HYPERSPACES OF SEPARABLE BANACH SPACES WITH THE WIJSMAN TOPOLOGY

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Abstract. Let X be a separable metric space. By $\mathrm{Cld}_W(X)$, we denote the hyperspace of non-empty closed subsets of X with the Wijsman topology. Let $\mathrm{Fin}_W(X)$ and $\mathrm{Bdd}_W(X)$ be the subspaces of $\mathrm{Cld}_W(X)$ consisting of all non-empty finite sets and of all non-empty bounded closed sets, respectively. It is proved that if X is an infinite-dimensional separable Banach space then $\mathrm{Cld}_W(X)$ is homeomorphic to ($\approx$) the Hilbert space ℓ_2 and $\mathrm{Fin}_W(X) \approx \mathrm{Bdd}_W(X) \approx \ell_2 \times \ell_2^f$, where

$$\ell_2^f = \{(x_i)_{i\in\mathbb{N}} \in \ell_2 \mid x_i = 0 \text{ except for finitely many } i \in \mathbb{N}\}.$$

Moreover, we show that if the complement of any finite union of open balls in X has only finitely many path-components, all of which are closed in X, then $\mathrm{Fin}_W(X)$ and $\mathrm{Cld}_W(X)$ are ANR's. We also give a sufficient condition under which $\mathrm{Fin}_W(X)$ is homotopy dense in $\mathrm{Cld}_W(X)$.

1. Introduction

Let $\mathrm{Cld}(X)$ be the set of all non-empty closed sets in a topological space X. By $\mathrm{Cld}_V(X)$ we denote the space $\mathrm{Cld}(X)$ with the *Vietoris topology*, which is the most typical topology. The Curtis-Schori-West Hyperspace Theorem is a celebrated result in Infinite-Dimensional Topology which states that $\mathrm{Cld}_V(X)$ is homeomorphic to ($\approx$) the Hilbert cube $Q = [-1,1]^\omega$ if and only if X is a non-degenerate, connected and locally connected compact metrizable space ([8] and [21]; cf. [14, Theorem 8.4.5]). For a non-compact metric space X, since $\mathrm{Cld}_V(X)$ is non-metrizable, we have to consider topologies different from the Vietoris topology. For a study of hyperspace topologies, we refer to the book [4] (cf. [13]).

In the paper [17], it is shown that the space $\mathrm{Cld}_F(X)$ with the *Fell topology* is homeomorphic to $Q \setminus \{0\}$ if and only if X is a locally compact, locally connected separable metrizable space with no compact components. In [2], it is shown that if X is an infinite-dimensional Banach space with weight $w(X)$, then the space $\mathrm{Cld}_{AW}(X)$ with the *Attouch-Wets topology* is homeomorphic to a Hilbert space with weight $2^{w(X)}$. It should be remarked that $\mathrm{Cld}_{AW}(X) = \mathrm{Cld}_F(X)$ for a finite-dimensional normed linear space X (cf.

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[4, p.144]). In [11], the first author proved that if X is an infinite-dimensional separable Banach space, then the space $\mathrm{Cld}_W(X)$ with the *Wijsman topology* is an AR. In this paper, we prove the following:

Theorem I. *If X is an infinite-dimensional separable Banach space, then $\mathrm{Cld}_W(X)$ is homeomorphic to the separable Hilbert space ℓ_2.*

Let $\mathrm{Fin}(X) \subset \mathrm{Cld}(X)$ be the set of all non-empty finite sets. By $\mathrm{Fin}_V(X)$, $\mathrm{Fin}_W(X)$, etc., we respectively denote the subspaces of $\mathrm{Cld}_V(X)$, $\mathrm{Cld}_W(X)$, etc. In [7], it is proved that $\mathrm{Fin}_V(X) \approx \ell_2^f$ if and only if X is a non-degenerate, strongly countable-dimensional,[1] connected, locally path-connected, σ-compact metrizable space, where

$$\ell_2^f = \{(x_i)_{i\in\mathbb{N}} \in \ell_2 \mid x_i = 0 \text{ except for finitely many } i \in \mathbb{N}\}.$$

In [17], it is shown that $\mathrm{Fin}_F(X) \approx \ell_2^f$ if and only if X is a strongly countable-dimensional, locally compact, locally connected, separable metrizable space with no compact components. We also consider the subspace $\mathrm{Bdd}_W(X) \subset \mathrm{Cld}_W(X)$ consisting of all non-empty bounded closed sets. In [16], it is proved that if X is an infinite-dimensional Banach space with weight $w(X)$ then

$$\mathrm{Fin}_{AW}(X) \approx \ell_2(w(X)) \times \ell_2^f \ \text{ and } \ \mathrm{Bdd}_{AW}(X) \approx \ell_2(2^{w(X)}) \times \ell_2^f.$$

The following is also shown in this paper:

Theorem II. *If X is an infinite-dimensional separable Banach space, then*

$$\mathrm{Fin}_W(X) \approx \mathrm{Bdd}_W(X) \approx \ell_2 \times \ell_2^f.$$

It is said that $Y \subset X$ is *homotopy dense* in X if there is a homotopy $h : X \times \mathbf{I} \to X$ such that $h_0 = \mathrm{id}$ and $h_t(X) \subset Y$ for every $t > 0$. It is well-known that every homotopy dense set in an AR (resp. an ANR) is also an AR (resp. an ANR).[2] In the way of the proof of Theorem II, we show the following:

Theorem III. *Let X be a separable metric space. If the complement of any finite union of open balls in X has only finitely many path-components all of which are closed and unbounded, then $\mathrm{Cld}_W(X)$ is an AR and $\mathrm{Fin}_W(X)$ is homotopy dense in $\mathrm{Cld}_W(X)$. In particular, $\mathfrak{H}$ is an AR if $\mathrm{Fin}_W(X) \subset \mathfrak{H} \subset \mathrm{Cld}_W(X)$.*

¿From Theorem III above, it follows that $\mathrm{Cld}_W(X)$ is an AR and $\mathrm{Fin}_W(X)$ is homotopy dense in $\mathrm{Cld}_W(X)$ for every infinite-dimensional separable Banach space X. Thus, we have an alternative proof of the result in [11]. To prove Theorem III, we give a condition on a metrizable Lawson semilattice X and its subsemilattice Y under which X is an ANR and Y is homotopy dense in X (Theorem 5.1). See §5 for the definition of Lawson semilattice.

[1] It is said that X is *strongly countable-dimensional* if X is a countable union of finite-dimensional closed subsets.

[2] This fact follows from [9, Chapter IV, Theorem 6.3].

Furthermore, we prove the following:

Theorem IV. *Let X be a separable metric space. If the complement of any finite union of open balls in X has only finitely many path-components, all of which are closed in X, then* $\mathrm{Fin}_W(X)$ *and* $\mathrm{Cld}_W(X)$ *are ANR's.*

For a closed subspace $Y \subset X$, we have $\mathrm{Cld}(Y) \subset \mathrm{Cld}(X)$ as sets. Due to [2, Proposition 2.1], the Attouch-Wets topology on $\mathrm{Cld}(Y)$ coincides with the subspace topologies of the Attouch-Wets topology on $\mathrm{Cld}(X)$ as well as the Vietoris and the Fell topologies. However, the Wijsman topology on $\mathrm{Cld}(Y)$ does not necessarily coincide with the subspace topology inherited from the Wijsman topology on $\mathrm{Cld}(X)$ (see Example in Preliminaries). We also discuss the space $\mathrm{Cld}(Y)$ with the subspace topology inherited from $\mathrm{Cld}_W(X)$, which is called the *relative Wijsman topology*. Theorem IV above is proved in this setting.

2. Preliminaries

Let $X = (X, d)$ be a metric space. The open ball and the closed ball centered at x with radius ε are denoted by $B(x, \varepsilon)$ and $\overline{B}(x, \varepsilon)$, respectively. By $C(X)$, we denote the set of all continuous real-valued functions on X. In this paper, by a 'map', we mean a continuous function. By identifying each $A \in \mathrm{Cld}(X)$ with the map $X \ni x \mapsto d(x, A) \in \mathbb{R}$, we can regard $\mathrm{Cld}(X) \subset C(X)$, whence $\mathrm{Cld}(X)$ has various topologies inherited from $C(X)$. The *Wijsman topology* on $\mathrm{Cld}(X)$ is the topology of point-wise convergence, which depends on the metric d for X. For each $x \in X$ and $r > 0$, we define

$$U^-(x, r) = \{A \in \mathrm{Cld}(X) \mid d(x, A) < r\};$$
$$U^+(x, r) = \{A \in \mathrm{Cld}(X) \mid d(x, A) > r\}.$$ [3]

It is easy to see that these are open sets in $\mathrm{Cld}_W(X)$ which form an open subbasis for $\mathrm{Cld}_W(X)$. Moreover, to generate the Wijsman topology, it suffices to take points x in a dense subset of X.

For each $k \in \mathbb{N}$, we denote

$$\mathrm{Fin}^k(X) = \{A \in \mathrm{Fin}(X) \mid \mathrm{card}\, A \leqslant k\}.$$

It is easily observed that X is homeomorphic to the subspace $\mathrm{Fin}^1_W(X)$ of $\mathrm{Cld}_W(X)$. Then, we can regard $X = \mathrm{Fin}^1_W(X) \subset \mathrm{Cld}_W(X)$.

It is well-known that $\mathrm{Cld}_W(X)$ is metrizable if and only if X is separable, whence we can define an admissible metric d_W by using a countable dense set $\{x_i \mid\in i \in \mathbb{N}\}$ in X as follows:

$$d_W(A, B) = \sup_{i \in \mathbb{N}} \min\{2^{-i},\ |d(x_i, A) - d(x_i, B)|\}.$$

[3]Although $d(x, A) < r \Leftrightarrow B(x, r) \cap A \neq \emptyset$ and $d(x, A) > r \Rightarrow \overline{B}(x, r) \cap A = \emptyset$, it should be noticed that $\overline{B}(x, r) \cap A = \emptyset \not\Rightarrow d(x, A) > r$. In fact, let $A = \bigcup_{n \in \mathbb{N}} [1/n, \infty) e_n \in \mathrm{Cld}(\ell_2)$, where $\{e_n \mid n \in \mathbb{N}\}$ is the canonical orthonormal basis for ℓ_2. Then, $\overline{B}(0, 1) \cap A = \emptyset$ but $d(0, A) = 1$.

If d is complete then d_W is complete [4, Theorem 2.5.4].[4] Thus,

> *the space* $\mathrm{Cld}_W(X)$ *is completely metrizable for every separable complete metric space* X.

For a closed subspace $Y \subset X$, the Wijsman topology on $\mathrm{Cld}(Y)$ is defined by using the metric $d_Y = d|Y^2$ inherited from X, and the space $\mathrm{Cld}_W(Y)$ admits this topology. On the other hand, as mentioned in Introduction, $\mathrm{Cld}(Y)$ has the subspace topology inherited from $\mathrm{Cld}_W(X)$, called the relative Wijsman topology. The following example shows that $\mathrm{Fin}^2_W(Y)$ is not the subspace of $\mathrm{Fin}^2_W(X)$.

Example. Let $X = \ell_2$ be the Hilbert space, $Y = \{x \in \ell_2 \mid \|x\| \leqslant 1\}$ the unit closed ball of X and $\{e_n \mid n \in \mathbb{N}\}$ the standard orthonormal base of X. Fix $\delta > 0$ and define

$$a_n = \frac{1}{\sqrt{1+\delta^2}}(e_{n+1} + \delta e_1),\ n \in \mathbb{N}.$$

Then, we have $A_n = \{0, a_n\} \in \mathrm{Fin}^2(Y)$. To see that $A_n \not\to \{0\}$ in $\mathrm{Cld}_W(X)$, consider $v = te_1 \in X\ (= \ell_2)$, where

$$t > \frac{\sqrt{1+\delta^2}}{2\delta}.$$

For each $n \in \mathbb{N}$, we have

$$\begin{aligned}\|v - a_n\|^2 &= \left\| \left(t - \frac{\delta}{\sqrt{1+\delta^2}}\right) e_1 - \frac{1}{\sqrt{1+\delta^2}} e_{n+1} \right\|^2 \\ &= t^2 - \frac{2t\delta}{\sqrt{1+\delta^2}} + 1 < t^2 = \|v\|^2,\end{aligned}$$

hence $d(v, A_n) = \|v - a_n\| \not\to \|v\| = d(v, \{0\})$.

Now, we should find $\delta > 0$ such that $A_n \to \{0\}$ in $\mathrm{Cld}_W(Y)$. In order to compute the upper estimate for δ, consider

$$x = (x_1, \dots, x_k, 0, 0, \dots) \in Y \cap \ell_2^f \subset X = \ell_2.$$

For each $n \geqslant k$,

$$\|x - a_n\|^2 = \left(x_1 - \frac{\delta}{\sqrt{1+\delta^2}}\right)^2 + \sum_{i=2}^k x_i^2 + \frac{1}{1+\delta^2}$$

$$\text{and}\quad \|x\|^2 = \sum_{i=1}^k x_i^2.$$

[4] In [4], the following metric is used but it is uniformly equivalent to ours:

$$d_W(A, B) = \sum_{i \in \mathbb{N}} 2^{-i} \min\{1,\ |d(x_i, A) - d(x_i, B)|\}.$$

Then, $d(x, A_n) \to d(x, \{0\})$ if and only if $\|x - a_n\|^2 \geqslant \|x\|^2$ for $n \geqslant k$. Since $|x_1| \leqslant \|x\| \leqslant 1$, it follows that

$$\begin{aligned}\|x - a_n\|^2 - \|x\|^2 &= \left(x_1 - \frac{\delta}{\sqrt{1+\delta^2}}\right)^2 + \frac{1}{1+\delta^2} - x_1^2 \\ &= 1 - \frac{2x_1\delta}{\sqrt{1+\delta^2}} \geqslant 1 - \frac{2\delta}{\sqrt{1+\delta^2}}.\end{aligned}$$

Choose $0 < \delta \leqslant 1/\sqrt{3}$. Then, $\|x - a_n\|^2 \geqslant \|x\|^2$, hence $d(x, A_n) \to d(x, \{0\})$ for every $x \in Y \cap \ell_2^f$. Since $Y \cap \ell_2^f$ is dense in Y, it follows that $A_n \to \{0\}$ in $\mathrm{Cld}_W(Y)$. □

To prove Theorems I and II, we need characterizations of ℓ_2 and $\ell_2 \times \ell_2^f$. The following characterization of ℓ_2 is due to Toruńczyk [19] (cf. [20]):

Theorem 2.1. *In order that $X \approx \ell_2$, it is necessary and sufficient that X is a separable completely metrizable AR which has the discrete approximation property, that is, each map $f : \bigoplus_{n\in\mathbb{N}} \mathbf{I}^n \to X$ is approximated by maps $g : \bigoplus_{n\in\mathbb{N}} \mathbf{I}^n \to X$ such that $\{g(\mathbf{I}^n) \mid n \in \mathbb{N}\}$ is discrete in X.* □

To state the characterization of $\ell_2 \times \ell_2^f$ due to Bestvina and Mogilski [5], we need some notions. A metrizable space X is *σ-completely metrizable* if X is a countable union of completely metrizable closed subsets. A closed set $A \subset X$ is a (*strong*) *Z-set* in X if there are maps $f : X \to X \setminus A$ arbitrarily close to id (such that $A \cap \mathrm{cl}\, f(X) = \emptyset$). A countable union of (strong) Z-sets is called a (*strong*) *Z_σ-set*. When X itself is a (strong) Z_σ-set in X, we call X a (*strong*) *Z_σ-space*. For a class $\mathcal{C}$ of spaces, X is *strongly universal* for $\mathcal{C}$ if given a map $f : A \to X$ of $A \in \mathcal{C}$ such that $f|B$ is a Z-embedding of a closed set $B \subset A$, there exist Z-embeddings $g : A \to X$ such that $g|B = f|B$ and which are arbitrarily close to f. In these definitions, the phrase 'arbitrarily close' is understood with respect to the *limitation topology*. In case $X = (X, d)$ is a metric space, given a collection M of maps from a space Y to X, a map $f : Y \to X$ is *arbitrarily close* to maps in M if for each $\alpha : X \to (0, 1)$ there is $g \in M$ such that $d(f(y), g(y)) < \alpha(f(y))$ for every $y \in Y$. The following is Corollary 6.3 in [5].

Theorem 2.2. *In order that $X \approx \ell_2 \times \ell_2^f$, it is necessary and sufficient that X is a separable σ-completely metrizable AR which is a strong Z_σ-space and it is strongly universal for separable completely metrizable spaces.* □

In case X is a homotopy dense set in ℓ_2, X has the discrete approximation property (cf. [3, Theorem 1.3.2] or [1]), hence every Z-set in X is a strong Z-set by Proposition 1.7 in [5]. Then, we have the following:

Theorem 2.3. *For a homotopy dense set X in ℓ_2, if X is a σ-completely metrizable Z_σ-space and it is strongly universal for separable completely metrizable spaces, then $X \approx \ell_2 \times \ell_2^f$.* □

3. Proof of Theorem I

It should be remarked that $\mathrm{Cld}_W(X)$ is separable (completely) metrizable if and only if X is separable (completely) metrizable. In [11], it has been proved that $\mathrm{Cld}_W(X)$ is an AR for an infinite-dimensional separable Banach space X. Thus, it remains to verify the discrete approximation property.

Let $X = (X, \|\cdot\|)$ be a normed linear space and d the metric induced by the norm $\|\cdot\|$ (i.e., $d(x,y) = \|x-y\|$). By $\mathbf{B}_X$ and $\mathbf{S}_X$, we denote the unit closed ball and the unit sphere of X, respectively. For $\mathrm{Cld}_W(X)$, the metric d_W is defined by a countable dense set $\{x_i \mid i \in \mathbb{N}\}$ in X, where $x_1 = 0$.

To prove Theorem I, we need the following lemma. Since it will be also used in the proof of Theorem II, it is formulated in a general setting.

Lemma 3.1. *Let $\mathfrak{H}$ be a subspace of $\mathrm{Cld}_W(X)$ and W an open set in $\mathfrak{H}$. For each map $\alpha : \mathfrak{H} \to (0,1)$, there exists a map $\gamma : W \to (0,\infty)$ such that*

$$(*)\quad A \in W,\ A' \in \mathfrak{H},\ A \cap \gamma(A)\mathbf{B}_X = A' \cap \gamma(A)\mathbf{B}_X \\ \Rightarrow A' \in W,\ d_W(A, A') < \alpha(A).$$

Proof. For each $A \in W$, we define

$$i(A) = \min\{i \in \mathbb{N} \mid 2^{-i} < \alpha(A) \text{ and } 2^{-i} < d_W(A, \mathfrak{H} \setminus W)\} \text{ and} \\ r(A) = \max\{\|x_i\| \mid i \leqslant i(A)\}.$$

Then, $r : \mathfrak{H} \to [0,\infty)$ is upper semi-continuous. Indeed, let $A \in \mathfrak{H}$ and $t > 0$ such that $r(A) < t$. Since α is continuous, we can choose $\delta > 0$ so that if $A' \in \mathfrak{H}$ and $d_W(A, A') < \delta$ then $2^{-i(A)} < \alpha(A')$, whence $i(A') \leqslant i(A)$ by the definition of $i(A')$, which implies that $r(A') \leqslant r(A) < t$ by the definition of $r(A')$. Thus, there exists a map $\gamma : \mathfrak{H} \to (0,\infty)$ such that

$$\gamma(A) > 3\max\{r(A),\ d(0,A)\} \text{ for each } A \in \mathfrak{H}.$$

To see that γ satisfies condition (*), suppose

$$A \in W,\ A' \in \mathfrak{H} \text{ and } A \cap \gamma(A)\mathbf{B}_X = A' \cap \gamma(A)\mathbf{B}_X.$$

Note that $A \cap \gamma(A)\mathbf{B}_X \neq \emptyset$ because $d(0,A) < \frac{1}{3}\gamma(A)$. For each $i \leqslant i(A)$, since $\|x_i\| \leqslant r(A) < \frac{1}{3}\gamma(A)$, it follows that

$$d(x_i, A) = d(x_i, A \cap \gamma(A)\mathbf{B}_X) \\ = d(x_i, A' \cap \gamma(A)\mathbf{B}_X) = d(x_i, A').$$

Then, $d_W(A, A') < 2^{-i(A)} < \alpha(A)$. Since $2^{-i(A)} < d_W(A, \mathfrak{H} \setminus W)$, we have also $A' \in W$. □

Theorem 3.2. *For every infinite-dimensional separable Banach space X, $\mathrm{Cld}_W(X)$ has the discrete approximation property. Hence, $\mathrm{Cld}_W(X) \approx \ell_2$.*

Proof. For each map $\alpha : \mathrm{Cld}_W(X) \to (0,1)$, let $\gamma : \mathrm{Cld}_W(X) \to (0,\infty)$ be the map obtained by Lemma 3.1. On the other hand, $\mathbf{S}_X$ has a countable-infinite $\frac{1}{2}$-discrete set $\{e_n \mid n \in \mathbb{N}\}$ (cf. [2]). Taking $v \in \mathbf{S}_X$, we define

$f : \mathrm{Cld}_W(X) \times \mathbb{N} \to \mathrm{Cld}_W(X)$ as follows:

$$f(A,n) = (A \cap \gamma(A)\mathbf{B}_X) \cup \gamma(A)\mathbf{S}_X \cup \{(\gamma(A)+2)v + e_n\}.$$

Then, $d_W(f(A,n),A) < \alpha(A)$ for every $A \in \mathrm{Cld}_W(X)$ and $n \in \mathbb{N}$, by (*) in Lemma 3.1.

To verify the continuity of f, observe that

$$d(x, f(A,n)) = \begin{cases} \min\{\gamma(A) - \|x\|,\ d(x,A)\} & \text{if } \|x\| \leqslant \gamma(A), \\ \min\{\|x\| - \gamma(A),\ \|(\gamma(A)+2)v + e_n - x\|\} & \text{if } \|x\| \geqslant \gamma(A). \end{cases}$$

For each $x \in X$, $(A,n) \mapsto d(x, f(A,n))$ is continuous, which implies that f is continuous.

It remains to show that

$$\{f(\mathrm{Cld}_W(X) \times \{n\}) \mid n \in \mathbb{N}\}$$

is discrete in $\mathrm{Cld}_W(X)$. On the contrary, assume that there exist $A_j \in \mathrm{Cld}_W(X)$, $j \in \mathbb{N}$, and $n_1 < n_2 < \cdots \in \mathbb{N}$ such that $(f(A_j,n_j))_{j\in\mathbb{N}}$ converges to $A \in \mathrm{Cld}_W(X)$.

In case $\sup_{j\in\mathbb{N}} \gamma(A_j) = \infty$, we may assume that $\gamma(A_j) \to \infty$ as $j \to \infty$. Observe that $d_W(f(A_j,n_j),A_j) \to 0$. Then, $A_j \to A$, hence $\gamma(A_j) \to \gamma(A)$ by the continuity of γ. This is a contradiction.

When $\sup_{j\in\mathbb{N}} \gamma(A_j) < \infty$, it can be assumed that $\lim_{j\to\infty} \gamma(A_j) = r$. We show that $A \subset r\mathbf{B}_X$. Suppose that $\|a\| > r$ for some $a \in A$ and choose $i_0 \leqslant j_0 \in \mathbb{N}$ so that $d(x_{i_0}, a) < s/6$ and

$$j \geqslant j_0 \Rightarrow |\gamma(A_j) - r| < s/2,\ d_W(f(A_j,n_j),A) < \min\{s/6,\ 2^{-i_0}\},$$

where $s = \min\{\|a\| - r,\ 1/4\}$. Then, for every $j \geqslant j_0$,

$$\begin{aligned} d(a, f(A_j,n_j)) &\leqslant d(a, x_{i_0}) + d(x_{i_0}, A) + |d(x_{i_0}, f(A_j,n_j)) - d(x_{i_0}, A)| \\ &< s/6 + s/6 + s/6 = s/2. \end{aligned}$$

On the other hand,

$$d(a, \gamma(A_j)\mathbf{B}_X) \geqslant \|a\| - \gamma(A_j) > \|a\| - r - s/2 \geqslant s/2.$$

Therefore, $d(a, (\gamma(A_j)+2)v + e_{n_j}) < s/2$. Then, we have

$$\begin{aligned} d(a, (r+2)v + e_{n_j}) &\leqslant d(a, (\gamma(A_j)+2)v + e_{n_j}) \\ &\quad + d((\gamma(A_j)+2)v + e_{n_j}, (r+2)v + e_{n_j}) \\ &< s/2 + |\gamma(A_j) - r| < s. \end{aligned}$$

Then, for $j \neq j' \geqslant j_0$,

$$\begin{aligned} 1/2 < d(e_{n_j}, e_{n_{j'}}) &= d((r+2)v + e_{n_j}, (r+2)v + e_{n_{j'}}) \\ &\leqslant d(a, (r+2)v + e_{n_j}) + d(a, (r+2)v + e_{n_{j'}}) < 2s \leqslant 1/2, \end{aligned}$$

which is a contradiction. Therefore, $A \subset r\mathbf{B}_X$.

Now, choose $i_1 \in \mathbb{N}$ so that $d(x_{i_1}, (r+2)v) < 1/4$, whence

$$\begin{aligned} d(x_{i_1}, A) &\geqslant d(x_{i_1}, r\mathbf{B}_X) \\ &\geqslant d((r+2)v, r\mathbf{B}_X) - d(x_{i_1}, (r+2)v) \\ &> 2 - 1/4 = 7/4. \end{aligned}$$

Moreover, choose $j_1 \in \mathbb{N}$ so that if $j \geqslant j_1$ then $|\gamma(A_{j_1}) - r| < 1/2$, whence

$$\begin{aligned} d(x_{i_1}, f(A_j, n_j)) &\leqslant d(x_{i_1}, (\gamma(A_j)+2)v) \\ &\leqslant d(x_{i_1}, (r+2)v) + d((r+2)v, (\gamma(A_j)+2)v) \\ &< 1/4 + |\gamma(A_j) - r| < 3/4. \end{aligned}$$

Then, for every $j \geqslant j_1$,

$$\begin{aligned} d_W(f(A_j, n_j)), A) &\geqslant \min\{2^{-i_1}, |d(x_{i_1}, f(A_j, n_j)) - d(x_{i_1}, A)|\} \\ &\geqslant \min\{2^{-i_1},\ 7/4 - 3/4\} = 2^{-i_1}, \end{aligned}$$

which contradicts to $f(A_j, n_j) \to A$. Consequently,

$$\{f(\mathrm{Cld}_W(X) \times \{n\}) \mid n \in \mathbb{N}\}$$

is discrete in $\mathrm{Cld}_W(X)$. □

4. Proof of Theorem II for $\mathrm{Bdd}_W(X)$

By Theorem I, we can apply Theorem 2.3 to prove that $\mathrm{Bdd}_W(X) \approx \ell_2 \times \ell_2^f$. We first show the following:

Lemma 4.1. *For every separable normed linear space X, $\mathrm{Bdd}_W(X)$ is homotopy dense in $\mathrm{Cld}_W(X)$.*

Proof. We define $\theta : \mathrm{Cld}_W(X) \times \mathbf{I} \to \mathrm{Cld}_W(X)$ by $\theta_0 = \mathrm{id}$ and

$$\theta(A, t) = \big(A \cap t^{-1}\mathbf{B}_X\big) \cup t^{-1}\mathbf{S}_X.$$[5]

To verify the continuity of θ, observe that

$$d(x, \theta(A,t)) = \begin{cases} \min\{t^{-1} - \|x\|,\ d(x, A)\} & \text{if } \|x\| \leqslant t^{-1}, \\ \|x\| - t^{-1} & \text{if } \|x\| > t^{-1}. \end{cases}$$

For each $x \in X$, $(A, t) \mapsto d(x, \theta(A, t))$ is continuous, which implies that θ is continuous. □

Notice $\mathrm{Bdd}_W(X) = \bigcup_{k\in\mathbb{N}} \mathrm{Cld}(k\mathbf{B}_X)$.

Lemma 4.2. *For every separable normed linear space X, each $\mathrm{Cld}(k\mathbf{B}_X)$ is closed in $\mathrm{Cld}_W(X)$. Thus, if X is a separable Banach space, then each $\mathrm{Cld}(k\mathbf{B}_X)$ is completely metrizable, hence $\mathrm{Bdd}_W(X)$ is σ-completely metrizable.*

[5] As compared with the contraction θ in [2], the parameter is different. Here, it is not necessary that θ_1 is the constant.

Proof. For each $A \in \mathrm{Cld}_W(X) \setminus \mathrm{Cld}(k\mathbf{B}_X)$, we have $a \in A \setminus k\mathbf{B}_X$. Let $r = \|a\| - k = d(a, k\mathbf{B}_X) > 0$. Then,

$$A \in U^-(a, r) \subset \mathrm{Cld}_W(X) \setminus \mathrm{Cld}(k\mathbf{B}_X).$$

Therefore, $\mathrm{Cld}_W(X) \setminus \mathrm{Cld}(k\mathbf{B}_X)$ is open in $\mathrm{Cld}_W(X)$, that is, $\mathrm{Cld}(k\mathbf{B}_X)$ is closed in $\mathrm{Cld}_W(X)$. □

Lemma 4.3. *For every separable normed linear space X, each $\mathrm{Cld}(k\mathbf{B}_X)$ is a Z-set in $\mathrm{Bdd}_W(X)$, hence $\mathrm{Bdd}_W(X)$ is a Z_σ-space.*

Proof. Let $\theta : \mathrm{Cld}_W(X) \times \mathbf{I} \to \mathrm{Cld}_W(X)$ be the homotopy defined in the proof of Lemma 4.1. Then, $\theta(\mathrm{Bdd}_W(X) \times \mathbf{I}) \subset \mathrm{Bdd}_W(X)$. For each map $\alpha : \mathrm{Bdd}_W(X) \to (0, 1)$, define $\xi : \mathrm{Bdd}_W(X) \to (0, 1)$ as follows:

$$\xi(A) = \sup\{t > 0 \mid \mathrm{diam}_{d_W} \theta(\{A\} \times [0, t]) < \alpha(A)\}.$$

Then, ξ is lower semi-continuous. Indeed, if $\xi(A) > s$ then we have $s < s' < \xi(A)$ and

$$\varepsilon = \alpha(A) - \mathrm{diam}_{d_W} \theta(\{A\} \times [0, s']) > 0.$$

By the continuity of θ and α, we have $\delta > 0$ such that if $A' \in \mathrm{Fin}_W(X)$ and $d_W(A, A') < \delta$ then $|\alpha(A) - \alpha(A')| < \frac{1}{3}\varepsilon$ and $d_W(\theta(A, t), \theta(A', t)) < \frac{1}{3}\varepsilon$ for all $t \in [0, s']$, whence

$$\begin{aligned}\mathrm{diam}_{d_W} \theta(\{A'\} \times [0, s']) &< \mathrm{diam}_{d_W} \theta(\{A\} \times [0, s']) + \tfrac{2}{3}\varepsilon \\ &= \alpha(A) - \tfrac{1}{3}\varepsilon < \alpha(A'),\end{aligned}$$

which means that $\xi(A') \geqslant s' > s$.

Thus, we have a map $f : \mathrm{Bdd}_W(X) \to \mathrm{Bdd}_W(X)$ defined by

$$f(A) = \theta(A, \beta(A)) = A \cup \beta(A)^{-1}\mathbf{S}_X,$$

where $\beta : \mathrm{Bdd}_W(X) \to (0, 1)$ is a map such that

$$\beta(A) < \min\{(k + 1)^{-1},\ \xi(A)\}.$$

Observe that $d_W(f(A), A) < \alpha(A)$ and $\beta(A)^{-1} > k + 1$ for each $A \in \mathrm{Bdd}_W(X)$. Then, it follows that $\mathrm{Cld}(k\mathbf{B}_X) \cap f(\mathrm{Bdd}_W(X)) = \emptyset$. Thus, $\mathrm{Cld}(k\mathbf{B}_X)$ is a Z-set in $\mathrm{Bdd}_W(X)$. □

It remains to prove the strong universality of $\mathrm{Bdd}_W(X)$. To this end, we use the following fact:

Proposition 4.4. *For every infinite-dimensional separable Banach space X, the unit sphere $\mathbf{S}_X$ is homeomorphic to ℓ_2.*

Proof. This is well-known but we give a proof for readers' convenience.

First note that $X \approx \ell_2$ (cf. [19, §6]). As an open set in $X \approx \ell_2$, $(1, 2)\mathbf{S}_X$ is an ℓ_2-manifold. It is easy to see that $\mathbf{S}_X \cup 2\mathbf{S}_X$ is a strong Z-set in $[1, 2]\mathbf{S}_X$. Then, $[1, 2]\mathbf{S}_X$ is an ℓ_2-manifold by [20, Theorem B1] (cf. [18, Theorem 5.2]). By using [19, Theorem 4.1], we can see that $\mathbf{S}_X$ is also an ℓ_2-manifold. On the other hand, $\mathbf{S}_X$ is contractible because $\mathbf{S}_X$ is a retract of $X \setminus \{0\}$ and $X \setminus \{0\} \approx \ell_2 \setminus \{0\}$ is homotopically trivial. Therefore, $\mathbf{S}_X \approx \ell_2$. □

Theorem 4.5. *For every infinite-dimensional separable Banach space X, $\mathrm{Bdd}_W(X)$ is strongly universal for separable completely metrizable spaces. Consequently,* $\mathrm{Bdd}_W(X) \approx \ell_2 \times \ell_2^f$.

Proof. First of all, note that the class of separable completely metrizable spaces is hereditary with respect to both closed subsets and open subsets. Due to the remark before Theorem 2.3, it follows from Lemma 4.1 that every Z-set in $\mathrm{Bdd}_W(X)$ is a strong Z-set. Then, by Proposition 2.2 in [5], it suffices to prove that each open set $W \subset \mathrm{Bdd}_W(X)$ is universal for separable completely metrizable spaces, that is, for every separable completely metrizable space Y, each map $f : Y \to W$ can be approximated by Z-embeddings.

For each map $\alpha : W \to (0,1)$, we apply Lemma 3.1 to obtain a map $\beta : W \to (0,1)$ such that

$$(*) \quad A \in W,\ A' \in \mathrm{Bdd}(X),\ A \cap \beta(A)^{-1}\mathbf{B}_X = A' \cap \beta(A)^{-1}\mathbf{B}_X \Rightarrow A' \in W,\ d_W(A, A') < \alpha(A).$$

Since the Hilbert space ℓ_2 is universal for separable completely metrizable spaces, we have a closed embedding $h : Y \to \mathbf{S}_X$ by Proposition 4.4. By taking any $v \in \mathbf{S}_X$ and using the homotopy $\theta : \mathrm{Cld}_W(X) \times \mathbf{I} \to \mathrm{Cld}_W(X)$ defined in the proof of Lemma 4.1, we can define a map $g : Y \to W$ by

$$g(y) = \theta(f(y), \beta(f(y))) \cup \{(\beta(f(y))^{-1} + 2)v + h(y)\}.$$

Then, it follows from (*) that $d_W(f(y), g(y)) < \alpha(f(y))$ for every $y \in Y$. As is easily observed, g is injective.

To see that g is a closed embedding, let $y_i \in Y$, $i \in \mathbb{N}$, such that $g(y_i) \to A \in W$. Then, $b = \inf_{i\in\mathbb{N}} \beta(f(y_i)) > 0$. Otherwise, by taking a subsequence, it can be assumed that $\beta(f(y_i)) \to 0$, that is, $\beta(f(y_i))^{-1} \to \infty$, whence $d_W(f(y_i), g(y_i)) \to 0$, so $f(y_i) \to A$. Hence, $\beta(f(y_i)) \to \beta(A) \neq 0$, which is a contradiction. Furthermore, $A \cap ((b^{-1} + 2)v + \frac{3}{2}\mathbf{B}_X) \neq \emptyset$. In fact, if $A \in U^+((b^{-1} + 2)v, \frac{3}{2})$ then $g(y_i) \in U^+((b^{-1} + 2)v, \frac{3}{2})$ for sufficiently large $i \in \mathbb{N}$. On the other hand, for sufficiently large $i \in \mathbb{N}$, $|\beta(f(y_i))^{-1} - b^{-1}| < \frac{1}{2}$, whence

$$\|(\beta(f(y_i))^{-1} + 2)v + h(y_i) - (b^{-1} + 2)v\| \leqslant |\beta(f(y_i))^{-1} - b^{-1}| + 1 < \tfrac{3}{2}.$$

This is a contradiction.

Now, let $c \in A \cap ((b^{-1} + 2)v + \frac{3}{2}\mathbf{B}_X)$. For each $0 < \varepsilon < \frac{1}{4}$, we can choose $i_0 \in \mathbb{N}$ so that if $i \geqslant i_0$ then $g(y_i) \in U^-(c, \varepsilon)$ and $|\beta(f(y_i))^{-1} - b^{-1}| < \varepsilon < \frac{1}{4}$, whence

$$\|(\beta(f(y_i))^{-1} + 2)v + h(y_i) - c\| = \|g(y_i) - c\| < \varepsilon$$

because $\|c\| \geqslant b^{-1} + \frac{1}{2} > \beta(f(y_i))^{-1} + \frac{1}{4}$. Thus, $(\beta(f(y_i))^{-1} + 2)v + h(y_i) \to c$. Since $\beta(f(y_i))^{-1} \to b^{-1}$, it follows that $h(y_i) \to c - (b^{-1} + 2)v$. Therefore, $(y_i)_{i\in\mathbb{N}}$ is convergent in Y. Hence, g is a closed embedding.

By Lemma 4.3, $g(Y)$ is a Z_σ-set in W. Since $g(Y) \approx Y$ is completely metrizable, $g(Y)$ is a Z-set in W by [6, Lemma 2.4]. Thus, $g : Y \to W$ is a Z-embedding. The proof is completed. □

5. Homotopy Dense Subsemilattices of Lawson Semilattices

A *topological semilattice* is a topological space S equipped with a continuous operator $\vee : S \times S \to S$ which is reflexive, commutative and associative (i.e., $x \vee x = x$, $x \vee y = y \vee x$, $(x \vee y) \vee z = x \vee (y \vee z)$). A topological semilattice S is called a *Lawson semilattice* if S admits an open basis consisting of subsemilattices [12].

In [2], it is shown that a metrizable Lawson semilattice X is an ANR (resp. an AR) if and only if it is locally path-connected (resp. connected and locally path-connected). Here, we introduce a relative version of local path-connectedness. A subset $Y \subset X$ is *relatively* LC^0 in X if for every $x \in X$, each neighborhood U of x in X contains a smaller neighborhood V of x such that every two points of $V \cap Y$ can be joined by a path in $U \cap X$. In this section, we shall prove the following theorem.

Theorem 5.1. *Let X be a metrizable Lawson semilattice with $Y \subset X$ a dense subsemilattice. If Y is relatively LC^0 in X (and Y is path-connected), then X is an ANR (an AR) and Y is homotopy dense in X, hence Y is also an ANR (an AR).*

To prove Theorem 5.1 above, we will use the following result in [15]:

Theorem 5.2. *Let X be a metric space with $Y \subset X$ a dense set. Assume that there exist a zero-sequence $\mathcal{U} = \{\mathcal{U}_n \mid n \in \mathbb{N}\}$ of open covers of X and a map $f : |TN(\mathcal{U})| \to Y$ such that $f(U) \in U$ for $U \in \bigcup_{n\in\mathbb{N}} \mathcal{U}_n = TN(\mathcal{U})^{(0)}$ and*

$$\lim_{n\to\infty} \operatorname{mesh}\{f(\sigma) \mid \sigma \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1})\} = 0.$$

Then, X is an ANR and Y is homotopy dense in X. □

Here, $\mathcal{U}$ is called a *zero-sequence* if $\lim_{n\to\infty} \operatorname{mesh} \mathcal{U}_n = 0$, and $TN(\mathcal{U})$ is a simplicial complex defined as the union $\bigcup_{n\in\mathbb{N}} N(\mathcal{U}_n \cup \mathcal{U}_{n+1})$ of the nerves of the covers $\mathcal{U}_n \cup \mathcal{U}_{n+1}$, where we regard $\mathcal{U}_n \cap \mathcal{U}_{n+1} = \emptyset$.

Lemma 5.3. *Every Lawson semilattice X is k-aspherical for any $k > 0$, that is, each map $f : \mathbf{S}^k \to X$ extends to $\tilde{f} : \mathbf{B}^{k+1} \to X$.*

Proof. Identify $\mathbf{S}^k = \operatorname{Fin}^1_V(\mathbf{S}^k) \subset \operatorname{Fin}_V(\mathbf{S}^k)$ and $X = \operatorname{Fin}^1_V(X) \subset \operatorname{Fin}_V(X)$. Then, f has the extension $f_V : \operatorname{Fin}_V(\mathbf{S}^k) \to \operatorname{Fin}_V(X)$. Since $\operatorname{Fin}_V(X)$ is a free Lawson semilattice over X, there exists a retraction $r : \operatorname{Fin}_V(X) \to X$ (see [2]). By [7], we have $\varphi : \mathbf{B}^{k+1} \to \operatorname{Fin}^3_V(\mathbf{S}^k)$ with $\varphi|\mathbf{S}^k = \mathrm{id}$. Then, $\tilde{f} = r f_V \varphi : \mathbf{B}^{k+1} \to X$ is an extension of f. □

Proof of Theorem 5.1. Fix an admissible metric d on X. For each $n \in \mathbb{N}$, let $\mathcal{V}_n$ be an open cover of X such that $\operatorname{mesh} V_n < 2^{-n}$ and each $V \in \mathcal{V}_n$ is a subsemilattice of X. Since Y is relatively LC^0 in X, each $\mathcal{V}_n$ has an open refinement $\mathcal{U}_n$ such that each $U \in \mathcal{U}_n$ is contained in some $V \in \mathcal{V}_n$ so that every pair of points in $U \cap Y$ can be connected by a path in V.

Now, for each $n \in \mathbb{N}$, we can define $f_n : |N(\mathcal{U}_n \cup \mathcal{U}_{n+1})| \to Y$ such that $f_n(U) \in U$ for $U \in \mathcal{U}_n \cup \mathcal{U}_{n+1}$ and for every simplex $\sigma \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1})$ the image $f_n(\sigma)$ is contained in a member of $\mathcal{V}_n$. This is possible because of Lemma 5.3. Each f_n can be inductively defined in such a way that $f_n|N(\mathcal{U}_{n+1}) = f_{n+1}|N(\mathcal{U}_{n+1})$. Then, we have the map $f : |TN(\mathcal{U})| \to Y$ defined by $f||N(\mathcal{U}_n \cup \mathcal{U}_{n+1})| = f_n$, which satisfies the assumptions of Theorem 5.2, hence X is an ANR and Y is homotopy dense in X.

Moreover, if X or Y is path-connected, then it is n-connected for every $n \in \mathbb{N}$ by Lemma 5.3. Then, X is an AR. □

The case $X = Y$ in Theorem 5.1 is [2, Proposition 3.2], that is,

Corollary 5.4. *Let X be a metrizable Lawson semilattice. Then, X is an ANR (an AR) if and only if X is locally path-connected (and path-connected).*

For an arbitrary subset $Y \subset X$, we have $\mathrm{Fin}(Y) \subset \mathrm{Comp}(Y) \subset \mathrm{Cld}(X)$. It should be noticed that $\mathrm{Cld}(Y) \not\subset \mathrm{Cld}(X)$ unless Y is closed in X. As saw in Preliminaries, $\mathrm{Fin}_W(Y)$ (nor $\mathrm{Cld}_W(Y)$) is not a subspace of $\mathrm{Cld}_W(X)$ even if Y is closed in X.

When Y is dense in X, we can use a countable dense subset of Y to define the metric d_W for $\mathrm{Cld}_W(X)$. Then, $\mathrm{Comp}_W(Y)$ and $\mathrm{Fin}_W(Y)$ are subspaces of $\mathrm{Cld}_W(X)$. Moreover, we have the following:

Proposition 5.5. *Let X be separable metric space and Y a dense subspace of X with the metric inherited from X. Then, the space $\mathrm{Cld}_W(Y)$ can be naturally embedded in $\mathrm{Cld}_W(X)$ by the closure operator $\mathrm{cl}_X : \mathrm{Cld}_W(Y) \to \mathrm{Cld}_W(X)$. Namely, the space $\mathrm{Cld}_W(Y)$ can be identified with the subspace $\{A \in \mathrm{Cld}(X) \mid \mathrm{cl}_X(A \cap Y) = A\}$ of $\mathrm{Cld}_W(X)$.* □

Proposition 5.6. *For a dense subset Y of a separable metric space X, $\mathrm{Fin}_W(Y)$ is a dense semilattice of $\mathrm{Cld}_W(X)$.*

Proof. Let $A \in \mathrm{Cld}_W(X)$. For each neighborhood $\mathcal{W}$ of A in $\mathrm{Cld}_W(X)$, we have $p_1, \dots, p_n \in X$, $r_1, \dots, r_n > 0$ and $m \leqslant n$ such that

$$A \in \bigcap_{i=1}^{m} U^-(p_i, r_i) \cap \bigcap_{j=m+1}^{n} U^+(p_j, r_j) \subset \mathcal{W}.$$

For each $1 \leqslant i \leqslant m$, $B(p_i, r_i) \setminus \bigcup_{j=m+1}^{n} \overline{B}(p_j, r_j) \neq \emptyset$ because it contains a point of A. Since Y is dense in X, we have

$$y_i \in Y \cap B(p_i, r_i) \setminus \bigcup_{j=m+1}^{n} \overline{B}(p_j, r_j).$$

Then, $A' = \{a_1, \dots, a_m\} \in \mathrm{Fin}_W(Y) \cap \mathcal{W}$. Therefore, $\mathrm{Fin}_W(Y)$ is dense in $\mathrm{Cld}_W(X)$. □

By applying Theorem 5.1 in the above setting, we have the following:

Corollary 5.7. *Let X be a separable metric space with $Y \subset X$ a dense subset. If $\mathrm{Fin}_W(Y)$ is relatively LC^0 in $\mathrm{Cld}_W(X)$ (and $\mathrm{Fin}_W(Y)$ is path-connected), then $\mathrm{Cld}_W(X)$ is an ANR (an AR) and $\mathrm{Fin}_W(Y)$ is homotopy dense in $\mathrm{Cld}_W(X)$, hence $\mathfrak{H}$ is an ANR (an AR) if $\mathrm{Fin}_W(Y) \subset \mathfrak{H} \subset \mathrm{Cld}_W(Y)$.* □

6. Proof of Theorem III

In this section, we prove Theorem III. Here the metric for a metric space is denoted by d. To apply Theorem 5.1, we first show the following:

Proposition 6.1. *For an arbitrary metric space X, $\mathrm{Cld}_W(X)$ is a Lawson semilattice with respect to the union operator $\cup$.*

Proof. For each $x \in X$,

$$(A, B) \mapsto d(x, A \cup B) = \min\{d(x, A),\ d(x, B)\}$$

is continuous, which implies that the union operator $\cup$ is continuous.

Observe that $U^-(x, r)$ and $U^+(x, r)$ are subsemilattices of $\mathrm{Cld}_W(X)$ for each $x \in X$ and $r > 0$. Since the intersection of subsemilattices is also a subsemilattice, $\mathrm{Cld}_W(X)$ has an open basis consisting of subsemilattices, hence $\mathrm{Cld}_W(X)$ is a Lawson semilattice. □

Lemma 6.2. *Let X be a separable metric space and Y a path-connected subset of X. Then, each $A, B \in \mathrm{Fin}_W(Y)$ can be connected by a path $\gamma : \mathbf{I} \to \mathrm{Fin}_W(Y)$ such that each $\gamma(t)$ contains A or B.*

Proof. For each $a \in A$ and $b \in B$, we have a path $\gamma_{a,b} : \mathbf{I} \to Y$ from a to b. Now, we define a path $\gamma : \mathbf{I} \to \mathrm{Fin}_W(Y)$ by

$$\gamma(t) = A \cup \bigcup\{\gamma_{a,b}(t) \mid a \in A,\ b \in B\}.$$

Then, γ is a path in $\mathrm{Fin}_W(Y)$ from A to $A \cup B$. By the same argument, we can find a path from B to $A \cup B$. □

To prove the next result, we shall use a well-known combinatorial fact, called König's Lemma: *Every finitely-branching infinite tree contains an infinite branch* (cf. [10, (4.12)]). A *tree* is a partially ordered set $(T, <)$ such that for every $x \in T$, the set $\{y \in T \mid y < x\}$ is well-ordered (in our case: finite, linearly ordered). A *branch* through T is a maximal linearly ordered subset of T. In the proof of the following theorem, we consider a tree of sets with the reversed inclusion as the partial order. It is said that $A, B \subset X$ are *strongly disjoint* if

$$\mathrm{dist}(A, B) = \inf\{d(x, y) \mid x \in A, y \in B\} > 0.$$

Theorem 6.3. *Let X be a separable metric space. If the complement of any finite union of open balls in X has only finitely many path-components, all of which are closed and unbounded or compact, then $\mathrm{Cld}_W(X)$ is an ANR and*

$\mathrm{Fin}_W(X)$ *is homotopy dense in* $\mathrm{Cld}_W(X)$. *Hence* $\mathfrak{H}$ *is an ANR whenever* $\mathrm{Fin}_W(X) \subset \mathfrak{H} \subset \mathrm{Cld}_W(X)$.

Proof. By Corollary 5.7, it suffices to show that $\mathrm{Fin}_W(X)$ is relatively LC^0 in $\mathrm{Cld}_W(X)$. Note that X itself has only finitely many path-components (the case the number of open balls is zero). We only consider the case X is not compact, hence X is unbounded by the assumption.

Let $A \in \mathrm{Cld}_W(X)$ and $\mathcal{U}$ a neighborhood of A in $\mathrm{Cld}_W(X)$. Then,

$$A \in \bigcap_{i<k} U^-(p_i, r_i) \cap \bigcap_{j<l} U^+(q_j, s_j) \subset \mathcal{U},$$

for some $p_i, q_j \in X$ and $r_i, s_j > 0$. Choose $\varepsilon > 0$ so that $A \in U^+(q_j, s_j + \varepsilon)$ for every $j < l$.

Let $X_0, \ldots, X_{m-1}$ be all path-components of $X \setminus \bigcup_{j<l} \mathrm{B}(q_j, s_j + \varepsilon)$, where $X_i \cap A \neq \emptyset$ for $i < m_0$ and $X_j \cap A = \emptyset$ for $j \geqslant m_0$. Moreover, we can assume that X_j is unbounded for $m_0 \leqslant j < m_1$ and X_j is compact for $j \geqslant m_1$. Note that each X_i is not only closed but also open in $X \setminus \bigcup_{j<l} \mathrm{B}(q_j, s_j + \varepsilon)$ and A is strongly disjoint from $\bigcup_{j<l} \mathrm{B}(q_j, s_j + \varepsilon)$. For each $i < k$, we can choose $a_i \in A \cap \mathrm{B}(p_i, r_i)$ and $\varepsilon_i > 0$ so that $\mathrm{B}(a_i, \varepsilon_i) \subset X_j \cap \mathrm{B}(p_i, r_i)$ for some $j < m_0$. For each $i < m_0$, we can choose $a_{k+i} \in X_i \cap A$ and $\varepsilon_{k+i} > 0$ so that $\mathrm{B}(a_{k+i}, \varepsilon_{k+i}) \subset X_i$. Since $\bigcup_{m_1 \leqslant j<m} X_j$ is compact, we can find $t_0, \ldots, t_{v-1} \in X$ and $\delta_0, \ldots, \delta_{v-1} > 0$ such that

$$\bigcup_{m_1 \leqslant j < m} X_j \subset \bigcup_{i<v} \mathrm{B}(t_i, \delta_i) \subset \bigcup_{i<v} \mathrm{B}(t_i, 2\delta_i) \subset X \setminus \bigcup_{i<m_0} X_i.$$

Since $A \subset \bigcup_{i<m_0} X_i$, it follows that $A \in \bigcap_{i<v} U^+(t_i, \delta_i)$. Thus, A has the following neighborhood:

$$\mathcal{V} = \bigcap_{i<k+m_0} U^-(a_i, \varepsilon_i) \cap \bigcap_{j<l} U^+(q_j, s_j + \varepsilon) \cap \bigcap_{i<v} U^+(t_i, \delta_i) \subset \mathcal{U}.$$

Let $A_0 = \{a_i \mid i < k + m_0\} \subset A$. Then, $A_0 \in \mathcal{V} \cap \mathrm{Fin}_W(X)$.

We show that each $B \in \mathcal{V} \cap \mathrm{Fin}_W(X)$ can be connected to A_0 by a path in $\mathcal{U} \cap \mathrm{Fin}_W(X)$. Let $B^* = B \cap \bigcup_{m_0 \leqslant j < m_1} X_j$. Applying Lemma 6.2 to $A_0 \cap X_i$ and $B \cap X_i$ for $i < m_0$, we can easily construct a path in $\mathcal{U} \cap \mathrm{Fin}_W(X)$ from B to $A_0 \cup B^*$. For each $z \in B^*$, choose $m_0 \leqslant j < m_1$ so that $z \in X_j$, If we can construct an infinite path $f_z : [1, \infty) \to X_j$ such that $f_z(1) = z$ and $\lim_{t\to\infty} d(z, f_z(t)) = \infty$ (whence $\lim_{t\to\infty} d(x, f_z(t)) = \infty$ for any $x \in X$), then we can define a path $\psi : \mathbf{I} \to \mathcal{V}$ from A_0 to $A_0 \cup B^*$ as follows: $\psi(0) = A_0$ and

$$\psi(t) = A_0 \cup \bigcup_{z \in B^*} f_z(t^{-1}) \text{ for } t > 0.$$

For any $x \in X$, since $\lim_{t\to\infty} d(x, f_z(t)) = \infty$, $d(x, \psi(t)) = d(x, A_0)$ for sufficiently small $t > 0$, which means that ψ is continuous at 0.

Let $z \in B^* \cap X_j$ $(m_0 \leqslant j < m_1)$. Enumerate as $B_1, B_2, \ldots$ all open balls of the form $\mathrm{B}(x_i, \alpha)$, where $0 < \alpha < d(x_i, A)$ and $\alpha \in \mathbb{Q}$. By the assumption, for each $n \in \mathbb{N}$, $X_j \setminus (B_1 \cup \cdots \cup B_n)$ has finitely many path-components $H_0^n, \ldots, H_{a(n)-1}^n$. Let $T = \{H_i^n \mid n \in \mathbb{N},\ i < a(n)\}$. Since X_j is unbounded, it follows that T is infinite. Thus, $(T, \supset)$ is a finitely-branching infinite tree (i.e. each element of T has only finitely many immediate successors). By König's lemma, T contains an infinite branch $X_j \supset H_{i(1)}^1 \supset H_{i(2)}^2 \supset \cdots$. For each $n \in \mathbb{N}$, pick $z_n \in H_{i(n)}^n$ and a path $f_n : [n-1, n] \to H_{i(n-1)}^{n-1}$ such that $f_n(n-1) = z_{n-1}$ and $f_n(n) = z_n$, where $H_{i(0)}^0 = X_j$ and $z_0 = z$. By joining all paths $f_1, f_2, \ldots$, we can obtain a path $f_z : [1, +\infty) \to Y_j$ with $f_z(1) = z$ and $\lim_{t\to\infty} d(z, f_z(t)) = \infty$. □

Theorem 6.4. *Let X be a separable metric space with $x_0 \in X$. Suppose that for arbitrarily large $r > 0$, $X \setminus \mathrm{B}(x_0, r)$ has only finitely many path-components, all of which are unbounded. Then,* $\mathrm{Fin}_W(X)$ *is path-connected.*

Proof. First, note that X itself has only finitely many path-components, say $X_0, \ldots, X_{m-1}$. For each $i < m$, take $a_i \in U_i$, and let $A_0 = \{a_i \mid i < m\} \in \mathrm{Fin}_W(X)$. We show that each $B \in \mathrm{Fin}_W(X)$ can be connected to A_0 by a path in $\mathrm{Fin}_W(X)$.

Each $x \in B$ is contained in some $X_{i(x)}$, whence we have a path $f_x : \mathbf{I} \to X_{i(x)}$ such that $f_x(0) = x$ and $f_x(1) = a_{i(x)}$. Then, B is connected to $A_1 = \{a_{i(x)} \mid x \in B\} \subset A_0$ by a path $\varphi : \mathbf{I} \to \mathrm{Fin}_W(X)$ defined by $\varphi(t) = \{f_x(t) \mid x \in A\}$. Let $A_0 \setminus A_1 = \{a_i \mid i \in S\}$, that is,

$$S = \{i \mid i \neq i(x) \ \text{ for any } x \in B\}.$$

For $i \in S$, we will construct a path $g_i : [0, \infty) \to X_i$ such that $g_i(0) = a_i$ and $\lim_{t\to\infty} d(a_i, g_i(t)) = \infty$. Then, A_0 and A_1 can be connected by the path $\varphi : \mathbf{I} \to \mathrm{Fin}_W(X)$ defined as follows:

$$\psi(t) = \begin{cases} A & \text{if } t = 0, \\ \{g_i(t^{-1}) \mid i \in S\} \cup A_1 & \text{if } t > 1. \end{cases}$$

Thus, it follows that $\mathrm{Fin}_W(X)$ is path-connected.

The path g_i above can be constructed as follows: By the assumption, there are $0 < r_1 < r_2 < \cdots$ such that $\lim_{n\to\infty} r_n = \infty$ and each $X \setminus \mathrm{B}(x_0, r_n)$ has only finitely many path-components. Then, $X_i \setminus \mathrm{B}(x_0, r_n)$ has also only finitely many path-components, say $H_1^n, \ldots, H_{k(n)}^n$. Let $T = \{H_j^n \mid n \in \mathbb{N},\ j < k(n)\}$. Since X_i is unbounded, T is infinite. Thus, $(T, \supset)$ is a finitely-branching infinite tree (i.e., each element of T has only finitely many immediate successors). By König's lemma, T contains an infinite branch $X_i \supset H_{j(1)}^1 \supset H_{j(2)}^2 \supset \cdots$. For each $n \in \mathbb{N}$, pick $v_n \in H_{j(n)}^n$ and a path $g_n : [n-1, n] \to H_{j(n-1)}^{n-1}$ such that $f_n(n-1) = v_{n-1}$ and $f_n(n) = v_n$, where $H_{j(0)}^0 = X_j$ and $v_0 = a_i$. By joining all paths $f_1, f_2, \ldots$, we can obtain a path $g_i : [1, +\infty) \to X_i$ with $g_i(1) = a_i$ and $\lim_{t\to\infty} d(a_i, g_i(t)) = \infty$. □

Due to Corollary 5.7, Theorem III follows from Theorems 6.3 and 6.4.

7. Proof of Theorem II for $\mathrm{Fin}_W(X)$

As in the case of $\mathrm{Bdd}_W(X)$, we apply Theorem 2.3 to prove that $\mathrm{Fin}_W(X)$ is homeomorphic to $\ell_2 \times \ell_2^f$. In the previous section, we have shown that $\mathrm{Fin}_W(X)$ is homotopy dense in $\mathrm{Cld}_W(X)$ for an infinite-dimensional separable Banach space X.

Now, notice $\mathrm{Fin}_W(X) = \bigcup_{k\in\mathbb{N}} \mathrm{Fin}^k(X)$.

Lemma 7.1. *For an arbitrary metric space X, each $\mathrm{Fin}^k(X)$ is closed in $\mathrm{Cld}_W(X)$. Thus, if X is complete, then each $\mathrm{Fin}^k(X)$ is completely metrizable, hence $\mathrm{Fin}_W(X)$ is σ-completely metrizable.*

Proof. For each $A \in \mathrm{Cld}_W(X) \setminus \mathrm{Fin}^k(X)$, choose distinct $k+1$ many points $a_1, \dots, a_{k+1} \in A$ and let $r = \frac{1}{2}\min\{d(a_i, a_j) \mid i \neq j\}$. Then,

$$A \in \bigcap_{i=1}^{k+1} U^-(a_i, r) \subset \mathrm{Cld}_W(X) \setminus \mathrm{Fin}^k(X).$$

Therefore, $\mathrm{Cld}_W(X) \setminus \mathrm{Fin}^k(X)$ is open in $\mathrm{Cld}_W(X)$, that is, $\mathrm{Fin}^k(X)$ is closed in $\mathrm{Cld}_W(X)$. □

Lemma 7.2. *For every separable normed linear space X, each $\mathrm{Fin}^k(X)$ is a Z-set in $\mathrm{Fin}_W(X)$, hence $\mathrm{Fin}_W(X)$ is a Z_σ-space.*

Proof. By using distinct $k+1$ many points $v_1, \dots, v_{k+1} \in \mathbf{S}_X$, we define a homotopy $\zeta : \mathrm{Fin}_W(X) \times \mathbf{I} \to \mathrm{Fin}_W(X)$ by $\zeta_0 = \mathrm{id}$ and

$$\zeta(A, t) = A \cup t^{-1}\{v_1, \dots, v_{k+1}\} \quad \text{for } t > 0.$$

For each map $\alpha : \mathrm{Fin}_W(X) \to (0,1)$, define $\gamma : \mathrm{Fin}_W(X) \to (0,1)$ by

$$\gamma(A) = \sup\{t > 0 \mid \mathrm{diam}_{d_W} \zeta(\{A\} \times [0,t]) < \alpha(A)\}.$$

Then, γ is lower semi-continuous. Indeed, if $\gamma(A) > s$ then we have $s < s' < \gamma(A)$ and

$$\varepsilon = \alpha(A) - \mathrm{diam}_{d_W} \zeta(\{A\} \times [0,s']) > 0.$$

By the continuity of ζ and α, we have $\delta > 0$ such that if $A' \in \mathrm{Fin}_W(X)$ and $d_W(A, A') < \delta$ then $|\alpha(A) - \alpha(A')| < \frac{1}{3}\varepsilon$ and $d_W(\zeta(A,t), \zeta(A',t)) < \frac{1}{3}\varepsilon$ for all $t \in [0, s']$, whence

$$\begin{aligned}\mathrm{diam}_{d_W} \zeta(\{A'\} \times [0,s']) &< \mathrm{diam}_{d_W} \zeta(\{A\} \times [0,s']) + \tfrac{2}{3}\varepsilon \\ &= \alpha(A) - \tfrac{1}{3}\varepsilon < \alpha(A'),\end{aligned}$$

which means that $\gamma(A') \geqslant s' > s$.

Now, we define a map $f : \mathrm{Fin}_W(X) \to \mathrm{Fin}_W(X)$ as follows:

$$f(A) = \zeta(A, \beta(A)) = A \cup \beta(A)^{-1}\{v_1, \dots, v_{k+1}\},$$

where $\beta : \mathrm{Fin}_W(X) \to (0,1)$ is a map such that $\beta(A) < \gamma(A)$. Observe that $d_W(f(A), A) < \alpha(A)$ for each $A \in \mathrm{Fin}_W(X)$. By the definition, $\mathrm{Fin}^k(X) \cap f(\mathrm{Fin}_W(X)) = \emptyset$. Thus, we have the result. □

Remark. Due to the remark before Theorem 2.3, it follows from Theorem III that every Z-set in $\mathrm{Fin}_W(X)$ is a strong Z-set. Therefore, each $\mathrm{Fin}^k(X)$ is a strong Z-set in $\mathrm{Fin}_W(X)$. However, this can be also obtained in the above proof by showing that

$$\mathrm{Fin}^k(X) \cap \mathrm{cl}\, f(\mathrm{Fin}_W(X)) = \emptyset.$$

To this end, assume that there exist $A_n \in \mathrm{Fin}_W(X)$, $n \in \mathbb{N}$, such that $A = \lim_{n\to\infty} f(A_n) \in \mathrm{Fin}^k(X)$. Then, we will find a contradiction in the both cases $\inf_{n\in\mathbb{N}} \beta(A_n) = 0$ and $\inf_{n\in\mathbb{N}} \beta(A_n) > 0$.

When $\inf_{n\in\mathbb{N}} \beta(A_n) = 0$, we may assume that $\lim_{n\to\infty} \beta(A_n) = 0$. For each $\varepsilon > 0$, choose $i_0 \in \mathbb{N}$ so that $2^{-i_0} < \varepsilon$. Since $\beta(A_n)^{-1} \to \infty$ as $n \to \infty$, we have $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &n \geqslant n_0 \Rightarrow d_W(f(A_n), A) < 2^{-i_0-1}, \\ &\qquad\qquad \beta(A_n)^{-1} > \max\{\|x_i\| + d(x_i, A) + 2^{-i_0-1} \mid i = 1, \dots, i_0\}. \end{aligned}$$

Then, for each $n \geqslant n_0$ and $i \leqslant i_0$,

$$\beta(A_n)^{-1} - \|x_i\| > d(x_i, A) + 2^{-i_0-1} > d(x_i, f(A_n)),$$

hence $d(x_i, f(A_n)) = d(x_i, A_n)$, which implies $d_W(f(A_n), A_n) \leqslant 2^{-i_0-1}$. Since $d_W(f(A_n), A) < 2^{-i_0-1}$ it follows that $d_W(A_n, A) < 2^{-i_0} < \varepsilon$. Thus, $(A_n)_{n\in\mathbb{N}}$ converges to A, hence $\lim_{n\to\infty} \beta(A_n) = \beta(A) > 0$, which is a contradiction.

When $\inf_{n\in\mathbb{N}} \beta(A_n) > 0$, we may assume that $\lim_{n\to\infty} \beta(A_n) = b > 0$. Then, we show that $z_j = b^{-1} v_j \in A$ for each $j = 1, \dots, k+1$. If $z_j \notin A$, choose x_{i_j} so that $d(x_{i_j}, z_i) < \frac{1}{4} d(z_j, A)$, whence

$$d(x_{i_j}, A) \geqslant d(z_j, A) - d(x_{i_j}, z_i) > \tfrac{3}{4} d(z_j, A).$$

For sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} |d(x_{i_j}, A) - d(x_{i_j}, f(A_n))| &< \min\{2^{-i_j}, \tfrac{1}{4} d(z_j, A)\} \text{ and} \\ |b^{-1} - \beta(A_n)^{-1}| &< \tfrac{1}{4} d(z_j, A), \end{aligned}$$

whence it follows that

$$\begin{aligned} d(z_j, A) &\leqslant d(z_i, x_{i_j}) + d(x_{i_j}, A) \\ &< \tfrac{1}{4} d(z_j, A) + |d(x_{i_j}, A) - d(x_{i_j}, A_n)| + d(x_{i_j}, A_n) \\ &< \tfrac{1}{2} d(z_j, A) + d(x_{i_j}, z_i) + d(z_i, A_n) \\ &< \tfrac{3}{4} d(z_j, A) + |b^{-1} - \beta(A_n)^{-1}| < d(z_j, A). \end{aligned}$$

This is a contradiction. □

Theorem 7.3. *For every infinite-dimensional separable Banach space X, $\mathrm{Fin}_W(X)$ is strongly universal for separable completely metrizable spaces. Consequently,* $\mathrm{Fin}_W(X) \approx \ell_2 \times \ell_2^f$.

Proof. By the same reason as in Theorem 4.5, it suffices to prove that each open set $W \subset \mathrm{Fin}_W(X)$ is universal for separable completely metrizable spaces. Let Y be a separable completely metrizable space, and $f : Y \to W$ a map. For each map $\alpha : W \to (0,1)$, similarly to Theorem 4.5, we apply Lemma 3.1 to obtain a map $\beta : \mathrm{Fin}_W(X) \to (0,1)$ such that

$$(*) \quad A \in W,\ A' \in \mathrm{Fin}(X),\ A \cap \beta(A)^{-1}\mathbf{B}_X = A' \cap \beta(A)^{-1}\mathbf{B}_X \Rightarrow A' \in W,\ d_W(A, A') < \alpha(A).$$

Let $\beta_0 = \beta$. For $n \in \mathbb{N}$, let $\beta_n : \mathrm{Fin}_W(X) \to \mathbf{I}$ be the map defined by

$$\beta_n(A) = \min\left\{(n+1)^{-1},\ \beta(A),\ \beta(A)d_W(A, \mathrm{Fin}^n(n\mathbf{B}_X))\right\}.$$

Note that $n+1 \leqslant \beta_n(A)^{-1}$ if $\beta_n(A) \neq 0$ and $\beta_n(A)^{-1} \leqslant \beta_{n+1}(A)^{-1}$ if $\beta_{n+1}(A) \neq 0$. For each $A \in \mathrm{Fin}_W(X)$, define $k(A) \in \mathbb{N}$ by

$$A \in \mathrm{Fin}^{k(A)}(k(A)\mathbf{B}_X) \setminus \mathrm{Fin}^{k(A)-1}((k(A)-1)\mathbf{B}_X).$$

Then, $\beta_n(A) \neq 0$ for $n < k(A)$ and $\beta_n(A) = 0$ for $n \geqslant k(A)$,.

By Proposition 4.4, there exists a closed embedding $h : Y \to \mathbf{S}_X$. Take $v_n \in \mathbf{S}_X$, $n \in \mathbb{N} \cup \{0\}$, so that $\|v_n - v_m\| > \frac{1}{2}$ for $n \neq m$. We define a map $g : Y \to \mathrm{Fin}_W(X)$ by

$$g(z) = f(z) \cup \left\{(\beta_n(f(z))^{-1} + 2)v_n,\ (\beta_n(f(z))^{-1} + 1)v_n + \tfrac{1}{8}h(z) \mid n < k(f(z))\right\}.$$

By (*), g is α-close to f. To see that g is injective, let $z \neq z' \in Y$. In case $k(f(z)) \neq k(f(z'))$, assume $k(f(z)) > k(f(z'))$. Then,

$$(\beta_{k(f(z))-1}(f(z))^{-1} + 2)v_{k(f(z))-1} \in g(z) \setminus g(z'),$$

so $g(z) \neq g(z')$. When $k(f(z)) = k(f(z')) = k$, it follows that

$$\left\{(\beta_{k-1}(f(z))^{-1} + 2)v_{k-1},\ (\beta_{k-1}(f(z))^{-1} + 1)v_{k-1} + \tfrac{1}{8}h(z)\right\} \neq \left\{(\beta_{k-1}(f(z'))^{-1} + 2)v_{k-1},\ (\beta_{k-1}(f(z'))^{-1} + 1)v_{k-1} + \tfrac{1}{8}h(z')\right\},$$

which implies $g(z) \neq g(z')$.

To see that g is a closed map, let $y_i \in Y$, $i \in \mathbb{N}$, and $G \in \mathrm{Fin}(X)$ such that $g(y_i) \to G$. For each $n \in \mathbb{N} \cup \{0\}$, let

$$b_n = \liminf_{i \to \infty} \beta_n(f(y_i)) \in [0, (n+1)^{-1}].$$

Then, $1 \geqslant b_0 \geqslant b_1 \geqslant \cdots \geqslant 0$. Moreover, $b_0 > 0$. Otherwise, by taking a subsequence, we can assume that $\beta(f(y_i)) \to 0$. Then, it follows that $f(y_i) \to G$, hence $\beta(f(y_i)) \to \beta(G) \neq 0$, which is a contradiction.

Let $b_m \neq 0$ and $b_{m+1} = 0$. Then, $(b_m^{-1} + 2)v_m \in G \subset k(G)\mathbf{B}_X$ because

$$d((b_m^{-1} + 2)v_m, G) = \liminf_{i \to \infty} d((b_m^{-1} + 2)v_m, g(y_i)) = 0.$$

Hence, $m + 3 \leqslant b_m^{-1} + 2 \leqslant k(G)$, that is, $m \leqslant k(G) - 3$. By taking a subsequence, we can assume that $\beta_{m+1}(f(y_i)) \to 0$. Since $\liminf_{i \in \mathbb{N}} \beta(f(y_i)) > 0$, it follows that

$$d_W(f(y_i), \mathrm{Fin}^{m+1}((m+1)\mathbf{B}_X)) \to 0.$$

Moreover, observe

$$\liminf_{i \to \mathbb{N}} d_W(f(y_i), \mathrm{Fin}^m(m\mathbf{B}_X)) \geqslant b_m > 0.$$

Then, we can choose

$$F_i \in \mathrm{Fin}^{m+1}((m+1)\mathbf{B}_X) \setminus \mathrm{Fin}^m(m\mathbf{B}_X),\ i \in \mathbb{N},$$

so that $d_W(F_i, f(y_i)) \to 0$. For each $i \in \mathbb{N}$, let

$$G_i = F_i \cup \big\{(\beta_n(f(y_i))^{-1} + 2)v_n, \\ (\beta_n(f(y_i))^{-1} + 1)v_n + \tfrac{1}{8}h(y_i) \mid n < k(f(y_i))\big\}.$$

Then, $G_i \to G$ as $i \to \infty$. Observe that

$$\begin{aligned} b_m^{-1} + 2 \geqslant \|(\beta_m(f(y_i))^{-1} + 2)v_n\| &= \beta_m(f(y_i))^{-1} + 2 \\ &> \beta_m(f(y_i))^{-1} + \tfrac{7}{8} \\ &\geqslant \|(\beta_m(f(y_i))^{-1} + 1)v_m + \tfrac{1}{8}h(y_i)\| \\ &\geqslant \beta_m(f(y_i))^{-1} + \tfrac{7}{8} > m + \tfrac{3}{2}. \end{aligned}$$

Since $F_i \subset (m+1)\mathbf{B}_X$, it follows that $(h(y_i))_{i \in \mathbb{N}}$ is convergent. Hence, $(y_i)_{i \in \mathbb{N}}$ is convergent because h is a closed embedding. □

8. The relative Wijsman topology

Let $X = (X, d)$ be a separable metric space. For $\mathfrak{H} \subset \mathrm{Cld}_W(X)$ and $Y \subset X$, we denote $\mathfrak{H}|Y = \{A \in \mathfrak{H} \mid A \subset Y\}$. Without any condition, we have $\mathrm{Fin}(X)|Y = \mathrm{Fin}(Y)$ and $\mathrm{Comp}(X)|Y = \mathrm{Comp}(Y)$ as sets. But, $\mathrm{Cld}(Y) = \mathrm{Cld}(X)|Y$ if and only if Y is closed in X. As saw in Example, even if Y is closed in X, $\mathrm{Fin}_W(X)|Y \neq \mathrm{Fin}_W(Y)$ nor $\mathrm{Cld}_W(X)|Y \neq \mathrm{Cld}_W(Y)$ as spaces.

In this section, we give some sufficient conditions in order that $\mathrm{Cld}_W(X)|Y$ is an ANR or $\mathrm{Fin}_W(X)|Y$ is homotopy dense in $\mathrm{Cld}_W(X)|Y$. Note that both $\mathrm{Cld}_W(X)|Y$ and $\mathrm{Fin}_W(X)|Y$ are Lawson semilattices of $\mathrm{Cld}_W(X)$.

Lemma 8.1. *Let X be a separable metric space. For any path-connected closed subset $Y \subset X$, each $A, B \in \mathrm{Cld}_W(X)|Y$ can be connected by a path $\gamma : \mathbf{I} \to \mathrm{Cld}_W(X)|Y$ such that each $\gamma(t)$ contains A or B.*

Proof. It suffices to show that each $A \in \mathrm{Cld}(Y)$ can be connected to Y by a path $\gamma : \mathbf{I} \to \mathrm{Cld}_W(X)$ such that $A \subset \gamma(t) \subset Y$ for every $t \in \mathbf{I}$. Let $\{y_n \mid n \in \mathbb{N}\}$ be dense in Y with $y_1 \in A$. For each $n \in \mathbb{N}$, we have a path $\lambda_n : [n, n+1] \to Y$ with $\lambda_n(n) = y_n$ and $\lambda(n+1) = y_{n+1}$. We define $\gamma : \mathbf{I} \to \mathrm{Cld}_W(X)$ by $\gamma(0) = Y$ and

$$\gamma_i(t) = A \cup \bigcup_{n<m} \lambda_n([n, n+1]) \cup \lambda_m([m, t^{-1}]) \ \text{ for } (m+1)^{-1} < t \leqslant m^{-1}.$$

It is easy to verify the continuity of γ. □

In the following, the case $Y = X$ is Theorem IV for $\mathrm{Cld}_W(X)$.

Theorem 8.2. *Let X be a separable metric space and Y a closed set of X such that for any finitely many open balls $B_1, \dots, B_n$ in X, $Y \setminus \bigcup_{i=1}^n B_i$ has only finitely many path-components, all of which are closed in X. Then, the space $\mathrm{Cld}_W(X)|Y$ is an ANR.*

Proof. By Corollary 5.4, it suffices to show that $\mathrm{Cld}_W(X)|Y$ is locally path-connected. Let $A \in \mathrm{Cld}_W(X)$ and $\mathcal{U}$ a neighborhood of A in $\mathrm{Cld}_W(X)$. Then,

$$A \in \bigcap_{i<k} U^-(p_i, r_i) \cap \bigcap_{j<l} U^+(q_j, s_j) \subset \mathcal{U},$$

for some $p_i, q_j \in X$ and $r_i, s_j > 0$. Choose $\varepsilon > 0$ so that $A \in U^+(q_j, s_j + \varepsilon)$ for every $j < l$.

Let $Y_0, \dots, Y_{m-1}$ be all path-components of $Y \setminus \bigcup_{j<l} \mathrm{B}(q_j, s_j + \varepsilon)$, where $Y_i \cap A \neq \emptyset$ for $i < m_0$ and $Y_j \cap A = \emptyset$ for $j \geqslant m_0$. Observe that each Y_i is open in $Y \setminus \bigcup_{j<l} \mathrm{B}(q_j, s_j + \varepsilon)$ and A is strongly disjoint from $\bigcup_{j<l} \mathrm{B}(q_j, s_j + \varepsilon)$. For each $i < m_0$, we can choose $p_{k+i} \in Y_i \cap A$ and $r_{k+i} > 0$ so that $\mathrm{B}(p_{k+i}, r_{k+i}) \cap Y \subset Y_i$. Since $A \subset \bigcup_{i<m_0} Y_i$, it follows that $\bigcup_{j \geqslant m_0} Y_i$ is covered by open balls which are strongly disjoint from A. It can be assumed that $m_0 \leqslant j < m_1$ if and only if Y_j cannot be covered by finite open balls which are strongly disjoint from A. We can find $t_0, \dots, t_{v-1} \in X$ and $\delta_0, \dots, \delta_{v-1} > 0$ such that

$$\bigcup_{m_1 \leqslant j < m} Y_j \subset \bigcup_{i<v} \mathrm{B}(t_i, \delta_i) \ \text{ and } \ A \in U^+(t_i, \delta_i) \ \text{ for each } i < v.^6$$

Thus, we have the neighborhood $\mathcal{V}$ of A defined by

$$\mathcal{V} = \bigcap_{i<k+m_0} U^-(p_i, r_i) \cap \bigcap_{j<l} U^+(q_j, s_j + \varepsilon) \cap \bigcap_{i<v} U^+(t_i, \delta_i) \subset \mathcal{U}.$$

It suffices to show that each $B \in \mathcal{V}|Y$ can be connected to A by a path in $\mathcal{U}|Y$. Then, $B \subset \bigcup_{j<m_1} Y_j$ and $B \cap Y_i \supset B \cap B(p_{k+i}, r_{k+i}) \neq \emptyset$ for every $i < m_0$. By Lemma 8.1, there are paths $\gamma_i : \mathbf{I} \to \mathrm{Cld}_W(Y_i)$, $i < m_0$, such that $\gamma_i(0) = A \cap Y_i$, $\gamma_i(1) = B \cap Y_i$ and each $\gamma_i(t)$ contains $A \cap Y_i$ or $B \cap Y_i$. Let $S = \{j \geqslant m_0 \mid B \cap Y_j \neq \emptyset\}$. For each $j \in S$, take $z_j \in B \cap Y_j$ and define $B^* = \{z_j \mid j \in S\}$. By Lemma 8.1, we have paths $\varphi_j : \mathbf{I} \to \mathrm{Cld}_W(Y_j)$, $j \in S$,

[6] See footnote 3.

such that $\varphi_j(0) = z_j$ and $\varphi(1) = B \cap Y_j$. Then, we have a path $\gamma : \mathbf{I} \to \mathcal{U}|Y$ from $A \cup B^*$ to B defined by

$$\gamma(t) = \bigcup_{i<m_0} \gamma_i(t) \cup \bigcup_{j\in S} \varphi_j(t).$$

For each $j \in S$, we will construct an infinite path $\psi_j : [1, \infty) \to Y_j$ with $\psi_j(1) = z_j$ and the following property:

(*) for each $n \in \mathbb{N}$, there is some $t_0 > 0$ such that

$$d(x_i, \psi_j(t)) > d(x_i, A) - 1/n \text{ for } i \leqslant n \text{ and } t \geqslant t_0,$$

where $\{x_i \mid i \in \mathbb{N}\}$ is a countable dense set in X to define the metric d_W. Then, we can define a path $\psi : \mathbf{I} \to \mathcal{V}$ from A to $A \cup B^*$ as follows: $\psi(0) = A$ and

$$\psi(t) = A \cup \bigcup_{j\in S} \psi_j(t^{-1}) \text{ for } t > 0.$$

The continuity of λ at 0 can be verified as follows: For each $n \in \mathbb{N}$, we use (*) to find $t_0 > 0$ so that

$$d(x_i, \psi_j(t)) > d(x_i, A) - 1/n \text{ for } i \leqslant n,\ t \geqslant t_0 \text{ and } j \in S.$$

Then, $t < t_0^{-1}$ implies $|d(x_i, \psi_j(t)) - d(x_i, A)| < 1/n$ for $i \leqslant n$, hence $d_W(\psi(t), A) < 1/n$.

Now, for $j \in S$, we construct an infinite path $\psi_j : [1, \infty) \to Y_j$. Enumerate as $B_1, B_2, \dots$ all open balls of the form $\mathrm{B}(x_i, \alpha)$, where $0 < \alpha < d(x_i, A)$ and $\alpha \in \mathbb{Q}$. By the assumption, for each $n \in \mathbb{N}$, $Y_j \setminus (B_1 \cup \dots \cup B_n)$ has finitely many path-components $H^n_0, \dots, H^n_{a(n)-1}$. Let $T = \{H^n_i \mid n \in \mathbb{N},\ i < a(n)\}$. Since every B_i is strongly disjoint from A, Y_j cannot be covered by finitely many B_i's, hence T is infinite. Thus, $(T, \supset)$ is a finitely-branching infinite tree (i.e., each element of T has only finitely many immediate successors). By König's lemma, T contains an infinite branch $Y_j \supset H^1_{i(1)} \supset H^2_{i(2)} \supset \cdots$. For each $n \in \mathbb{N}$, pick $y_n \in H^n_{i(n)}$ and a path $f_n : [n-1, n] \to H^{n-1}_{i(n-1)}$ such that $f_n(n-1) = y_{n-1}$ and $f_n(n) = y_n$, where $H^0_{i(0)} = Y_j$ and $y_0 = z_j$. By joining all paths $f_1, f_2, \dots$, we can obtain a path $\psi_j : [1, +\infty) \to Y_j$ with $f_y(1) = z_j$.

For each $i \leqslant n$, choose $\alpha_i \in \mathbb{Q}$ so that $d(x_i, A) - 1/n < \alpha_i < d(x_i, A)$. Then we have $m \in \mathbb{N}$ such that all balls $\mathrm{B}(x_1, \alpha_1), \dots, \mathrm{B}(x_n, \alpha_n)$ appear in $B_0, \dots, B_{m-1}$. For $t < 1/m$, we have $l \geqslant m$ such that $l < t^{-1} \leqslant l + 1$, whence

$$\psi_j(t) = f_{l+1}(t^{-1}) \in H^n_{i(l)} \subset Y_j \setminus (\mathrm{B}(x_1, \alpha_1) \cup \dots \cup \mathrm{B}(x_n, \alpha_n)).$$

Therefore, $d(x_i, \psi_j(t)) \geqslant \alpha_i > d(x_i, A) - 1/n$ for every $i \leqslant n$. Thus ψ satisfies (*). □

In the above, we use Lemma 6.2 instead of Lemma 8.1 to obtain the following, in which the case $Y = X$ is Theorem IV for $\mathrm{Fin}_W(X)$:

Theorem 8.3. *Let X be a separable metric space and Y a path-connected subset of X such that for any finitely many open balls $B_1, \dots, B_n$ in X, $Y \setminus \bigcup_{i=1}^n B_i$ has only finitely many path-components which are open in $Y \setminus \bigcup_{i=1}^n B_i$. Then, the space $\mathrm{Fin}_W(X)|Y$ is an ANR.* □

In the same setting as Theorem 8.2, Theorem 6.3 (or Theorem III) can be generalized as follows:

Theorem 8.4. *Let X be a separable metric space and Y a path-connected subset of X such that for any finitely many open balls $B_1, \dots, B_n$ in X, $Y \setminus \bigcup_{i=1}^n B_i$ has finitely many path-components which are compact or unbounded (resp. all are unbounded). Then, $\mathrm{Fin}_W(X)|Y$ is homotopy dense in $\mathrm{Cld}_W(X)|Y$ and $\mathrm{Cld}_W(X)|Y$ is an ANR (resp. an AR).*

The proof of this theorem is left to the readers.

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