Exercises for Mathematical Logic (Fall 2023/24)

We have seen in the lecture that the De Morgan language $\{\wedge, \vee, \neg, \top, \bot\}$ is functionally complete, and specifically, that every Boolean function can be represented by a CNF or DNF of size $O(2^n n)$.

1. Prove that $\{\vee, \neg\}$, $\{\to, \bot\}$, and $\{\uparrow\}$ are functionally complete, where $x \uparrow y$ denotes the Sheffer stroke $\neg(x \wedge y)$.

2. Prove that $\{\to\}$, $\{\wedge, \vee, \top, \bot\}$, and $\{\leftrightarrow, \top, \bot\}$ are not functionally complete.
[Hint: Find a nontrivial property of Boolean functions which is preserved by composition, and holds for functions in the given basis.]

3. For any Boolean function $f\colon \{0,1\}^n \to \{0,1\}$, the following are equivalent:

(i) $\{f\}$ is functionally complete.

(ii) $f(0,\dots,0) = 1$, $f(1,\dots,1) = 0$, and there exists an assignment $\langle e_0,\dots,e_{n-1}\rangle \in \{0,1\}^n$ such that $f(e_0,\dots,e_{n-1}) = f(\neg e_0,\dots,\neg e_{n-1})$.

[Hint: For (ii) $\to$ (i), look at functions obtained from f by identifying some of the variables.]

4. For any $n \in \mathbb{N}$, the *parity* function $\bigoplus_{i<n} x_i\colon \{0,1\}^n \to \{0,1\}$ is defined as $\sum_{i<n} x_i \bmod 2$. Show that any DNF or CNF representing $\bigoplus_{i<n} x_i$ has size $\Omega(2^n n)$. [Hint: What terms of the form $\bigwedge_{i\in I} x_i^{e_i}$ can imply one of $\bigoplus_{i<n} x_i = 0$ or $\bigoplus_{i<n} x_i = 1$? Here, $I \subseteq [n]$, $e_i \in \{0,1\}$, $x^1 = x$, $x^0 = \neg x$.]

5. There are formulas representing $\bigoplus_{i<n} x_i$ of size $O(n^c)$ for some constant c.
[Hint: Consider a balanced tree of binary parities. You may get it down to $c = 2$.]

6. Any DNF equivalent to the CNF $\bigwedge_{i<n}(x_i \vee y_i)$ has size $\Omega(2^n n)$.

7. Prove the propositional soundness theorem: for all $\Gamma \subseteq \mathrm{Prop}(A)$ and $\varphi \in \mathrm{Prop}(A)$, if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.

8. Let $\Gamma, \Delta \subseteq \mathrm{Prop}(A)$ and $\varphi, \psi \in \mathrm{Prop}(A)$. Show that if $\Gamma \vdash \varphi$ and $\Delta, \varphi \vdash \psi$, then $\Gamma, \Delta \vdash \psi$.

9. For every $\varphi \in \mathrm{Prop}(A)$, we define its *De Morgan dual* $\varphi^{\mathrm{d}} \in \mathrm{Prop}(A)$ by induction on the complexity of φ:

$$
\begin{aligned}
a^{\mathrm{d}} &= a, \quad a \in A, & (\neg\varphi)^{\mathrm{d}} &= \neg(\varphi^{\mathrm{d}}),\\
\top^{\mathrm{d}} &= \bot, & \bot^{\mathrm{d}} &= \top,\\
(\varphi \wedge \psi)^{\mathrm{d}} &= (\varphi^{\mathrm{d}} \vee \psi^{\mathrm{d}}), & (\varphi \vee \psi)^{\mathrm{d}} &= (\varphi^{\mathrm{d}} \wedge \psi^{\mathrm{d}}).
\end{aligned}
$$

Show that for all assignments $v\colon A \to \{0,1\}$, $v(\varphi^{\mathrm{d}}) = v_\neg(\neg\varphi)$, where $v_\neg\colon A \to \{0,1\}$ is the assignment defined by $v_\neg(a) = 1 - v(a)$ for each $a \in A$.

10. Let $\varphi, \psi \in \mathrm{Prop}(A)$.

(i) $\varphi \equiv \psi$ if and only if $\varphi^{\mathrm{d}} \equiv \psi^{\mathrm{d}}$.

(ii) $\varphi \vDash \psi$ if and only if $\psi^{\mathrm{d}} \vDash \varphi^{\mathrm{d}}$.

11. (If you are familiar with topology.) Give a direct proof of the propositional compactness theorem, not using the completeness theorem.
[Hint: Consider the product topology on the set $\{0,1\}^A$ of all assignments.]

In the lecture, we have proved completeness of a proof system using connectives $\{\to, \bot\}$. A complete system using the De Morgan language $\{\wedge, \vee, \neg, \bot, \top\}$ is given in the van den Dries lecture notes, but the next exercise shows how to construct one mechanically.

12. For any $\{\to,\bot\}$-formula φ, let φ^* denote the De Morgan formula such that $p^* = p$ for atoms p, $\bot^* = \bot$, and $(\varphi \to \psi)^* = (\neg\varphi^* \vee \psi^*)$. Similarly, given a De Morgan formula ψ, let $\psi^\#$ be its translation to a $\{\to,\bot\}$-formula using fixed $\{\to,\bot\}$-translations of all De Morgan connectives. Let $\vdash_0$ denote a sound and complete Hilbert-style proof system for $\{\to,\bot\}$-formulas such as the one given in the lecture, and let $\vdash_1$ be the Hilbert-style proof system in the De Morgan language that has inference rule schemata $\varphi_1^*, \dots, \varphi_k^* \ / \ \varphi_0^*$ for each rule schema $\varphi_1, \dots, \varphi_k \ / \ \varphi_0$ of $\vdash_0$ (where axioms are treated as rules with $k = 0$), and axiom schemata $\neg c(\varphi_0, \dots, \varphi_{k-1}) \vee c^{\#*}(\varphi_0, \dots, \varphi_{k-1})$, $\neg c^{\#*}(\varphi_0, \dots, \varphi_{k-1}) \vee c(\varphi_0, \dots, \varphi_{k-1})$ for each k-ary De Morgan connective c. Then $\vdash_1$ is a sound and complete proof system in the De Morgan language. [Hint: You will need to show $\vdash_1 \neg\psi \vee \psi^{\#*}$, $\vdash_1 \neg\psi^{\#*} \vee \psi$ for all De Morgan formulas ψ.]

13. A set of propositional or first-order sentences S is *independent* if S is not equivalent to S' for any proper subset $S' \subsetneq S$.

(i) S is independent iff $S \smallsetminus \{\varphi\} \nvDash \varphi$ for all $\varphi \in S$.

(ii) Show that every countable theory T has an independent axiomatization, i.e., an independent set of sentences S equivalent to T. [Hint: Try to generalize the fact that $\{\varphi, \psi\} \equiv \{\varphi, \psi \vee \neg\varphi\}$.]

(Uncountable theories have independent axiomatizations as well by a theorem of I. Reznikoff, but this is more difficult to prove.)

14. Prove that if a term $t(x_0, \dots, x_{n-1}, y)$ is free for y in a formula $\varphi(x_0, \dots, x_{n-1}, y)$, then for all terms $s_0, \dots, s_{n-1}, r$, the formula $\big(\varphi(t/y)\big)(s_0/x_0, \dots, s_{n-1}/x_{n-1}, r/y)$ is syntactically identical to the formula $\varphi\big(s_0/x_0, \dots, s_{n-1}/x_{n-1}, t(s_0/x_0, \dots, s_{n-1}/x_{n-1}, r/y)/y\big)$.

15. Consider a modification of the first-order proof system given in the lecture such that the axioms of equality are replaced with the axiom $x = x$ and the axiom schema $t = s \wedge \varphi(t/s) \to \varphi(s/x)$ for all formulas φ and terms t, s free for x in φ. Show that this is equivalent to the original proof system.

16. For any formula $\varphi(x)$ and variable y free for x in φ, show that the formula $\exists y\, (\exists x\, \varphi(x) \to \varphi(y))$ is provable.

17. Let $\mathcal{A}$ be an L-structure, t a closed L-term such that $t^{\mathcal{A}} = a \in A$, and $\varphi(x)$ an L-formula. Show that $\mathcal{A} \vDash \phi(t)$ iff $\mathcal{A} \vDash \phi(\underline{a})$.

18. Using Vaught's test, show the completeness of the theory of a successor: it has a language with one unary function symbol s, and axioms $s(x) = s(y) \to x = y$, $\forall x\, \exists y\, s(y) = x$, and $s^n(x) \neq x$ for each $n \in \mathbb{N}_{>0}$, where s^n denotes the n-fold iteration of s (i.e., $s^0(x)$ is x, and s^{n+1} is $s(s^n(x))$).

19. For each $n \in \mathbb{N}$, let P_n denote the path graph of length n, i.e., the structure $\langle [n], E_n \rangle$, where $[n] = \{0, \dots, n-1\}$ and $E_n = \{\langle i, j \rangle \in [n]^2 : |i - j| = 1\}$. Show that there is no sentence φ such that for all $n \in \mathbb{N}$, $P_n \vDash \varphi$ iff n is odd. [Hint: Adapt the previous exercise.]

20. Fix a field F. The theory of vector spaces over F has a language consisting of the language $\{+, -, 0\}$ of abelian groups and unary functions $a \cdot x$ for each $a \in F$; it has the usual algebraic axioms (axioms of abelian groups, $ab \cdot x = a \cdot (b \cdot x)$, $1 \cdot x = x$, $(a+b) \cdot x = a \cdot x + b \cdot x$, $a \cdot (x + y) = a \cdot x + a \cdot y$). Show that the theory of infinite vector spaces over F (i.e., with additional axioms $\exists x_0 \dots \exists x_n \bigwedge_{i<j} x_i \neq x_j$ for $n \in \mathbb{N}$) is complete and κ-categorical for all infinite $\kappa > |F|$. [Hint: Every vector space has a basis.]

21. An *atom* in a Boolean algebra $\mathbf{A} = \langle A, 0, 1, \wedge, \vee, -, \leq \rangle$ is an element $a \in A$ such that $a > 0$, but $0 < x < a$ for no $x \in A$; $\mathbf{A}$ is *atomless* if $0 \neq 1$ and $\mathbf{A}$ has no atoms. Show that the theory of atomless Boolean algebras is $\aleph_0$-categorical, hence complete.
[Hint: Construct an isomorphism between two countable atomless Boolean algebras $\mathbf{A}$ and $\mathbf{B}$ by a back-and-forth argument, as a union of a sequence of isomorphisms between finite subalgebras. It might help to observe that if $\mathbf{A}_0$ is a finite subalgebra of $\mathbf{A}$, and $\mathbf{A}_1$ is the algebra generated by $A_0 \cup \{b\}$ for some $b \in A$, then each atom of $\mathbf{A}_0$ either remains an atom in $\mathbf{A}_1$, or splits into two atoms.]

22. Let L be a finite first-order language. Show that the following sets and functions are computable:

(i) The set of L-terms.

(ii) The set of L-formulas.

(iii) The set of pairs (φ, x) where x is a free variable of an L-formula φ.

(iv) The substitution function: given an L-formula φ, a variable x, and an L-term t, compute $\varphi(t/x)$.

(v) The set of triples (Γ, φ, π) where π is a proof of an L-formula φ from a finite set of L-formulas Γ.

23. Prove $\mathsf{Q} \vdash \forall x\,(x \leq \overline{n} \vee \overline{n} \leq x)$ for each $n \in \mathbb{N}$.

24. Q proves $x \cdot y = 0 \to x = 0 \vee y = 0$, and more generally, $x \cdot y = \overline{n} \to x = 0 \vee y \leq \overline{n}$ for each $n \in \mathbb{N}$.

25. The standard model $\mathbb{N}$ extends to an L_{PA}-structure $\mathbb{N}^\infty$ with domain $\mathbb{N} \cup \{\infty\}$, $\infty \notin \mathbb{N}$, so that $\mathbb{N}^\infty \vDash \mathsf{Q}$. Moreover, we are free to choose $(0 \cdot \infty)^{\mathbb{N}^\infty}$ in an arbitrary way (while the rest of the model is uniquely determined by the axioms of Q). Conclude that Q does not prove any of the formulas $S(x) \nleq x$, $x \cdot y = y \cdot x$, or $0 \cdot x \neq 1$.

26. Q does not prove $x + y = y + x$ or $0 + (x + y) = (0 + x) + y$.
[Hint: Modify the previous exercise to a model with two "infinities".]

27. All Σ_1-definable sets are semidecidable.

28. (Craig's trick.) Every semidecidable theory is recursively axiomatizable. [Hint: Express $\mathrm{Thm}(T)$ as $\exists y\, P(x, y)$ with P decidable. Given $x = \ulcorner \varphi \urcorner$ and y, devise a sentence equivalent to φ that encodes y.]

29. Show that every decidable consistent theory T has a decidable completion. [Hint: Consider a completion procedure that enumerates sentences φ one by one, and extends the current list of axioms with φ or $\neg\varphi$, whichever maintains consistency with T.]

In the next three exercises, you will develop an alternative sequence encoding scheme due to Edward Nelson.

30. The set $\{x : \exists n \in \mathbb{N}\, x = 2^n\}$ of powers of 2 is definable by a Δ_0 formula, not using the 2^n function. [Hint: Consider the divisors of x.]

31. Consider an encoding of finite sets $X \subseteq \mathbb{N}$ by pairs $\langle r, w \rangle$ where the binary expansion of r acts as a "ruler" with marks at positions of 1s, and the binary expansion of w is a concatenation of binary expansions of elements of X such that each element occupies the position between two ruler marks. Show that the predicate "x is in the set encoded by $\langle r, w \rangle$" is Δ_0-definable.

32. Construct a Δ_0 encoding of finite sequences based on the previous exercise.