \mathcal{P}(A) \colon \sigma \subseteq x \text{ and } x \cap \tau = \emptyset\}$$

defined for every σ, τ finite subsets of A. The resulting topology is Hausdorff, compact, and 0-dimensional.

Since the intersection (and unions) of a family of final segments is again a final segment, $Fs(P)$ is a lattice (a sublattice of $\mathcal{P}(P)$) in addition to being a topological space. In fact it is a topological lattice as we shall see. Denote by τ^P the Cantor topology on $Fs(P)$ and define

$$L(P) := \langle Fs(P), \tau^P, \cup, \cap \rangle.$$

The main goal of this section is to prove the following theorem.

Theorem 2.6. (a) *Let P be a poset. Then $L(P)$ is a Hausdorff compact 0-dimensional topological distributive lattice.*

(b) *Let $\langle X, \tau, \vee, \wedge \rangle$ be a Hausdorff compact 0-dimensional topological distributive lattice. Then there is a poset P such that $\langle X, \tau, \vee, \wedge \rangle \cong L(P)$.*

(c) *If $\varphi : L(P) \cong L(Q)$, then there is $\alpha : P \cong Q$ such that α induces φ. Actually, it is enough to assume that φ is a lattice isomorphism of $Fs(P)$ and $Fs(Q)$; this already implies that φ is continuous.*

We recall first some notions on lattices. Let L be a lattice. A subset $F \subseteq L$ is a *filter* if (i) F is closed under meet and (ii) for every $a \in F$ and $b \geq a$, $b \in F$. F is called a *proper filter* if $F \neq L$. $I \subseteq L$ is an *ideal* if (i) F is closed under join and (ii) for every $a \in I$ and $b \leq a$, $b \in I$.

An element e of a lattice L is *compact* if, for every $D \subseteq L$, $e \leq \bigvee D$ implies $e \leq \bigvee D_0$ for some finite $D_0 \subseteq D$. An element e is *join irreducible* if $e = a \vee b$ implies $e = a$ or $e = b$. An element e of a lattice is *completely join prime* if $e \leq \bigvee D$ implies that there exists $d \in D$ such that $e \leq d$. Observe that the minimal element of a lattice L (if it exists) is not completely join prime. (For if 0 is the minimal element of L then $0 = \bigvee \emptyset$.) Similarly, we say that e is *completely meet prime* if $e \geq \bigwedge D$ implies that there exists $d \in D$ such that $e \geq d$. Observe, again, that 1 (if it exists) is not completely meet prime. If e is completely join prime then it is both compact and join irreducible. The converse also holds in distributive lattices.

If $a < b$ and there is no c in the lattice with $a < c < b$, then we way that b *covers* a (and b is said to be a *cover*).

Let e and f be two elements of a complete lattice L. We shall say that $\langle e, f\rangle$ is a *prime pair* iff L is a disjoint union of the sets $L^{\geq e}$ and $L^{\leq f}$. In this case e is obviously completely join prime, and f is completely meet prime.

If e is any completely join prime in a complete lattice L, then there exists a (uniquely determined) completely meet prime element f such that $\langle e, f\rangle$ is a prime pair. Indeed, define $f = \bigvee(L - L^{\geq e})$. Then $e \not\leq f$, or else the fact that e is completely join prime implies that $e \leq d$ for some $d \notin L^{\geq e}$ which is impossible. Similarly, if f is completely meet prime and we define $e = \bigwedge(L - L^{\leq f})$, then $\langle e, f\rangle$ is a prime pair.

Prime pairs and covers are intimately connected. If $\langle e, f\rangle$ is a prime pair, then e covers $e \wedge f$, and $e \vee f$ covers f. Let L be a distributive, complete lattice, and suppose that b covers a. Define

$$I = \{x \in L \colon x \wedge b \leq a\}$$
$$F = \{x \in L \colon x \vee a \geq b\}.$$

Then I is an ideal, F a filter, $I \cap F = \emptyset$, and $I \cup F = L$. To prove this last equation, let x be an arbitrary element of L, and define $c = a \vee (b \wedge x)$. Then $a \leq c \leq b$, and hence there are two possibilities: $c = a$ which implies that $c \in I$, and $c = b$ which implies that $c \in F$.

Now, since L is complete we can define $e = \bigwedge F$, $f = \bigvee I$. Then $\langle e, f\rangle$ is a prime pair. Thus we have proved the following

Lemma 2.7. *Suppose that L is a distributive complete lattice, and b covers a. Then there exists a prime pair $\langle e, f\rangle$ such that $e \leq b$ and $f \geq a$.*

A *topological lattice* is a structure $\langle L, \tau, \vee, \wedge\rangle$ such that (L, τ) is a Hausdorff topological space, $(L, \vee, \wedge)$ a lattice, and the join and meet functions are continuous (from $L \times L$ to L). Observe that the set $\leq$ (as a subset of $L \times L$) is closed. Observe also that a compact (Hausdorff) topological lattice L is necessarily complete. (In particular, 0^L and 1^L exist.)

Let P be a poset. We have defined $L(P)$ as the topological lattice of final segments $Fs(P)$ of P. We note that part (a) of Theorem 2.6 is an easy exercise. To prove that $L(P)$ is a Hausdorff, compact, 0-dimensional topological distributive lattice, recall that $Fs(P)$ is closed in $\mathcal{P}(P)$ and hence it inherits these topological properties. It is obvious that the set of all sets of the form $Fs(P) \cap U_{\sigma,\tau}$ is a basis for the topology. It follows now that the lattice operations are continuous.

What are the covers and the completely prime elements of $L(P)$? The covers of $L(P)$ are those final segments x that possess a minimal element p (that is, $p \in x$ and there is no $q < p$ in x). Such x covers the initial segment $x - \{p\}$ (and possibly other initial segments if it has other minimal elements).

For any $p \in P$, $e = P^{\geq p}$ is completely join prime in $L(P)$, $f = P - P^{\leq p}$ is completely meet prime, and $\langle e, f\rangle$ forms a prime pair. For the other direction, every completely join prime member of $Fs(P)$ is some $P^{\geq p}$. To see this, suppose that $e \in Fs(P)$ is completely join prime. Since $e = \bigvee\{P^{\geq p} : p \in e\}$, $e = P^{\geq p}$ for some $p \in P$. Similarly, if $f \in Fs(P)$ is completely meet prime, then $f = P - P^{\leq p}$ for some $p \in P$, and $\langle P^{\geq p}, f\rangle$ is a prime pair.

Thus $e : P \to Fs(P)$ defined by $e(p) = P^{\geq p}$ is an order-inversion bijections between P and the set of completely join prime elements of $L(P)$.

We return to the general discussion and prove that prime pairs are connected to clopen filters and ideals.

Lemma 2.8. *Let L be a distributive compact topological lattice.*

(1) *If F is a non-empty closed filter, then $e = \bigwedge F \in F$ and $F = L^{\geq e}$.*

(2) *If F is a proper, clopen, prime filter, then $e = \bigwedge F$ is completely join prime. Conversely, if e is any completely join prime, then $L^{\geq e}$ is a non-empty, non-trivial, clopen, prime filter.*

Proof. Given that F is a closed filter, the closed sets $F_a = \{x \in F : x \leq a\}$, $a \in F$, have the finite intersection property, and hence their non-empty intersection gives the desired $e \in F$.

Assume now that F is a proper, clopen, prime filter. Then $e = \bigwedge F \in F$. Define $I = L - F$. Since F is a proper, prime filter, I is a non-empty ideal, and I is closed, since F is open. It follows, as above, that $f = \bigvee I \in I$ exists. Hence $e \not\leq f$, and it follows that e is completely join prime. (If $e \leq \bigvee D$ then $D \cap F \neq \emptyset$, or else $D \subseteq I$ and then $\bigvee D \leq f$ contradicts the assumption that $e \leq \bigvee D$.) $\square$

We have identified above the completely meet prime elements of $Fs(P)$. So this lemma yields the following characterization.

Lemma 2.9. *Let P be a poset. In $Fs(P)$ every clopen, prime, proper filter has the form $V_p = \{X \in Fs(P) : p \in X\}$ for some $p \in P$.*

We are interested in this section in compact, 0-dimensional, distributive lattices, and our first result concerning these lattices is the following.

Lemma 2.10. *Let X be a compact, 0-dimensional, distributive topological lattice. Then for every $a < b$ in L there are a' and b' such that $a \leq a' < b' \leq b$ and b' covers a'.*

Proof. Let C be a maximal chain (linearly ordered set) such that $a \leq \bigwedge C$ and $\bigvee C \leq b$. Since X is 0-dimensional, there exists a clopen set B such that $b \in B$ and $a \notin B$. Any maximal chain between a and b is a closed set. Thus $B \cap C$ is closed and trivially closed under meets. Hence $b' = \bigwedge(B \cap C) \in B \cap C$. So $b' > a$, and we observe that $C^{<b'} \cap B = \emptyset$. Thus $C^{<b'} = C^{\leq b'} - B$ is closed, and we define $a' = \bigvee C^{<b'}$. Then $a' < b'$ and b' covers a'. $\square$

Lemma 2.11. *Let X be a compact, 0-dimensional, distributive topological lattice. Then:*

(1) *Every element $x \in X$ is the join of the set of completely join prime elements g such that $g \leq x$.*

(2) *The proper, clopen, prime filters of X separate points, and hence generate the algebra of clopen subsets of X.*

Proof. Let $x \in X$ be an arbitrary element and define x_0 as the join of all $g \leq x$ that are completely meet prime. We shall obtain a contradiction from the assumption that $x_0 < x$. By the previous lemma there are a', b' in X such that $x \leq a' < b' \leq x_0$ and b' covers a'. By Lemma 2.7, there is a prime pair $\langle e, f\rangle$ with $e \leq b'$ and $f \geq a'$. Since $e \leq x$ and e is completely join prime, $e \leq x_0$. But then $e \leq f$, which contradicts the fact that $\langle e, f\rangle$ is a prime pair.

Let $a, b \in X$ be any two points, and assume that $b \not\leq a$. We argue first that we can assume that $a < b$. Indeed, $a \wedge b < b$, and if we find a proper, clopen prime filter F such that $a \wedge b \notin F$ and $b \in F$, then $a \notin F$ as well since F is a filter. So we assume that $a < b$. By the previous lemma there are a', b' in X such that $x \leq a' < b' \leq x_0$ and b' covers a'. By Lemma 2.7, there is a prime pair $\langle e, f\rangle$ with $e \leq b'$ and $f \geq a'$. The filter $X^{\geq e}$ contains b, excludes a and is as required.

Thus the algebra of clopen subsets of X is generated by the clopen filters of X. (In any compact space a family of clopen sets that separate points generates all clopen sets.) $\square$

We are now ready to prove part (b) of Theorem 2.6. So let $L = \langle X, \tau, \vee, \wedge\rangle$ be a Hausdorff compact 0-dimensional topological distributive lattice. Let $<^L$ denote its lattice ordering. Let $P \subseteq X$ be the set of completely join prime elements of X, and define $a <^P b$ iff $b <^L a$. That is reverse the lattice ordering. Then $P = \langle P, <^P\rangle$ is the required poset for which we prove that $L(\langle P, <^P\rangle) \cong L$. To define the required isomorphism $\psi : L(P) \to L$, let $x \in L(P)$ be any element. Then $x \in Fs(P)$ is a final segment in P, and hence it is an initial segment of completely join prime members of X. Anyhow, $x \subseteq X$ and we define

$$\psi(x) = \bigvee x \in X.$$

It is obvious that x is an order-homomorphism, and Lemma 2.11(1) immediately implies that ψ is onto X. We prove that ψ is an embedding. Suppose that $x_1 \not\leq x_2$ and we shall prove that $\psi(x_1) \not\leq \psi(x_2)$. There exists $p \in x_1 - x_2$, and since p is completely join prime, $p \not\leq \bigvee x_2$. Thus $\psi(x_1) \not\leq \psi(x_2)$. Hence ψ is a lattice isomorphism. But this implies, as we are going to prove in the following lemma, that ψ preserves the topological structure as well, namely that it is an isomorphism between the topological lattices $L(P)$ and L.

Lemma 2.12. *Suppose that L_1 and L_2 are Hausdorff compact 0-dimensional topological distributive lattices, and $\psi : L_1 \to L_2$ is a lattice isomorphism. Then ψ is continuous.*

Proof. Since the notion of being a completely join prime element is lattice-theoretic, Lemma 2.8 implies that ψ establish a correspondence between the clopen prime filters of L_1 and L_2. But, by Lemma 2.11, it follows that ψ

establishes a correspondence between the clopen sets of L_1 and L_2. Since the topology of our 0-dimensional spaces is generated by the clopen sets, it is clear that ψ is a homemomorphism.

We can now also prove part (c) of Theroem 2.6. Suppose that $\varphi : L(P) \cong L(Q)$, namely that φ is a lattice isomorphism (and thus an isomorphism of the topological structures as well). Let $J_{L(P)}$ be the set of completely join prime elements of $L(P)$, and $J_{L(Q)}$ be the corresponding set in $L(Q)$. Then φ is an order isomorphism between $J_{L(P)}$ and $J_{L(Q)}$. Now, $e_P : P \to J_{L(P)}$ defined by $e_P(x) = P^{\geq x}$ is an order anti-iomorphism, and so is e_Q. Thus $\alpha : P \to Q$ defined by

$$\alpha(p) = e_Q^{-1} \circ \varphi \circ e_P(p)$$

is an order isomorphism between P and Q. We have $\varphi(P^{\geq p}) = Q^{\geq \alpha(p)}$. Hence α induces φ in the sense that for every final segment $s \in Fs(P)$, $\varphi(s) = \alpha[s]$. $\square$

Our results reveal that the category of posets with order-preserving maps is (anti) isomorphic to the category of compact, 0-dimensional, distributive topological lattices, with continuous lattice homomorphisms that preserve 0 and 1. For this, we need the following lemma.

Lemma 2.13. *Let X and Y be compact 0-dimensional distributive lattices. A map $f : X \to Y$ is a continuous lattice homomorphism iff the f-preimage of every clopen prime filter of Y is a clopen prime filter of X.*

Proof. Clearly, the condition is necessary: if f is a continuous lattice homomorphism then the preimage of every clopen set is clopen, and the preimage of every prime filter is a prime filter.

Assume that $f : X \to Y$ is such that $f^{-1}[P]$ is a clopen prime filter in X, whenever P is a clopen prime filter in Y. Since the complement of a clopen prime ideal is a clopen prime filter, we get that the preimage of any clopen prime ideal is also a clopen prime ideal. Then f is continuous, because by Lemma 2.11 the collection of clopen prime filters and ideals generates the topology.

Suppose $x, y \in X$ are such that $a = f(x \wedge y) \neq f(x) \wedge f(y) = b$. There exists a clopen prime filter F which separates a, b. In case $a \in F$ and $b \notin F$, $x \wedge y \in f^{-1}[F]$ (since $f^{-1}[F]$ is a filter) and hence $x, y \in f^{-1}[F]$ and therefore $f(x), f(y) \in F$, which is a contradiction. In case $a \notin F$ and $b \in F$, then we get a contradiction again. Thus f is meet preserving. By a dual argument, f is join preserving.

$\square$

For every order preserving map $f : P \to Q$ there is a continuous homomorphism $g : L(Q) \to L(P)$ that maps 0 to 0 and 1 to 1, and satisfies

$$g^{-1}[V_p] = V_{f(p)} \tag{3}$$

for every $p \in P$. (Simply define $g(X) = f^{-1}[X]$.)

Lemma 2.13 implies that if f and g are any two functions connected as in (3), then f is order-preserving and g is a continuous homomorphism (preserving 0 and 1).

Now, if $g : L(Q) \to L(P)$ is any continuous homomorphism that maps 0 to 0 and 1 to 1, then f that satisfies (3) can be defined by referring to Lemma 2.9.

3 The well-generatedness of poset algebras

In this section we prove Theorem 1.3. It says that the poset algebra of a narrow scattered poset is well-generated.

Recall that $\mathcal{W}$ is the class of posets which are either well-ordered or anti well-ordered.

Definition 3.1. We define by induction on the ordinal α the class of posets $\mathcal{H}_\alpha$. Let $\mathcal{H}_0$ be the class of all posets whose universe is a singleton.
If α is a limit ordinal, then $\mathcal{H}_\alpha = \bigcup\{\mathcal{H}_\beta : \beta < \alpha\}$.
If $\alpha = \beta+1$, then $\mathcal{H}_\alpha$ is the class of all posets P such that there is $W \in \mathcal{W}$ and an indexed family of posets $\{P_w : w \in W\} \subseteq \mathcal{H}_\beta$ such that P is isomorphic to $\sum\{P_w : w \in W\}$. Define

$$\mathcal{H} = \bigcup\{\mathcal{H}_\alpha : \alpha \in Ord\}.$$

We shall prove by induction on α that if $P \in \mathcal{H}_\alpha$, then $F(P)$ is well-generated. The rough idea of the proof is as follows. Let $P \in \mathcal{H}_{\alpha+1}$, and suppose that $P = \sum_{w\in W} P_w$, where $W \in \mathcal{W}$ and each P_w belongs to $\mathcal{H}_\alpha$. For every $w \in W$ let L_w be a well-founded lattice which generates $F(P_w)$. If the L_w are appropriately chosen, then $\bigcup_{w\in W} L_w$ generates a well-founded lattice, and certainly, this lattice generates $F(P)$.

The following key lemma from [BR] is reproved here for the reader's convenience.

Proposition 3.2. *Let B be a well-generated algebra and I be a maximal ideal of B. Then there is a sublattice $G \subseteq I$ such that G generates B, and G is well-founded.*

Proof. We prove the following claim.

Claim 1. *Let G' be a well-founded sublattice of B and $b \in B$. Let H be the sublattice of B generated by $G' \cup \{b\}$. Then H is well-founded.*

Proof. We may assume that $0^B, 1^B \in G'$. Then every member of H has the form $g + h \cdot b$, where $g, h \in G'$.

Let $\{g_n + h_n \cdot b : n \in \omega\}$ be a decreasing sequence in H. Let $d_n = g_n + h_n \cdot b$ and $e_n = g_n + h_n$. Suppose by contradiction that $\{d_n \cdot b : n \in \omega\}$ is not eventually constant. We have $d_n \cdot b = e_n \cdot b$. So $\{e_n \cdot b : n \in \omega\}$ is not eventually constant. Then $p_n := \prod_{i\leq n} e_i$, $n \in \omega$, is a decreasing sequence of

members of G' which is not eventually constant. So
(i) $\{d_n \cdot b \ : \ n \in \omega\}$ is eventually constant.

Suppose by contradiction that $\{d_n - b \ : \ n \in \omega\}$ is not eventually constant. $d_n - b = g_n - b$. So $\{g_n - b \ : \ n \in \omega\}$ is a decreasing sequence, which is not eventually constant. Then $q_n := \prod_{i \leq n} g_i, \ \ n \in \omega$, is a decreasing sequence of members of G' which is not eventually constant. So
(ii) $\{d_n - b \ : \ n \in \omega\}$ is eventually constant.
It follows from (i) and (ii) that $\{d_n \ : \ n \in \omega\}$ is eventually constant. So H is well-founded. We have proved Claim 1.

We prove the proposition. Let G' be a well-founded sublattice of B which generates B. If $G' \subseteq I$, then there is nothing to prove. Assume that $G' \not\subseteq I$. Let $G_1 := G' - I$. By the maximality of I, G_1 is a lattice, and since in addition, G_1 is well-founded and not empty, it has a minimum. Denote the minimum by a. Let G be the sublattice of B generated by
$(G' \cap I) \cup \{b - a \ : \ b \in G_1\} \cup \{-a\}$. Clearly, G is contained in the lattice generated by $G' \cup \{-a\}$. So by Claim 1, G is well-founded. By the maximality of I, $\ -a \in I$, so $G \subseteq I$. Let $b \in G'$. If $b \in I$, then $b \in G$. If $b \notin I$, then $a \leq b$. So $b = (b - a) + a$. This means that b belongs to the subalgebra generated by G. So G generates B. $\square$

Let $\langle T, \vee, \wedge \rangle$ be a lattice. $x \vee y$ and $x \wedge y$ denote the join and the meet of x and y in T. The partial ordering of T is denoted by $\leq$. So $x \leq y$ if $x \wedge y = x$.

We shall need the following lemma. It appears in [BR] Lemma 2.8, and the proof is brought here for the readers's convenience.

Lemma 3.3. *Let T be a distributive lattice.*

(a) *If T is well-founded, then every nonempty subset L of T closed under meet has a minimum m.*

(b) *Let S be a subset of T such that:*
(1) *S generates T.*
(2) *The meet of two elements of S is a finite join of elements of S.*
(3) *S is well-founded with respect to the partial ordering of T.*

Then T is well-founded.
In particular, if $S \subseteq T$ is closed under meet, generates T, and is well-founded, then T is well-founded.

Proof. (a) Let m_1 and m_2 be minimal elements of L. Then $m := m_1 \wedge m_2 \in L$, and thus $m = m_1 = m_2$.

(b) Clearly, every element of T is a finite join of elements of S. It is easy to check that the following holds.

$(*)$ If $w_0 > w_1 > \cdots$ is a strictly decreasing sequence in T, and $w_0 = \bigvee_{i<n} v_i$, then there is $\ell < n$ such that $\{w_j \wedge v_\ell : j < \omega\}$ contains a strictly decreasing infinite subsequence.

The proof uses the distributivity of T.

Next suppose by contradiction that $u_0 > u_1 > \cdots$ is a strictly decreasing sequence in T. We define by induction a strictly decreasing sequence $\{v_n : n < \omega\}$ in S. Assume by induction that v_n has the following property.

$(**)$ There is a strictly decreasing sequence $w_0 > w_1 > \cdots$ in T such that $w_0 < v_n$.

Let U be a finite subset of S such that $u_0 = \bigvee U$. By $(*)$, there is $v_0 \in U$ such that $\{u_j \wedge v_0 : j < \omega\}$ contains a strictly decreasing subsequence. Hence v_0 satisfies the induction hypothesis. Suppose that v_n has been defined, and let $v_n > w_0 > w_1 > \cdots$ be as in the induction hypothesis. Let W be a finite subset of S such that $\bigvee W = w_0$. By $(*)$, there is $v_{n+1} \in W$ such that $\{w_j \wedge v_{n+1} : j < \omega\}$ contains a strictly decreasing subsequence. So v_{n+1} satisfies the induction hypothesis and $v_n > w_0 \geq v_{n+1}$. The sequence $\{v_n : n < \omega\} \subseteq S$ is strictly decreasing. This contradicts the well-foundedness of S, so T is well-founded.

Theorem 3.4. *Let $\langle T, \vee, \wedge\rangle$ be a distributive lattice. We assume that T has a minimum 0 and a maximum 1. Let $\langle I, \leq\rangle$ be a poset, and $\{T_i : i \in I\}$ be a family of sublattices of T. Suppose that:*

(W1) *$\langle I, \leq\rangle$ is a well-ordered poset.*

(W2) *For every $i \in I$, T_i is well-founded.*

(W3) *$\bigcup\{T_i : i \in I\}$ generates the lattice T.*

(W4) *For every $i, j \in I$, $s \in T_i$ and $t \in T_j$: if $i < j$ and $s \neq 1$, then either $s < t$ or $s \wedge t = 0$.*

(W5) *Let $\sigma \subseteq I$, $\ell \in I$, and for every $i \in \sigma \cup \{\ell\}$, $u_i \in T_i$. Suppose that the following properties hold:*
(1) *σ is finite;*
(2) *for every $i \in \sigma$, $i \not\leq \ell$;*
(3) *$\bigwedge_{i\in\sigma} u_i \neq 0$ and $u_\ell \neq 1$.*
Then $\bigwedge_{i\in\sigma\cup\{\ell\}} u_i < \bigwedge_{i\in\sigma} u_i$.

Then T is a well-founded lattice.

Proof. We may assume that for every $i \in I$, $1 \in T_i$. Let $\mathcal{T}$ be the set of all functions $\vec{t}$ such that:
(1) $Dom(\vec{t})$ is a finite subset of I,
(2) for every $i \in Dom(\vec{t})$, $\vec{t}(i) \in T_i$.
Let $\bigwedge \vec{t} := \bigwedge\{\vec{t}(i) : i \in Dom(\vec{t})\}$, and

$$L := \{\textstyle\bigwedge \vec{t} : \vec{t} \in \mathcal{T}\}.$$

Let $\vec{t} \in \mathcal{T}$ and $i \in Dom(\vec{t})$. We denote $\vec{t}(i)$ by t_i.

Claim 1. Let $u \in L - \{0,1\}$. Let $\vec{t} \in \mathcal{T}$ be such that $u = \bigwedge\{t_i : i \in Dom(\vec{t})\}$. Let σ be the set of minimal elements of $Dom(\vec{t})$. Then σ is a finite antichain in I and $u = \bigwedge\{t_i : i \in \sigma\}$.

Proof. For each $j \in Dom(\vec{t}) - \sigma$, let $m(j) \in \sigma$ be such that $m(j) < j$. Since $u \neq 0$, $t_{m(j)} \wedge t_j \neq 0$. By (W4), $t_{m(j)} < t_j$. Hence
$u = \bigwedge \vec{t} \leq \bigwedge\{t_j : j \in \sigma\} \leq u$. So $u = \bigwedge\{t_i : i \in \sigma\}$.

Claim 2. For $u \in L - \{0,1\}$, there is $\vec{t}^u \in \mathcal{T}$ satisfying the following properties.

(1) $Dom(\vec{t}^u)$ is an antichain.

(2) $u = \bigwedge \vec{t}^u$.

(3) If $i \in Dom(\vec{t}^u)$, then $t^u_i \neq 0$, 1.

(4) For every $\vec{t} \in \mathcal{T}$ such that $u = \bigwedge \vec{t}$: $Dom(\vec{t}^u) \subseteq Dom(\vec{t})$, and if $i \in Dom(\vec{t}^u)$ then $t^u_i \leq t_i$.

Proof. Let $i \in I$. The set

$$T^u_i := \{\, t' \in T_i : \text{there is } \vec{t} \in \mathcal{T} \text{ such that } i \in Dom(\vec{t}),\ t_i = t' \text{ and } u = \bigwedge \vec{t}\,\}$$

is nonempty and closed under $\wedge$. By (W2), T_i is well-founded, and it follows from Lemma 3.3(a), that $t^*_i := \min(T^u_i)$ exists.

Let $\vec{r} \in \mathcal{T}$ be such that $u = \bigwedge \vec{r}$. Since $u \neq 1$, we may assume that for every $k \in Dom(\vec{r})$, $r_k \neq 1$. Let σ be the set of minimal elements of $Dom(\vec{r})$ and $\vec{s} = \vec{r} \restriction \sigma$. By Claim 1 and since $u \neq 0$, $\bigwedge \vec{s} = u$. Also, σ is an antichain. For every $k \in \sigma$, $t^*_k \leq s_k$, so $\bigwedge_{k\in\sigma} t^*_k \leq \bigwedge_{k\in\sigma} s_k$. That is,

(i) $$\bigwedge\{t^*_k : k \in \sigma\} \leq u.$$

For every $k \in \sigma$, let $\vec{t}^k \in \mathcal{T}$ be such that $u = \bigwedge \vec{t}^k$ and $t^k_k = t^*_k$. Let $\rho = \{\langle k, j\rangle : k \in \sigma, j \in Dom(\vec{t}^k)\}$. Obviously,

$$u = \bigwedge\{\bigwedge \vec{t}^k : k \in \sigma)\} = \bigwedge\{t^k_j : \langle k, j\rangle \in \rho\}.$$

Since for every $k \in \sigma$, $t^*_k \in \{t^k_j : \langle k, j\rangle \in \rho\}$,
$\bigwedge\{t^k_j : \langle k, j\rangle \in \rho\} \leq \bigwedge\{t^*_k : k \in \sigma\}$. That is,

(ii) $$u \leq \bigwedge\{t^*_k : k \in \sigma\}.$$

It follows from (i) and (ii) that $u = \bigwedge\{t^*_k : k \in \sigma\}$.

Let $\vec{t}^u$ be defined as follows: $Dom(\vec{t}^u) = \sigma$ and for every $i \in Dom(\vec{t}^u)$, $\vec{t}^u(i) = t^*_i$, that is $\vec{t}^u_i = t^*_i$.

We show that $\vec{t}^u$ is as required.

(1) $Dom(\vec{t}^u) = \sigma$ is an antichain.

(2) We have proved that $u = \bigwedge \vec{t}^u$.

(3) Since for every $i \in \sigma$, $t^*_i \leq s_i < 1$, (3) holds.

(4) Let $\vec{t} \in \mathcal{T}$ be such that $u = \bigwedge \vec{t}$, and suppose by contradiction that $\ell \in Dom(\vec{t}^u) - Dom(\vec{t})$. By Claim 1, we may assume that $Dom(\vec{t})$ is an antichain. Suppose by contradiction that for some $k \in Dom(\vec{t})$, $k < \ell$. Since $Dom(\vec{t}^u)$ is an antichain and $k < \ell \in Dom(\vec{t}^u)$, for every $i \in Dom(\vec{t}^u)$, $i \not\leq k$. Since $u \neq 1$, we may assume that for every $k \in Dom(\vec{t})$, $t_k \neq 1$. We apply (W5) to $Dom(\vec{t}^u)$, k, $\{t^u_i : i \in Dom(\vec{t}^u)\}$, and t_k, and conclude that

$$u \wedge t_k = (\bigwedge\{t^u_i : i \in Dom(\vec{t}^u)\}) \wedge t_k < \bigwedge\{t^u_i : i \in Dom(\vec{t}^u)\} = u.$$

Hence $u \wedge t_k < u$, and so $u \not\leq t_k$. A contradiction. So there is no $k \in Dom(\vec{t})$ such that $k < \ell$.

We now apply (W5) to $Dom(\vec{t})$, ℓ, $\{t_i : i \in Dom(\vec{t})\}$, and t^u_ℓ, and conclude that

$$u \wedge t^u_\ell \;=\; (\bigwedge\{t_i : i \in Dom(\vec{t})\}) \wedge t^u_\ell \;<\; \bigwedge\{t_i : i \in Dom(\vec{t})\} \;=\; u.$$

Hence $u \wedge t^u_\ell < u$, and so $u \not\leq t^u_\ell$. A contradiction. So $Dom(\vec{t}^{\,u}) \subseteq Dom(\vec{t})$.

Let $i \in Dom(\vec{t}^{\,u})$, and we show that $t^u_i \leq t_i$. Since $\bigwedge \vec{t}^{\,u} \wedge \bigwedge \vec{t} = u$, $t^u_i \wedge t_i \in T^u_i$. Also, $t^u_i = \min(T^u_i)$. So $t^u_i \leq t^u_i \wedge t_i$. That is, $t^u_i \leq t_i$. We have proved Clause (4) of Claim 2. So Claim 2 is proved.

It is obvious that the object $\vec{t}^{\,u}$ which satisfies Clauses (1) - (4) of Claim 2 is unique. For every $u \in L - \{0, 1\}$ we denote

$$\sigma(u) := Dom(\vec{t}^{\,u}).$$

Claim 3. Let $u, v \in L - \{0, 1\}$ be such that $u \leq v$.

(a) For every $i \in \sigma(u) \cap \sigma(v)$, $t^u_i \leq t^v_i$.

(b) For every $i \in \sigma(v)$ there is $j \in \sigma(u)$ such that $j \leq i$.

Proof. (a) For $w \in L - \{0, 1\}$ and $i \in I - \sigma(w)$ we set $t^w_i = 1$. Hence

$$\begin{aligned}\bigwedge\{t^u_i : i \in \sigma(u)\} &= u = u \wedge v = \bigwedge\{t^u_i : i \in \sigma(u)\} \wedge \bigwedge\{t^v_i : i \in \sigma(v)\}\\ &= \bigwedge\{t^u_i \wedge t^v_i : i \in \sigma(u) \cup \sigma(v)\}.\end{aligned}$$

So by Claim 2, for every $i \in \sigma(u) \cap \sigma(v)$, $t^u_i \leq t^u_i \wedge t^v_i$, and thus $t^u_i \leq t^v_i$.

(b) Let $i \in \sigma(v)$, and suppose by contradiction that there is no $j \in \sigma(u)$ such that $j \leq i$. We apply (W5) to $\sigma(u)$, i, $\{t^u_j : j \in \sigma(u)\}$ and t^v_i. Then

$$u \wedge t^v_i \;=\; (\bigwedge\{t^u_j : j \in \sigma(u)\}) \wedge t^v_i \;<\; \bigwedge\{t^u_j : j \in \sigma(u)\} \;=\; u.$$

So $u \not\leq t^v_i$. But $u \leq v \leq t^v_i$. A contradiction.

So there is $j \in \sigma(u)$ such that $j \leq i$. Claim 3 is proved.

Let Q be a poset. A subset J of Q is an *initial segment* of Q if for every $q \in Q$ and $p \in J$: if $q \leq p$, then $q \in J$.

Claim 4. Let Q be a well-ordered poset. Then there is no strictly decreasing sequence of initial segments of Q with respect to set inclusion.

Proof. Claim 4 is due to Higman [H]. For completeness, we sketch a proof. Assume by contradiction that $\{J_n : n \in \omega\}$ is a strictly decreasing sequence of initial segments of Q. For every $n \in \omega$, let $q_n \in J_n - J_{n+1}$. Then $\{q_n : n \in \omega\}$ has no increasing pair. Hence Q is not well-ordered. A contradiction. We have proved Claim 4.

We define a partial ordering on the set $Ant(Q)$ of antichains of a poset Q: let $\sigma \leq^m \tau$, if for every $i \in \tau$ there is $j \in \sigma$ such that $j \leq i$.

Claim 5. If Q is a well-ordered poset, then $\langle Ant(Q), \leq^m \rangle$ is well-founded.

Proof. For $\sigma \in Ant(Q)$ let $Q^{\geq\sigma} = \{q \in Q : (\exists p \in \sigma)(p \leq q)\}$ and $I(\sigma) = Q - Q^{\geq\sigma}$. Since $Q^{\geq\sigma}$ is a final segment of Q, its complement $I(\sigma)$ is an inital segment. The function $\sigma \mapsto Q^{\geq\sigma}$ is one-to-one, and if $\sigma \leq^m \tau$, then $Q^{\geq\sigma} \supseteq Q^{\geq\tau}$, and so $I(\sigma) \subseteq I(\tau)$. If $\{\sigma_n : n \in \omega\}$ is a strictly decreasing sequence in $Ant(Q)$, then $\{I(\sigma_n) : n \in \omega\}$ is a strictly decreasing sequence in the set of initial segments of Q, which by Claim 4 is impossible. So $\langle\, Ant(Q), \leq^m \rangle$ is well-founded.

Claim 6. L is well-founded.

Proof. We prove that L is well-founded.

Note that by Claim 3(b), for every $u, v \in L - \{0,1\}$: if $u \leq v$, then $\sigma(u) \leq^m \sigma(v)$.

Let $\{u_n : n < \omega\}$ be a decreasing sequence in L. Then for every $m > n$, $\sigma(u_m) \leq^m \sigma(u_n)$. So by Claim 5, we may assume that $\{\sigma(u_n) : n \in \omega\}$ is a constant sequence. Let $\sigma = \sigma(u_0)$. By Claim 3(a), if $m > n$ then $t_i^m \leq t_i^n$. Recall that $t_i^n \in T_i$ and that T_i is well-founded. Since σ is finite, there is $n_0 \in \omega$ such that for every $i \in \sigma$ and $m > n_0$, $t_i^m = t_i^{n_0}$. So $u_m = u_{n_0}$. Claim 6 is proved.

We now prove the theorem. By the Claim 6, L is well-founded. It is obvious that L is closed under $\wedge$. Since T is a distributive lattice, and T is generated by $\bigcup\{T_i : i \in I\}$, every member of T is a finite sum of members of L. So L generates T. By Lemma 3.3(b), T is well-founded. $\square$

Definition 3.5. Let P be a poset. An element $b \in F(P)$ is said to be *bounded* if there is a nonempty finite subset σ of P such that $b \leq \sum\{x_p : p \in \sigma\}$. Let $I^{bnd}(P)$ be the set of bounded elements of $F(P)$.

Observe the following trivial fact.

Proposition 3.6. *Let $F(P)$ be a poset algebra. Then $I^{bnd}(P)$ is a maximal ideal of $F(P)$.*

Proof of Theorem 1.3. (3) $\Rightarrow$ (2). The easy proof that every well-generated Boolean algebra is superatomic appears in [BR] Proposition 2.7(b).

(2) $\Rightarrow$ (1). Suppose that it is not true that P is narrow and scatterd, and we show that $F(P)$ is not superatomic. If $\mathbb{Q}$ is embeddable in P, then $F(\mathbb{Q})$ (which is atomless) is embeddable in $F(P)$. So $F(P)$ is not superatomic.

If A is an infinite antichain in P, then $F(A)$ (which is an infinite free Boolean algebra) is embeddable in $F(P)$. So $F(P)$ is not superatomic.

(1) $\Rightarrow$ (3). We prove by induction on α, that for every $P \in \mathcal{H}_\alpha$, $F(P)$ is well-generated. There is nothing to prove for $\alpha = 0$ and for limit ordinals. Suppose that the claim is true for α, and let $P \in \mathcal{H}_{\alpha+1} - \mathcal{H}_\alpha$.

Note that if $\langle P, \leq \rangle \in \mathcal{H}_\alpha$, then $\langle P, \geq \rangle \in \mathcal{H}_\alpha$. Hence by Proposition 2.5(a), we may assume that $P = \sum\{P_v : v \in V\}$, where V is a well-ordered poset, and for every $v \in V$, $P_v \in \mathcal{H}_\alpha$. By the induction hypothesis, for every $v \in V$, $F(P_v)$ is well-generated. By Proposition 3.6, $I^{bnd}(P_v)$ is a maximal ideal in $F(P_v)$. So by Proposition 3.2, there is a well-founded sublattice G_v of $F(P_v)$ such that G_v generates $F(P_v)$ and $G_v \subseteq I^{bnd}(P_v)$. Let G be the sublattice of $F(P)$ generated by $\bigcup\{G_v : v \in V\}$. We verify that G and $\{G_v : v \in V\}$ satisfy the hypotheses of Theorem 3.4.

By the definition, V is well-ordered, and thus (W1) holds.

Since each G_v is a well-founded lattice, (W2) holds.

(W3) follows from the definition of G.

(W4) Let $v < w$ in V, $g \in G_v - \{1^{F(P)}\}$ and $h \in G_w$. We show that either $g < h$ or $g \cdot h = 0$. Let γ be a finite subset of P_v such that $g \leq \sum\{x_p : p \in \gamma\}$. Let $b = \sum\{x_p : p \in \gamma\}$. It suffices to show that either $b < h$ or $b \cdot h = 0$. Note that h has a representation of the following form: there is $\ell \in \omega$ and for every $i < \ell$ there are finite disjoint subsets $\eta(i)$ and $\tau(i)$ of P_w such that $\eta(i) \neq \emptyset$ and

$$h = \sum_{i<\ell} \Big(\prod\{x_p : p \in \eta(i)\} \cdot \prod\{-x_q : q \in \tau(i)\} \Big).$$

Case 1. There is $i_0 < \ell$ such that $\tau(i_0) = \emptyset$.
Hence $b < \prod\{x_p : p \in \eta(i_0)\} \leq h$, and thus $b < h$.
Case 2. For every $i < \ell$, $\tau(i) \neq \emptyset$.
For every $i < \ell$, $b \cdot \prod\{-x_q : q \in \tau(i)\} = \emptyset$, and thus
$b \cdot \prod\{x_p : p \in \eta(i)\} \cdot \prod\{-x_q : q \in \tau(i)\} = 0$. Hence $b \cdot h = 0$.
We have proved that (W4) holds.

(W5) Let $\rho \subseteq V$, $w \in V$, and $\{g_v : v \in \rho \cup \{w\}\} \subseteq G$. We assume that: (1) ρ is finite; (2) for every $v \in \rho$, $v \not\leq w$; (3) for every $v \in \rho \cup \{w\}$, $g_v \in G_v$; (4) $\prod\{g_v : v \in \rho\} \neq 0$ and $g_w \neq 1$.

It needs to be shown that

$$g_w \cdot \prod\{g_v : v \in \rho\} < \prod\{g_v : v \in \rho\}.$$

By (W4), arguing as in Claim 1 of Theorem 3.4, we may assume that ρ is an antichain.

We rely on the fact that $F(P)$ and $\widehat{F}(P)$ are isomorphic, and argue in $\widehat{F}(P)$ rather than in $F(P)$. Recall that the above isomorphism takes x_p to V_p. Let σ, τ be finite subsets of P. Define

$$T_{\sigma,\tau} = \{x \in Fs(P) : \sigma \subseteq x \text{ and } x \cap \tau = \emptyset\}.$$

Clearly, for every $T \subseteq Fs(P)$: $T \in \widehat{F}(P)$ iff T is a finite union of $T_{\sigma,\tau}$'s. Also,

(i) $\qquad T_{\sigma,\tau} \neq \emptyset$ iff for every $p \in \sigma$ and $q \in \tau$, $p \not\leq q$.

Let $T = \bigcup_{i<n} T_{\sigma_i,\tau_i}$ and assume that each T_{σ_i,τ_i} is nonempty. Then T is bounded iff for every $i < n$, $\sigma_i \neq \emptyset$.

We regard the g_v's as members of $\widehat{F}(P)$. For every $v \in \rho \cup \{w\}$ let $g_v = \bigcup_{i<n(v)} T_{\sigma(v,i),\tau(v,i)}$, where $\sigma(v,i), \tau(v,i) \subseteq P_v$. We may assume that each $T_{\sigma(v,i),\tau(v,i)}$ is nonempty. Let $v \in \rho$. Since $g_v \neq 0$, $n(v) > 0$, Let $v \in \rho \cup \{w\}$. Since g_v is bounded, $\sigma(v,i) \neq \emptyset$ for every $i < n(v)$.

Let $\sigma = \bigcup_{v\in\rho} \sigma(v,0)$ and $x = P^{\geq\sigma}$. So $x \in Fs(P)$. We shall show that

$$x \in \prod_{v\in\rho} g_v \;\text{ and }\; x \notin g_w.$$

For $v \in \rho \cup \{w\}$ let $\eta(v) = \bigcup_{i<n(v)} \sigma(v,i) \cup \bigcup_{i<n(v)} \tau(v,i)$.

(ii) If $u,v \in \rho$ are distinct, $p \in \eta(u)$ and $q \in \eta(v)$, then p,q are incomparable. This follows from the fact that ρ is an antichain and $\eta(v) \subseteq P_v$. We show that for every $v \in \rho$, $x \in T_{\sigma(v,0),\tau(v,0)}$. By the definition of x, $\sigma(v,0) \subseteq x$.

Let $q \in \tau(v,0)$. By (i), $q \notin P^{\geq\sigma(v,0)}$. By (ii), $q \notin P^{\geq\sigma(u,0)}$, for every $u \neq v$. So $q \notin x$. Hence $\tau(v,0) \cap x = \emptyset$. We have shown that $x \in T_{\sigma(v,0),\tau(v,0)} \subseteq g_v$.

Hence $x \in \prod_{v\in\rho} g_v$.

Recall that for every $v \in \rho$, $v \not\leq w$. It follows that for every $v \in \rho$, $p \in \eta(v)$ and $q \in \eta(w)$, $p \not\leq q$. That is, denoting $\eta = \bigcup_{v\in\rho} \eta(v)$,

$$P^{\geq\eta} \cap \eta(w) = \emptyset.$$

But $\sigma \subseteq \eta$, and for every $i < n(w)$, $\sigma(w,i) \subseteq \eta(w)$. So $\sigma(w,i) \cap P^{\geq\sigma} = \emptyset$. Recalling that $x = P^{\geq\sigma}$ and that $\sigma(w,i) \neq \emptyset$, one concludes that for every $i < n(w)$, $x \notin T_{\sigma(w,i),\tau(w,i)}$. So $x \notin \bigcup_{i<n(w)} T_{\sigma(w,i),\tau(w,i)} = g_w$.

We have shown that G and $\{G_v : v \in V\}$ satisfy the hypotheses (W1) - (W5). So by Theorem 3.4, G is well-founded. For every $v \in V$, G_v generates $F(P_v)$ and $\bigcup\{F(P_v) : v \in V\}$ generates $F(P)$. So $G = \bigcup\{G_v : v \in V\}$ generates $F(P)$. □

4 The embeddability of poset algebras in well-ordered poset algebras

In this section we prove Theorems 1.6 and 1.4. In fact, we prove the following slight strengthening of 1.6.

Theorem 4.1. *Let $P \in \mathcal{H}$. Then there is a well-ordered poset W and an embedding φ of $F(P)$ in $F(W)$ such that $\varphi(I^{bnd}(P)) \subseteq I^{bnd}(W)$.*

We shall prove by induction on α that the claim of the theorem is true for every $P \in \mathcal{H}_\alpha$.

For a poset Q let m_Q be an element which does not belong to Q and let $Q^+ = \{m_Q\} + Q$. That is, Q^+ is the lexicographic sum of a singleton and Q over a chain with two elements. Let $X := \{x_p : p \in P\}$ be the set of generators of P. Denote by $\{z_q : q \in Q\}$ the generators of $F(Q)$ and by $\{z^+_q : q \in Q^+\}$ the generators of $F(Q^+)$. Let $a \mapsto a^+$ be the embedding of $F(Q)$ into $F(Q^+)$ which sends z_q to z^+_q. Let $\varphi : F(P) \to F(Q)$ be an embedding. Define $\hat{\varphi} : X \to F(Q^+)$ as follows: $\hat{\varphi}(x_p) = x^+_{m_Q} + (\varphi(x_p))^+$. The following notation will be used below. If σ, τ are finite subsets of Q, then $z_{\sigma,\tau} := \prod_{p\in\sigma} z_p \cdot \prod_{q\in\tau} -z_q$. $z^+_{\sigma,\tau}$ is defined similarly.

Lemma 4.2. *Let P, Q, φ and $\hat{\varphi}$ be as in the preceding paragraph.*

(a) *Let σ be a finite subset of P. Then:*

(1) $\sum\{\hat{\varphi}(x_p) : p \in \sigma\} = z^+_{m_Q} + \sum\{\varphi(x_p)^+ : p \in \sigma\}$.

(2) $\prod\{\hat{\varphi}(x_p) : p \in \sigma\} = z^+_{m_Q} + \prod\{\varphi(x_p)^+ : p \in \sigma\}$.

(b) *$\hat{\varphi}$ can be extended to an embedding of $F(P)$ in $F(Q^+)$.*

Proof. (a) The proof of is trival.

(b) Let $p, q \in P$ and $p \leq q$. Then

$$\hat{\varphi}(x_p) \cdot \hat{\varphi}(x_q) = z^+_{m_Q} + \varphi(x_p)^+ \cdot \varphi(x_q)^+ = z^+_{m_Q} + \varphi(x_p)^+ = \hat{\varphi}(x_p).$$

By Theorem 1.2, $\hat{\varphi}$ can be extended to a homomorphism $\tilde{\varphi}$ from $F(P)$ to $F(Q)$.

We show that $\tilde{\varphi}$ is an embedding. Notice first that for every $a \in F(P)$, either $\tilde{\varphi}(a) = z^+_{m_Q} + \varphi(a)^+$ or $\tilde{\varphi}(a) = -z^+_{m_Q} \cdot \varphi(a)^+$. To see this, notice that if $\tilde{\varphi}(a)$ and $\tilde{\varphi}(b)$ have one of the above forms, then so do $\tilde{\varphi}(-a)$ and $\tilde{\varphi}(a \cdot b)$. Let $a \in F(P) - \{0\}$. We show that $\tilde{\varphi}(a) \neq 0$. Write $\varphi(a)$ as $\varphi(a) = \sum_{i<n} z_{\sigma_i,\tau_i}$, where all the summands are different from 0. Then $n > 0$ and $\varphi(a)^+ = \sum_{i<n} z^+_{\sigma_i,\tau_i}$. If $\tilde{\varphi}(a) = z^+_{m_Q} + \varphi(a)^+$, then $\tilde{\varphi}(a) \neq 0$. Suppose that $\tilde{\varphi}(a) = -z^+_{m_Q} \cdot \varphi(a)^+$. Then $\tilde{\varphi}(a) = \sum_{i<n} z^+_{\sigma_i,\tau_i \cup \{m_Q\}}$. Since m_Q is the minimum of Q^+ and $m_Q \notin \sigma_i$, for every $p \in \sigma_i$ and $q \in \tau_i \cup \{m_Q\}$, $p \not\leq q$. By Proposition 2.5(b), for every $i < n$, $z^+_{\sigma_i,\tau_i \cup \{m_Q\}} \neq 0$. So $\tilde{\varphi}(a) \neq 0$. $\square$

Let $\{x_i : i \in I\}$ be a set of generators for a Boolean algebra B. Let $\sigma, \tau \subseteq I$ be finite. We denote $x_{\sigma,\tau} = \prod\{x_p : p \in \sigma\} \cdot \prod\{-x_q : q \in \tau\}$. If φ is a function from $\{x_i : i \in I\}$ to a Boolean algebra C, then we denote $x^{\varphi}_{\sigma,\tau} = \prod\{\varphi(x_p) : p \in \sigma\} \cdot \prod\{-\varphi(x_q) : q \in \tau\}$.

The following claim is part of a lemma due to Sikorski. See [Ko] Theorem 5.5.

Proposition 4.3. *Let $\{x_i : i \in I\}$ be a set of generators for a Boolean algebra B and φ be a homomorphism from B to a Boolean algebra C. Suppose that for every finite $\sigma, \tau \subseteq I$: if $x^{\varphi}_{\sigma,\tau} = 0$, then $x_{\sigma,\tau} = 0$. Then φ is an embedding of B into C.*

Lemma 4.4. *Let $R = \sum\{R_i : i \in I\}$ be a lexicographic sum of posets. Suppose that $\sigma, \tau \subseteq I$ are finite, and for every $i \in \sigma$ and $j \in \tau$, $i < j$. For every $i \in \sigma$ let $a_i \in I^{bnd}(R_i)$. For every $j \in \tau$ let $r_j \in R_j$. Then $\sum_{i\in\sigma} a_i < \prod_{j\in\tau} x_{r_j}$.*

Proof. We rely on the fact that $F(R)$ and $\widehat{F}(R)$ are isomorphic, and argue in $\widehat{F}(R)$ rather than in $F(R)$. Recall that the isomorphism between $F(R)$ and $\widehat{F}(R)$ takes each x_r to V_r, where $V_r = \{x \in Fs(R) : r \in x\}$. As in the proof Theorem 1.3 we denote $T_{\sigma,\tau} = \{x \in Fs(R) : \sigma \subseteq x \text{ and } x \cap \tau = \emptyset\}$. Now, each x_{r_i} is replaced by V_{r_i} and a_i has the form $a_i = \bigcup_{\ell<n(i)} T_{\sigma(i,\ell),\tau(i,\ell)}$. We may assume that for every i, ℓ, $T_{\sigma(i,\ell),\tau(i,\ell)} \neq \emptyset$. By the boundedness of the a_i's, for every i and ℓ, $\sigma(i,\ell) \neq \emptyset$. Let $s(i,\ell) \in \sigma(i,\ell)$. Clearly, for every $i \in \sigma$, $\ell < n(i)$ and $j \in \tau$, $s(i,\ell) < r_j$. Let $x \in \bigcup_{i\in\sigma} a_i$. Then for some i and ℓ, $s(i,\ell) \in x$. Hence for every $j \in \tau$, $r_j \in x$. So $x \in \bigcap_{j\in\tau} V_{r_j}$. That is, $\bigcup_{i\in\sigma} a_i \subseteq \bigcap_{j\in\tau} V_{r_j}$.

Let $x = \bigcup_{j\in\tau} P^{\geq r_j}$. Then $x \in \bigcap_{j\in\tau} V_{r_j}$. Since for every $i \in \sigma$ and $\ell < n(i)$ and $j \in \tau$, $r_j \not< s(i,\ell)$, $s(i,\ell) \notin x$. So $x \notin \bigcup_{i\in\sigma} a_i$. That is, $\bigcup_{i\in\sigma} a_i \subsetneqq \bigcap_{j\in\tau} V_{r_j}$. $\square$

The next lemma contains the main claim in the proof of Theorem 4.1. It will be used in the inductive step.

Lemma 4.5. *Let V be a poset. For every $v \in V$ let P_v, Q_v be posets and $\varphi_v : F(P_v) \to F(Q_v)$ be an embedding such that $\varphi_v(I^{bnd}(P_v)) \subseteq I^{bnd}(Q_v)$.*

Let $P = \sum\{P_v : v \in V\}$. For every $v \in V$ let Q_v^+ be as in Lemma 4.2(b), and let $R = \sum\{Q_v^+ : v \in V\}$. For every $v \in V$ denote m_{Q_v} by $m(v)$. We regard each $F(P_v)$ as a subalgebra of $F(P)$ and each $F(Q_v^+)$ as a subalgebra of $F(R)$. Let $Z = \{z_r : r \in R\}$ be the set of generators of $F(R)$, and for every $v \in V$ let $X_v = \{x_p : p \in P_v\}$ be the set of generators of P_v. For $v \in V$ define $\hat{\varphi}_v : X_v \to F(R)$ as follows:

$$\hat{\varphi}_v(x_p) = z_{m(v)} + \varphi_v(x_p),$$

and let

$$\varphi^+ = \bigcup_{v \in V} \hat{\varphi}_v.$$

Then φ^+ can be extended to an embedding ψ of $F(P)$ in $F(R)$, and $\psi(I^{bnd}(P)) \subseteq I^{bnd}(R)$.

Proof. Claim 1. Let $p \in P_v$ and $q \in P_w$. Suppose that $v < w$. Then $\varphi^+(x_p) < x_{m(w)} < \varphi^+(x_q)$.
Proof. Claim 1 follows trivially from Lemma 4.4.

We show that φ^+ can be extended to a homomorphism from $F(P)$ to $F(R)$. By Theorem 1.2 it suffices to show that if $p, q \in P$ and $p \leq q$, then $\varphi^+(x_p) \leq \varphi^+(x_q)$. Suppose that $p \in P_v$ and $q \in P_w$.
Case 1 $v = w$.
By Proposition 4.2(b), φ_v^+ can be extended to an embedding $\chi : F(P_v) \to F(Q_v^+)$. So

$$\varphi^+(x_p) \cdot \varphi^+(x_q) = \chi(x_p) \cdot \chi(x_q) = \chi(x_p \cdot x_q) = \chi(x_p) = \varphi^+(x_p).$$

Hence $\varphi^+(x_p) \leq \varphi^+(x_q)$.
Case 2 $v \neq w$.
Since $p \leq q$, $v < w$. By Claim 1,
$\varphi^+(x_p) < x_{m(w)} < \varphi^+(x_q)$.

So φ^+ can be extended to a homomorphism.

We prove that the homomorphism extending φ^+ is an embedding.

By Proposition 4.3, it suffices to show that:

$(\star)$ if $x_{\sigma,\tau}^{\varphi^+} = 0$, then $x_{\sigma,\tau} = 0$.

Let σ, τ be finite subsets of P. Denote $\rho^+(\sigma,\tau) = \{v \in V : \sigma \cap P_v \neq \emptyset\}$, $\rho^-(\sigma,\tau) = \{v \in V : \tau \cap P_v \neq \emptyset\}$ and $\rho(\sigma,\tau) = \rho^+(\sigma,\tau) \cup \rho^-(\sigma,\tau)$.

Claim 2. If $\rho^+(\sigma,\tau)$ and $\rho^-(\sigma,\tau)$ are antichains and $x_{\sigma,\tau}^{\varphi^+} = 0$, then $x_{\sigma,\tau} = 0$.

Proof. By induction on $|\rho(\sigma,\tau)|$. Let $n = |\rho(\sigma,\tau)|$. We show that the induction holds for $n = 1$. Suppose that $x_{\sigma,\tau}^{\varphi^+} = 0$ and that $|\rho(\sigma,\tau)| = 1$. Then there is $v \in V$ such that $\sigma \cup \tau \subseteq P_v$. By Proposition 4.2(b), there is an embedding $\chi : F(P_v) \to F(Q_v^+)$ which extends φ_v^+. Then
$0 = x_{\sigma,\tau}^{\varphi^+} = x_{\sigma,\tau}^{\varphi_v^+} = \chi(x_{\sigma,\tau})$. Since χ is an embedding, $x_{\sigma,\tau} = 0$.

Suppose that the induction hypothesis holds for n, and let σ, τ be such that $x_{\sigma,\tau}^{\varphi^+} = 0$, $\rho^+(\sigma,\tau)$ and $\rho^-(\sigma,\tau)$ are antichains and $|\rho(\sigma,\tau)| = n + 1$. Denote $\rho^+ = \rho^+(\sigma,\tau)$, $\rho^- = \rho^-(\sigma,\tau)$ and $\rho = \rho(\sigma,\tau)$.
Case 1 There is $v \in \rho^+$ and $w \in \rho^-$ such that $v < w$.
Let $p \in P_v \cap \sigma$ and $q \in P_w \cap \tau$. Then $p < q$. Hence $x_p \cdot -x_q = 0$. So $x_{\sigma,\tau} = 0$.

For the other cases we need some additional notations. For $u \in P$ let
$b_u^+ := b_u^+(\sigma,\tau) = \prod\{\varphi^+(x_p) : p \in P_u \cap \sigma\}$,
$c_u^+ := c_u^+(\sigma,\tau) = \prod\{\varphi^+(x_p) : p \in \sigma - P_u\}$,
$b_u^- := b_u^-(\sigma,\tau) = \prod\{-\varphi^+(x_q) : q \in P_u \cap \tau\}$,
$c_u^- := c_u^-(\sigma,\tau) = \prod\{-\varphi^+(x_q) : q \in \tau - P_u\}$.

Case 2 $\rho^+ \cap \rho^- \neq \emptyset$.

Let $u \in \rho^+ \cap \rho^-$. Then $0 = x_{\sigma,\tau}^{\varphi^+} = (b_u^+ \cdot b_u^-) \cdot (c_u^+ \cdot c_u^-)$. Since ρ^+ and ρ^- are antichains, u is incomparable with every member of $\rho - \{u\}$. Let $\eta = Q_u^+$ and $\zeta = \bigcup_{v \in \rho - \{u\}} Q_v^+$. Then for every $p \in \eta$ and $q \in \zeta$, p, q are incomparable. Also, $b_u^+ \cdot b_u^-$ belongs to the algebra generated by $\{z_r^+ : r \in \eta\}$, and $c_u^+ \cdot c_u^-$ belongs to the algebra generated by $\{z_r^+ : r \in \zeta\}$. By Proposition 2.5(c), either $b_u^+ \cdot b_u^- = 0$, or $c_u^+ \cdot c_u^- = 0$.

Suppose that $b_u^+ \cdot b_u^- = 0$. Let $\sigma' = \sigma \cap P_u$ and $\tau' = \tau \cap P_u$, then $b_u^+ \cdot b_u^- = x_{\sigma',\tau'}^{\varphi^+}$ and $\rho(\sigma',\tau') = \{u\}$. So by the induction claim for $n = 1$, $x_{\sigma',\tau'} = 0$. Obviously, $x_{\sigma,\tau} \leq x_{\sigma',\tau'} = 0$. Thus $x_{\sigma,\tau} = 0$.

Suppose next that $c_u^+ \cdot c_u^- = 0$. Let $\sigma' = \sigma - P_u$ and $\tau' = \tau - P_u$. Then $c_u^+ \cdot c_u^- = x_{\sigma',\tau'}^{\varphi^+}$, $\rho^+(\sigma',\tau') = \rho^+ - \{u\}$, and $\rho^-(\sigma',\tau') = \rho^- - \{u\}$. Clearly, $|\rho(\sigma',\tau')| = |\rho| - 1$. So by the induction hypothesis, $x_{\sigma',\tau'} = 0$. Obviously, $x_{\sigma,\tau} \leq x_{\sigma',\tau'} = 0$. Thus $x_{\sigma,\tau} = 0$.

Case 3 Cases 1 and 2 do not occur.

So

(1) For every $v \in \rho^+$ and $w \in \rho^-$, either $w < v$ or w and v are incomparable. We shall show that in this case $x_{\sigma,\tau}^{\varphi^+} \neq 0$. This will contradict our assumption. We shall thus conclude that Case 3 cannot happen.

We compute $x_{\sigma,\tau}^{\varphi^+}$. For $v \in \rho^+$ let $\sigma_v = \sigma \cap P_v$. For $w \in \rho^-$ let $\tau_w = \tau \cap P_w$.

Let $v \in \rho^+$. Then $x_{\sigma_v,\emptyset}^{\varphi^+} = \prod\{\varphi_v^+(x_p) : p \in \sigma_v\}$. So by Lemma 4.2($a$)(1), $x_{\sigma_v,\emptyset}^{\varphi^+}$ has the form $z_{m(v)} + d_v$, where $d_v \in I^{bnd}(Q_v^+)$. So

(2) $x_{\sigma,\emptyset}^{\varphi^+} = \prod\{x_{\sigma_v,\emptyset}^{\varphi^+} : v \in \rho^+\} = \prod\{x_{z(v)} + d_v : v \in \rho^+\} \geq \prod\{z_{m(v)} : v \in \rho^+\}$.

Let $w \in \rho^-$. Then

$$x_{\emptyset,\tau_w}^{\varphi^+} = \prod\{-\varphi_w^+(x_q) : q \in \tau_w\} = -\sum\{\varphi_w^+(x_q) : q \in \tau_w\}.$$

Then by Lemma 4.2(a)(2), $x_{\emptyset,\tau_w}^{\varphi^+}$ has the form $-(z_{m(w)} + e_w)$, where $e_w \in I^{bnd}(Q_w^+)$. So

(3) $$x_{\emptyset,\tau}^{\varphi^+} = \prod\{x_{\emptyset,\tau_w}^{\varphi^+} : w \in \rho^-\} = \prod\{-(z_{m(w)} + e_w) : w \in \rho^-\} = -(\sum\{z_{m(w)} + e_w : w \in \rho^-\}).$$

For every $w \in \rho^-$ there is a finite set $\eta_w \subseteq Q_w$ such that

(4) $$e_w \leq \sum\{z_q : q \in \eta_w\}.$$

By (2), (3) and (4),

(5) $$\begin{aligned} x_{\sigma,\tau}^{\varphi^+} = (x_{\sigma,\emptyset}^{\varphi^+}) \cdot (x_{\emptyset,\tau}^{\varphi^+}) &\geq \prod\{z_{m(v)} : v \in \rho^+\} - \sum\{z_{m(w)} + e_w : w \in \rho^-\} \\ &\geq \prod\{z_{m(v)} : v \in \rho^+\} - \sum\{z_q : w \in \rho^-,\ q \in \{m(w)\} \cup \eta_w\} \\ &= \prod\{z_{m(v)} : v \in \rho^+\} \cdot \prod\{-z_q : w \in \rho^-,\ q \in \{m(w)\} \cup \eta_w\}. \end{aligned}$$

For every $v \in \rho^+$ and $q \in \{m(w) : w \in \rho^-\} \cup \bigcup\{\eta_w : w \in \rho^-\}$, $m(v) \not\leq q$. So by Proposition 2.5(b), the last expression in (5) is different from 0. So $x^{\varphi^+}_{\sigma,\tau} \neq 0$. A contradiction, so Case 3 does not happen.

It follows that $x_{\sigma,\tau} = 0$. So Claim 2 is proved.

Claim 3. Let σ, τ be such that $x^{\varphi^+}_{\sigma,\tau} = 0$. Then there are $\sigma' \subseteq \sigma$ and $\tau' \subseteq \tau$ such that $x^{\varphi^+}_{\sigma',\tau'} = 0$ and $\rho^+(\sigma', \tau'), \rho^-(\sigma', \tau')$ are antichains.

Proof. Denote $\rho^+ = \rho^+(\sigma, \tau)$ and $\rho^- = \rho^-(\sigma, \tau)$. Let η^+ be the set of minimal elements of ρ^+, and η^- be the set of maximal elements of ρ^-. Let $\sigma' = \sigma \cap \bigcup\{P_v : v \in \eta^+\}$ and $\tau' = \tau \cap \bigcup\{P_v : v \in \eta^-\}$.

Clearly, $\rho^+(\sigma', \tau') = \eta^+$ and $\rho^-(\sigma', \tau') = \eta^-$. So $\rho^+(\sigma', \tau')$ and $\rho^-(\sigma', \tau')$ are antichains.

Let $p \in \sigma - \sigma'$. Let $v \in \rho^+$ be such that $p \in P_v$. There is $w \in \eta^+$ such that $w < v$. Let $q \in P_w \cap \sigma$. By the definition of σ', $q \in \sigma'$. By Claim 1 $\varphi^+(x_q) < \varphi^+(x_p)$. It follows that
$\prod\{\varphi^+(x_p) : p \in \sigma'\} \leq \prod\{\varphi^+(x_p) : p \in \sigma\}$. Since $\sigma' \subseteq \sigma$,
$\prod\{\varphi^+(x_p) : p \in \sigma'\} \geq \prod\{\varphi^+(x_p) : p \in \sigma\}$. So $x^{\varphi^+}_{\sigma',\emptyset} = x^{\varphi^+}_{\sigma,\emptyset}$.

An identical argument shows that $x^{\varphi^+}_{\emptyset,\eta'} = x^{\varphi^+}_{\emptyset,\eta}$. So $x^{\varphi^+}_{\sigma',\eta'} = x^{\varphi^+}_{\sigma,\eta} = 0$. We have proved Claim 3.

We now prove $(\star)$. Suppose that $x^{\varphi^+}_{\sigma,\tau} = 0$. Let σ', τ' be as assured by Claim 3. By Claim 2, $x_{\sigma',\tau'} = 0$. Since $\sigma' \subseteq \sigma$ and $\tau' \subseteq \tau$, $x_{\sigma,\tau} = 0$. We have proved $(\star)$.

We have shown that φ^+ fulfills the condition of Proposition 4.3. So the homomorphism ψ, which extends φ^+ is an embedding of $F(P)$ into $F(Q^+)$. It remains to show that $\psi(I^{bnd}(P)) \subseteq I^{bnd}(Q^+)$. Since $\{x_p : p \in P\}$ generates $I^{bnd}(P)$, it suffices to show that for every $p \in P$, $\psi(x_p) \in I^{bnd}(R)$. But this follows from the definition of φ^+. $\square$

Proof of Theorem 4.1. We prove by induction on α, that for every $P \in \mathcal{H}_\alpha$, $F(P)$ is well-generated. There is nothing to prove for $\alpha = 0$ and for limit ordinals. Suppose that the claim is true for α, and let $P \in \mathcal{H}_{\alpha+1} - \mathcal{H}_\alpha$.

By Proposition 2.5(a), $F(P^*) \cong F(P)$. It follows from the definition of $\mathcal{H}_\alpha$ that if $P \in \mathcal{H}_{\alpha+1} - \mathcal{H}_\alpha$, then $P^* \in \mathcal{H}_{\alpha+1} - \mathcal{H}_\alpha$.

Hence we may assume that $P = \sum\{P_v : v \in V\}$, where V is a well-ordered poset, and for every $v \in V$, $P_v \in \mathcal{H}_\alpha$. By the induction hypothesis, for every $v \in V$ there is a well-ordered poset Q_v and an embedding φ_v of $F(P_v)$ in $F(Q_v)$ such that $\varphi_v(I^{bnd}(P_v)) \subseteq I^{bnd}(Q_v)$.

We may apply Lemma 4.5 to V, $\{P_v : v \in V\}$, $\{Q_v : v \in V\}$ and $\{\varphi_v : v \in V\}$.

Let Q^+_v, $v \in V$ and ψ be as assured by that lemma and $R = \sum_{v \in V} Q^+_v$. Since Q^+_v is obtained from Q_v by adding only one element, Q^+_v is well-ordered. Recall that V is well-ordered.

It is easy to check that if in a lexicographic sum the index poset is well-ordered and each summand is well-ordered, then the sum is well-ordered. So R is well-ordered.

Finally, ψ is an embedding of $F(P)$ in $F(R)$, and $\psi(I^{bnd}(P)) \subseteq I^{bnd}(R)$. So R and ψ are as required. $\square$

Proof of Theorem 1.4. Let $\langle P, \leq\rangle$ be narrow and scattered. By Theorem 1.5 we may assume that there is $\leq' \subseteq \leq$ such that $\langle P, \leq'\rangle \in \mathcal{H}$. By Theorem 1.6, $F(\langle P, \leq'\rangle)$ is embeddable in $F(W)$, where W is a well-ordered poset, and obviously $F(\langle P, \leq\rangle)$ is a homomorphic image of $F(\langle P, \leq'\rangle)$. $\square$

Remark. The Boolean algebra $F(\omega^* \cdot \omega_1)$ is not a homomorphic image of a poset algebra of a well-ordered poset. This is proved in [ABK].

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